Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
March 2007

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Let X be a topological space and $\sim$ an equivalence relation on X such that the natural projection $\pi : X \to X/\sim$ is an open map. Show that the quotient space $X/\sim$ is Hausdorff *iff* the set

$$R = \{(p, q) \in X \times X \mid p \sim q\}$$

is closed.

(2) (a) Show that there exists a metric d on $\mathbb{R}^n$ determining the usual topology but such that $(\mathbb{R}^n, d)$ is not complete.
(b) Show that there exists a metric δ on $\mathbb{R}^n \setminus \{0\}$ determining the usual topology but such that $(\mathbb{R}^n \setminus \{0\}, \delta)$ is complete.

(3) For each $n \in \mathbb{N}$ indicate a manifold whose fundamental group is $\mathbb{Z}_n$.

(4) Show that $S^m \times S^n$ is homeomorphic to $S^p \times S^q$ *iff* $m = p$ and $n = q$ or $m = q$ and $n = p$.

(5) Compute $H_*(S^3 \setminus (F_1 \cup F_2))$, where F_1 and F_2 are two distinct fibers of the Hopf fibration.

(6) Let M be a simply connected n-dimensional differentiable manifold, and $N \subset M$ a closed embedded $(n-1)$-dimensional submanifold. Show that if N is connected and orientable then $M \setminus N$ has two connected components.

(7) Let M be a compact 2-dimensional manifold with boundary, with ∂M diffeomorphic to S^1. Let $\omega \in \Omega^1(M)$ be a 1-form whose restriction to ∂M is the 1-form $\sigma = \cos\theta d\theta$, where θ is the usual angular coordinate in S^1. Show that $d\omega$ has at least one zero in $M \setminus \partial M$.

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(8) Compute the degree of the map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ given by

$$f\left(\left[z^0, \ldots, z^n\right]\right) = \left[(z^0)^k, \ldots, (z^n)^k\right] \qquad (k \in \mathbb{N}).$$

(9) Let M be a 2-dimensional differentiable manifold, $\alpha \in \Omega^1(M)$ a 1-form and $p \in M$ such that $\alpha_p \neq 0$. Show that there exists an open set $U \ni p$ and functions $f, g \in C^\infty(U)$ such that $\alpha|_U = f dg$.

(10) Let $S \subset \mathbb{R}^3$ be a revolution surface. A *parallel* of S is the intersection of S with a plane which is perpendicular to the revolution axis.

(a) Let $\gamma : I \subset \mathbb{R} \to S$ be a geodesic, $r : \mathbb{R}^3 \to \mathbb{R}$ the function distance to the revolution axis and $\beta(t)$ the angle between the geodesic and the parallel through $\gamma(t)$. Prove the *Clairaut relation*:

$$r(\gamma(t)) \cos(\beta(t)) = \text{constant}.$$

(b) Use the Clairaut relation to show that any geodesic of a revolution ellipsoid either is periodic or has an infinite number of self-intersections.

(11) Prove Schur's Theorem: any connected isotropic Riemannian manifold with dimension $n \geq 3$ has constant curvature.

(12) Show that any compact connected 2-dimensional Lie group G admits a flat metric. Conclude that G is diffeomorphic to the torus T^2.