Instituto Superior Técnico
Departamento de Matemática

Qualifying Exam
(Algebra)

Please solve 8 out of 10 problems

Exam lentgh: 4 hours.
Important Note: For each problem, even if you don't achive a complete solution, present any partial results you may have found.

1a. Let $p \geq 2$ be a prime number and let G be a group in which every element satisfies $g^p = 1$. Is G abelian?

1b. Find the Sylow subgroups of the symmetric group S_5. Show that S_5 has no subgroups of order 15 or 30.

1c. Recall that a group H is called an *extension* of G with kernel an abelian group A if there exists a short exact sequence of groups:

$$0 \longrightarrow A \longrightarrow H \xrightarrow{\pi} G \longrightarrow 0$$

We say that the extension *splits* if there exists a group homomorphism $\sigma : G \to H$ such that $\pi \circ \sigma$ is the identity homomorphism of G. Show that every extension of $\mathbb{Z}_2$ with kernel $\mathbb{Z}_3$ splits.

2a. Let $T : \mathbb{Q}^4 \to \mathbb{Q}^4$ be a $\mathbb{Q}$-linear transformation which, relative to some basis, is represented by the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

Find the Jordan normal form for T.

(see back cover)

2b. Give an example of a ring A such that A and $A \oplus A$ are isomorphic as A-modules.

2c. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} A = \{0\}$ for any abelian group A in which every element has torsion.

3a. Find the lattice of subgroups of the Galois group of the extension $\mathbb{Q}(\sqrt{-1}, \sqrt{5})$ of $\mathbb{Q}$. Find also generators for all intermediate extensions

$$\mathbb{Q}(\sqrt{-1}, \sqrt{5}) \subset E \subset \mathbb{Q}.$$

3b. Is the polynomial $p(x) = x^4 + x^3 + x^2 + x + 1$ irreducible over $\mathbb{Q}$? And over $\mathbb{Z}_{31}$?

4a. Let f_1 and f_2 be reduced polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ which are *not* multiples of each other. Show that the algebraic varieties in $\mathbb{C}^n$ determined by these polynomials are distinct.

(Recall that the algebraic variety in $\mathbb{C}^n$ determined by a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is

$$V(f) \equiv \{(z_1, \ldots, z_n) \in \mathbb{C}^n : f(z_1, \ldots, z_n) = 0\}$$

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4b. Let A be a noetherian ring (comutative with identity) and let $\phi : A \to A$ be a surjective homomorphism. Show that ϕ is an isomorphism.