QUALIFICATION EXAM IN MATHEMATICS (MATHEMATICAL ANALYSIS)

April 28, 2005

Solve, at least partially, 8 of the 10 proposed problems. Duration: 4 hours

---

1. Answer "Correct" or "Incorrect" in each proposed item (with justification in the case "Incorrect"):

   (a) Let f be a continuous mapping from a metric space into a compact metric space. Then f is uniformly continuous.

   (b) If X is a measure space and $f : X \to \mathbb{C}$ measurable, then $|f|$ is measurable as well.

   (c) Let E be a Banach space and F a linear subspace. Then F is closed if and only if there exists a bounded linear operator $P : E \to E$ such that

   $$P^2 = P \ , \ \text{im}P = F$$

   (d) If $T : H \to H$ is a self-adjoint compact operator in a Hilbert space, then $\lambda = \|T\|$ is an eigenvalue of T.

   (e) If a complex function is bounded and holomorphic in $\mathbb{C}$ then it is constant.

   (f) For any open ball in $\mathbb{C}$ there is a conformal mapping onto the half-plane $\Re ez > 0$.

2. Formulate the Egoroff Theorem about convergence of sequences of measurable functions explaining the meaning of terms and symbols used in this formulation.

3. Give an example of a bounded closed subset of l^∞ that is not sequentially compact and justify your answer. Could it be that this set contains a subset which is not totally bounded?

4. Prove the following version of the Fundamental Theorem of Calculus: If f is a complex-valued function of a real variable and Lebesgue integrabel in the interval $]a, b[\subset \mathbb{R}$, with constant $a < b$ and $c \in \mathbb{C}$, then the indefinite integral

   $$g(x) = \int_a^x f(t)dt + c \ \ , \ \ x \in [a, b]$$

is differentiable a.e. and $g'(x) = f(x)$ holds a.e.

Suggestion: Consider real functions, a decomposition into the positive and the negative part and use the monotony of the indefinite integrals of these parts.

5. Formulate the Baire Theorem and explain its role in Functional Analysis through an application (without giving a detailed proof).

6. Give three examples of bounded linear operators $A : L^p(\Omega) \to L^p(\Omega)$, $\Omega = [0, 1]$, $p \in [1, \infty[$ such that the spectrum of A is

$$\sigma(A) = \{0\}, \{0, 1\} \text{ or } [0, 1],$$

respectively.

7. For $1 \leq p < \infty$ and $x = (x_1, ..., x_n) \in \mathbb{C}^n$ define

$$\|x\|_p = (\sum_{j=1}^{n} |x_j|^p)^{1/p}$$

$$\|x\|_\infty = \sup\{|x_j| : 1 \leq j \leq n\}$$

Show that all these norms are equivalent. For $1 \leq p, q \leq \infty$, determine constants c e C such that, for each $x \in \mathbb{C}^n$,

$$c\|x\|_p \leq \|x\|_q \leq C\|x\|_p \quad .$$

8. Explain the principal idea of the Functional Calculus of Riesz: Convergence of Mac-Laurin series with argument in a unital Banach algebra, the Cauchy Theorem and the Cauchy Integral Formula in the corresponding scenery.

9. Formulate the Maximum Modulus Principle for holomorphic functions and give an example of its application.

10. Determine the values of the complex integrals

$$I_1 = \int_\gamma \frac{dz}{z^4 + 1} \quad , \quad I_2 = \int_\gamma \frac{dz}{(z^2 + 1)^2}$$

where γ denotes a semi-circle

$$\gamma = \{z \in \mathbb{C} : \Re ez \in [-R, R]\} \cup \{z \in \mathbb{C} : |z| = R, \Im mz > 0\}$$

with R > 1.