QUALIFYING EXAM IN MATEMATICAL ANALYSIS

Model Exam

Duration: 4 hours

I. Real Analysis

Solve 4 of the 5 proposed problems.

1. Formulate the Theorem of Tietze in metric spaces (or even in normal spaces) and explain the idea of its proof.

2. Give an example of a bounded closed subset of l^∞ that is not sequentially compact and justify your answer. Could it be that this set contains a subset which is not totally bounded?

3. Formulate the Theorem of Radon-Nikodym in a measure space $(X, \mathcal{M}, \mu)$ explaining the meaning of occurring terms and symbols.

4. Prove the following version of the Fundamental Theorem of Calculus: If f is a complex-valued function of a real variable and Lebesgue integrabel in the interval $]a, b[\subset \mathbb{R}$, with constant $a < b$ and $c \in \mathbb{C}$, then the indefinite integral

$$g(x) = \int_a^x f(t)dt + c \quad , \quad x \in [a, b]$$

is differentiable a.e. and $g'(x) = f(x)$ holds a.e.

5. Give an example of a function defined in $\mathbb{R}^2$ which is Lebesgue integrable but not Riemann integrable and justify your answer. Conversely: Do Riemann integrable functions exist (in the improper sense) which are not Lebesgue integrable?

II. Functional Analysis

1. H denotes a complex separable Hilbert space with inner product $<\cdot,\cdot>$; $L(H)$ represents the space of bounded linear operators defined and with values on H and $L_0(H)$ denotes the linear subspace of H of compact operators.

   i. Given a set of vectors $\Phi = \{\varphi_n : n \in J \subset \mathbb{N}\} \subset H$ is it possible to obtain another set of vectors $\Psi = \{\psi_n : n \in J\} \subset H$ such that

   $$<\psi_j, \varphi_k>= \delta_{jk} \quad \forall_{j,k \in J} \; ?$$

   ii. Show that H is a reflexive Banach space.

   iii. a) Show that any operator $K \in L_0(H)$ is the limit (in $L(H)$) of a sequence of finite rank operators. [Sugestion: Use the spectral representation of K^*K, where K^* denotes the adjoint of K.]

   b1) If $K \in L_0(H)$ is a positive operator, show that $\sigma_p(T) \subset [0, \infty)$ and that the equation $L^2 = K$ has a unique solution with $L \in L_0(H)$ positive.

   b2) Suppose that $T \in L_0(H)$ is selfadjoint. Show that T can be represent as $T = T_+ - T_-$, where T_+ and T_- are positive compact operators such that $T_+T_- = T_-T_+$.

   (*Choose only one of the last two itens).*

2. Let X be a Banach space for the norm $||\cdot||_X$; $L(X)$ represents the space of bounded linear operators defined and with values on X and $L_0(X)$ denotes the linear subspace of $L(X)$ of all compact operators.

   i. For $K \in L_0(X)$ is it possible that K is invertible in $L(X)$?

   ii. Show that $L_0(X)$ is a closed ideal of $L(X)$.

   iii. Let $(T_n)_{n\in\mathbb{N}} \subset L(X)$ be such that $T_n x$ converges (in X) for each $x \in X$. Show that there exist $T \in L(X)$ such that $T_n x$ converges to Tx, for each $x \in X$. Does this results holds true if X is not complete?

   iv. Let $T \in L(X)$, $(T_n)_{n\in\mathbb{N}} \subset L(X)$ and $K \in L_0(X)$. Show that, if $||T_n x - Tx||_X \to 0$ for each $x \in X$, then $||T_n K - TK||_{L(X)} \to 0$.

III. Complex Analysis

Solve 2 of the 3 proposed problems.

1. Let $T(z)$ be a Möbius transformation with a unique fixed point on the Riemann sphere.

   i. If this fixed point is $\alpha \in \mathbb{C}$, show that there exists $\beta \in \mathbb{C} \setminus \{0\}$ such that

   $$\frac{1}{T(z) - \alpha} = \frac{1}{z - \alpha} + \beta, \qquad \forall z \in \mathbb{C} \setminus \{\alpha\}.$$

   ii. Writing $T(z)$ as $\dfrac{az+b}{cz+d}$, show that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a unique eigenvector and the corresponding eigenspace is one-dimensional.

2. Consider the entire function $f(z) = z^{n+1} - e^{\frac{1}{2}z-1}$, for an integer $n \geq 1$.

   i. Show that f has $n+1$ zeros (counted with their multiplicities) in the annulus $\left\{z \in \mathbb{C} : \frac{1}{2} < |\, z| < 1\right\}$ (note that $e^{-\frac{5}{4}} > \frac{1}{4}$).

   ii. Compute the index around the origin of the path given by $\gamma(t) = \frac{f(e^{it})}{e^{2it}}$, $t \in [0, 4\pi]$.

3. Let g be a conformal map from the region $\Omega = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| < 2\}$ to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

   i. Determine an explicit expression for g.

   ii. Let $z_0 \in \Omega$ and let h be another conformal map $h : \Omega \to \mathbb{D}$. If $h(z_0) = g(z_0) = 0$, what is the general expression for $g \circ h^{-1} : \mathbb{D} \to \mathbb{D}$?