Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Logic and Computation
Model - March 2009

Solve 8 of the 12 problems.
Justify all your answers.
Partial solutions will be graded.
Duration: 4 hours.

(1) Let

$$h = \lambda\, x, y\, .\, \begin{cases} 4y & \text{if } x \text{ is even} \\ 0 & \text{otherwise} \end{cases} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

and

$$\Psi = \lambda\, f\, .\, (\lambda\, x\, .\, f(h(x, x))) : \mathcal{F}_1 \to \mathcal{F}_1.$$

Show that :

(a) h is Turing computable;

(b) there is a computable $s : \mathbb{N} \to \mathbb{N}$ such that $\Psi(\phi_k) = \phi_{s(k)}$.

(2) Sketch the proof of existence of computable universal functions with the *s-m-n* property.

(3) Let $C, D \subseteq \mathbb{N}$. Show that if C is computable enumerable and D productive then $C^{\mathsf{c}} \leq_{\mathsf{m}} D$. Explain how this result is used in proving Myhill's theorem.

(4) Establish a polynomial-time reduction from SAT to 3-SAT.

(5) State and prove the metatheorem of deduction. Illustrate its repeated application in a formal derivation of

$$(\forall x\, (\varphi \Rightarrow \psi)) \Rightarrow ((\forall x\, \varphi) \Rightarrow (\forall x\, \psi)).$$

(continues in the next page)

(6) State and prove the cut elimination theorem for quantifier free formulas.

(7) Describe the main steps of Henkin's construction. Prove the main lemma.

(8) Using Vaught's test, show that the theory of dense orders without left endpoint has quantifier elimination.

(9) Show that $\lambda k \,.\, k+1$ is representable in the theory $\mathsf{Th}(\mathbb{N})$. Explain the role of representability in incompleteness.

(10) State and prove Church's theorem and explain how it can be used for obtaining Gödel's first incompleteness theorem.

(11) Let Θ be a computably enumerable and true theory of arithmetic where computable maps are representable. Show that $\alpha \in \Theta$ iff $\Box_\Theta \alpha \in \mathsf{Th}(\mathbb{N})$. What can be said about $\Box_\Theta \alpha \in \Theta$?

(12) Prove or refute the following assertion. If every element of a set is constructible, then the set is included in a constructible set.