Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
March 2008

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) A topological space X is said to be *locally path-connected* if each point $p \in X$ belongs to a path-connected open set. Show that if X connected and locally path-connected then it is path-connected.

(2) Let (X, d) be a complete metric space, and $f : X \to X$ a continuous surjective map such that

$$d(f(p), f(q)) \geq \alpha\, d(p, q)$$

for some $\alpha > 1$. Show that f has exactly one fixed point. Show also that X cannot be compact.

(3) Show that any finite group is the fundamental group of some compact differentiable manifold. (Hint: You may want to use the fact that $SU(2n)$ is simply connected).

(4) Show that there are no continuous maps $f : S^2 \to S^1$ satisfying $f(-p) = -f(p)$.

(5) Let M be a connected differentiable manifold and $N \subset M$ a connected embedded submanifold of codimension $k \geq 2$. Show that $M \setminus N$ is connected.

(6) For each $n \in \mathbb{Z}$ give an example of a smooth map $f_n : M \to M$ with degree n, where:

(a) $M = S^2$;
(b) $M = S^3$;
(c) $M = S^1 \times S^1$.

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(7) Let $\Phi : G \to H$ be a surjective Lie group homomorphism whose derivative at the identity $d_e\Phi : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism. Show that if G is connected then Φ is a covering map.

(8) Let M and N be differentiable manifolds with dimensions m and n, respectively, and ω a closed m-form on N. Show that if two smooth maps $f, g : M \to N$ are homotopic by a smooth homotopy then $\int_M f^*\omega = \int_M g^*\omega$.

(9) Let M be a compact differentiable manifold with dimension $n \geq 2$. A *Lorentzian metric* on M is a symmetric 2-tensor field which determines on each tangent space a quadratic form equivalent to $\operatorname{diag}(-1, 1, \ldots, 1)$. Show that M admits a Lorentzian metric *iff* $\chi(M) = 0$.

(10) Let (M, g) be a homogeneous Riemannian manifold (i.e. such that the isometry group acts transitively). Show that (M, g) is geodesically complete (that is, every geodesic is defined in $\mathbb{R}$).

(11) In appropriate units, the horizon of a maximally spinning black hole is the sphere S^2 with the Riemannian metric given in the usual spherical coordinates (θ, φ) by

$$g = (1 + \cos^2\theta)\, d\theta \otimes d\theta + \frac{4 \operatorname{sen}^2\theta}{1 + \cos^2\theta}\, d\varphi \otimes d\varphi.$$

(a) Compute the Gauss curvature of the horizon.

(b) Show that it is not possible to isometrically embed the horizon in $\mathbb{R}^3$ as a surface of revolution (you may use, without proof, the fact that $\frac{\partial}{\partial\varphi}$ is the only Killing vector field of the horizon).

(12) Let S be a compact surface with boundary whose boundary is the disjoint union of $k \geq 3$ geodesic circles. Show that there exists a point in S with negative Gauss curvature.