QUALIFICATION EXAM IN MATHEMATICS
(MATHEMATICAL ANALYSIS)

October 27, 2006

Solve, at least partially, 8 of the 10 proposed problems. Duration: 4 hours

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1. Answer "Correct" or "Incorrect" in each proposed item (with justification in the case "Incorrect"):

   (a) If f is a continuous function from a compact metric space X into a metric space Y, then f is uniformly continuous.

   (b) If f is a continuous function from a metric space X into a compact metric space Y, then its image $f(X) = \{f(x) : x \in X\}$ is compact.

   (c) If $A : X \to Y$, $B : Y \to Z$ are Fredholm operators acting in Banach spaces, then $BA : X \to Z$ is also Fredholm and

   $$\texttt{ind}\, BA \leq \texttt{ind}\, A + \texttt{ind}\, B.$$

   (d) Let $\mathcal{A}$ be a unital C*-algebra and $a \in \mathcal{A}$ normal. Then a is right invertible iff a is left invertible.

   (e) Let $f : \Omega \to \mathbb{C}$ be holomorfic in a domain $\Omega \subset \mathbb{C}$ which contains $\overline{B_r(a)} = \{x \in \mathbb{C} : |x - a| \leq r\}$ where $a \in \mathbb{C}, r > 0$. Then

   $$|f(a)| \leq \sup\{|f(z)| : |z - a| < r\}.$$

   (f) If $u : \Omega \to \mathbb{R}$ is a function defined in a domain $\Omega \subset \mathbb{C}$ such that, for every $z \in \Omega$, there exists a sequence $r_n \to 0$ which satisfies

   $$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z + r_n e^{it})\, dt\,,\ z \in \Omega\,,\ n = 1, 2, 3, ...,$$

   then $\Delta u = 0$ in Ω .

2. Formulate the Theorem of Tietze in metric spaces (or even in normal spaces) and explain the idea of its proof.

3. Formulate the Theorem of Radon-Nikodym in a measure space $(X, \mathcal{M}, \mu)$ explaining the meaning of occurring terms and symbols.

4. Give an example of a function defined in $\mathbb{R}^2$ which is Lebesgue integrable but not Riemann integrable and justify your answer. Conversely: Do Riemann integrable functions exist (in the improper sense) which are not Lebesgue integrable?

5. Formulate the Cauchy Integral Formula for functions with values in a Banach space explaining the meaning of occurring terms and symbols.

6. Let X, Y be Banach spaces and $A : X \to Y, B : Y \to X$ two bounded linear operators. Show that the compositions AB e BA are compact if A or B is compact. Is the inverse conclusion valid, as well?

7. Decide which of the following spaces are (a) separable, (b) compact, (c) reflexive:

$$\begin{aligned} l^1 &= \{(x_n) : \sum |x_n| < +\infty\}, \\ l^\infty &= \{(x_n) : \sup |x_n| < +\infty\}, \\ c_0 &= \{(x_n) : x_n \to 0 \texttt{ if } n \to +\infty\} \ ? \end{aligned}$$

8. Formulate the Schwarz Reflection Principle and give a sketch of its proof.

9. Decide which of the following functions are analytical in the point $z_0 = 0$ and determine the Taylor series, if existing, in the same point:

   (a) $f(z) = 1 - \frac{1}{e^{z^2}}$,

   (b) $g(z) = 1 - e^{-1/z^2}$, if $z \neq 0$ e $g(0) = 1$,

   (c) $h(z) = \log(z^2 - 1) = \log|z^2 - 1| + i \arg(z^2 - 1)$,

   where $\arg \zeta \in [0, 2\pi[$ for $\zeta = z^2 - 1$.

10. Give an example of a meromorphic function F such that:

    (a) F has a unique singularity in $z = 1 + i$, which is a pole of order 2,

    (b) $|F(z)| = 1$ for $|z| = 1$,

    (c) $|F(z)| = \mathcal{O}(z^{-m})$ as $z \to \infty$ where m is a given integer.