Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
October 2007

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Let X be a Hausdorff topological space. Show that if $K \subset X$ is compact then K is closed. Give an example of a topological space X and a compact subset $K \subset X$ such that K is not closed.

(2) Prove the Heine-Borel Theorem: the compact subsets of $\mathbb{R}^n$ (with the usual topology) are the closed bounded subsets.

(3) Let $\mathbb{C}P^2$ be the complex projective plane.

  (a) Show that $\pi_1(\mathbb{C}P^2) = \{e\}$;
  (b) Compute $H_*(\mathbb{C}P^2; \mathbb{Z})$.

(4) Let $M = \mathbb{R}^3 \setminus (C_1 \cup C_2)$, where

$$C_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\};$$
$$C_2 = \{(0, y, z) \in \mathbb{R}^3 \mid (y-1)^2 + z^2 = 1\}.$$

Compute $H_*(M; \mathbb{Z})$.

(5) Let M and N be n-dimensional compact connected differentiable manifolds, and $f : M \to N$ an immersion.

  (a) Show that $f : M \to N$ is a covering map.
  (b) Show that there are no immersions $f : S^1 \times S^1 \to S^2$.

(6) Let $f : S^d \to S^d$ be a diffeomorphism. Show that:

  (a) If f has no fixed points then f is homotopic to the antipodal map.
  (b) If d is even and f preserves orientations then f has at least one fixed point.

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(7) Let G be a Lie group, $X \in \mathfrak{X}(G)$ a left-invariant vector field and $Y \in \mathfrak{X}(G)$ a right-invariant vector field. Show that X and Y commute (i.e. $[X, Y] = 0$).

(8) Let M be an n-dimensional differentiable manifold, $\{X_1, \ldots, X_n\}$ a local frame and $\{\omega^1, \ldots, \omega^n\}$ the dual co-frame. Show that

$$d\omega^i + \frac{1}{2} \sum_{j,k=1}^{n} C^i_{jk} \omega^j \wedge \omega^k = 0,$$

where the n^3 structure functions C^i_{jk} are defined by

$$[X_j, X_k] = \sum_{i=1}^{n} C^i_{jk} X_i.$$

(9) Show that there are no rank 1 distributions on S^2.

(10) Determine all the affine connections on $\mathbb{R}^n$ whose geodesics are the affine maps $c : \mathbb{R} \to \mathbb{R}^n$ (i.e. $c(t) = at + b$ for $a, b \in \mathbb{R}^n$).

(11) Compute the Gauss curvature of $\mathbb{R}^2$ with the metric

$$g = \frac{1}{\cosh^2(y)} \left(dx \otimes dx + dy \otimes dy \right).$$

What is the relation of this Riemannian manifold with S^2 (equipped with the standard metric)?

(12) Let (M, g) be a 2-dimensional compact orientable Riemannian manifold whose Gauss curvature K is everywhere positive. Prove that any two geodesics whose images are simple closed curves must intersect.