Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
March 2009

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Let $p : X \to Y$ be a surjective closed map and $U \subset X$ be an open subset. Show that $\partial(p(\overline{U})) \subset p(\overline{U}) \cap p(X \setminus U)$.

(2) Let Y be a complete metric space, X be a compact metric space and $A \subset X$ a dense subset. Show that a continuous map $f : A \to Y$ extends to a continuous map from X to Y, if and only if f is uniformly continuous.

(3) Consider the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$. Let X_n denote the orbit space (with the quotient topology).

(a) Is X_n compact?
(b) Is X_n a manifold?

(4) Let $f : S^2 \to \mathbb{T}^2$ be a continuous map. Show that there is no continuous map $g : \mathbb{T}^2 \to S^2$ such that $f \circ g$ is homotopic to the identity map on $\mathbb{T}^2$.

(5) Suppose $f, g : S^n \to S^n$ are maps with $f(x) \neq g(x)$ for all $x \in S^n$.

(a) Show g is homotopic to $A \circ f$, where A is the antipode map on S^n.
(b) What is the relationship between the degrees of f and g?

(6) Suppose the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$ acts freely on S^3, with quotient $Y = S^3/Q$. Compute $\pi_1(Y)$ and $H_1(Y, \mathbb{Z})$.

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(7) Let ω be a closed 2-form on S^4. Prove that $\omega \wedge \omega$ vanishes at some point.

(8) Consider the real 4-manifold Z consisting of all 2×2 upper-triangular complex matrices with determinant 1. Discuss the existence on Z of n-forms which are closed but not exact.

(9) Let M be a smooth $2n$-dimensional orientable manifold and let $X = M \sharp \mathbb{C}P^n$ be the connected sum of M with $\mathbb{C}P^n$. [1] Compute the deRham cohomology of X in terms of the cohomology of M.

(10) Let $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ be a vector field and let D denote the 2-dimensional distribution on $\mathbb{R}^3$ given by the planes orthogonal to $\vec{F}$. Show that D is integrable if and only if $\nabla \times \vec{F}$ is orthogonal to $\vec{F}$.

(11) Consider the Riemannian metric

$$g = e^{Q(x)} \sum_{1 \le i \le n} dx^i \otimes dx^i$$

defined on $\mathbb{R}^n$, where $Q(x)$ is the quadratic form on $\mathbb{R}^n$ associated to a symmetric $n \times n$ matrix A.

(a) Calculate the explicit form of the gedesic equation for this metric.

(b) Calculate the Riemannian curvature tensor of this metric at the origin in $\mathbb{R}^n$.

(12) Let M be a manifold equipped with an affine connection ∇. Given a vector field X on M and $\gamma(t)$ an integral curve of X with $\gamma(0) = p$ let

$$P_{\gamma,t}\colon T_pM \to T_{\gamma(t)}M$$

denote parallel transport along γ.

Show that for any vector field Y,

$$\nabla_X Y(p) = \left.\frac{d}{dt}\right|_{t=0} P_{\gamma,t}^{-1}(Y(\gamma(t))).$$

[1] If M is a complex manifold then X is called the blow-up of M at a point.