Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
February 2006

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Let X be a topological space. A map $f : X \to \mathbb{R}$ is said to be *upper semicontinuous* if it is continuous for the topology

$$\mathcal{O} = \{\,]-\infty, a\,[: a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$$

of $\mathbb{R}$. Show that if X is compact and f is upper semicontinuous then f has a maximum. Must f also have a minimum?

(2) Let (X, d) be a metric space and H the set of closed, bounded, nonempty subsets of X. The *Hausdorff metric* on H is the map $d_H : H \times H \to \mathbb{R}$ given by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\}$$

Show that if (X, d) is complete then (H, d_H) is also complete.

(Hint: If $\{A_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, assume without loss of generality that $d_H(A_n, A_{n+1}) < \frac{1}{2^n}$, and consider the closure of the set of limits of sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \in A_n$ and $d(a_n, a_{n+1}) < \frac{1}{2^n}$).

(3) Given that the binary icosahedral group $I \subset S^3$ is perfect (i.e. coincides with its commutator subgroup), indicate a differentiable manifold M such that $H_1(M; \mathbb{Z}) = \{0\}$ but $\pi_1(M) \neq \{e\}$.

(4) Show that $\chi(M \times N) = \chi(M)\chi(N)$ for any two compact differentiable manifolds M and N.

(5) Let M be a compact non-orientable 3-dimensional differentiable manifold. Show that $H^1_{dR} \neq \{0\}$.

(6) Let Σ be a compact surface and $f : \Sigma \to S^2$ an immersion. Show that f is a diffeomorphism.

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(7) Let M be a compact differentiable manifold and $\omega \in \Omega^1(M)$ a closed 1-form without zeros. Show that $H^1_{dR}(M) \neq \{0\}$.

(8) Show that any smooth map $f : S^d \to T^d$ has degree zero if $d \geq 2$ ($T^d = S^1 \times \ldots \times S^1$ stands for the d-dimensional torus). Give an example of a map $g : T^2 \to S^2$ with nonzero degree.

(9) Use the Frobenius Theorem to show that any closed 1-form is locally exact.

(10) If $(G, \langle \cdot, \cdot \rangle)$ is a Lie group with a bi-invariant metric then the exponential map $\exp : \mathfrak{g} \to G$ coincides with the geodesic exponential $\exp_e : T_eG \to G$.

   (a) Show that if X, Y are left-invariant vector fields then $\nabla_X Y = \frac{1}{2}[X, Y]$, where ∇ is the Levi-Civita connection.

   (b) Show that if X, Y, Z are left-invariant vector fields then $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.

(11) (a) Compute the Gauss curvature of a surface of revolution in $\mathbb{R}^3$.

   (b) Determine all the surfaces of revolution with constant Gauss curvature.

(12) Let $\Sigma \subset \mathbb{R}^3$ be an embedded, connected, compact 2-dimensional submanifold (hence orientable). Let $g : \Sigma \to S^2$ be the Gauss map for a given choice of orientation (i.e. $g(p)$ is the unit normal to Σ at point p compatible with the orientation). Show that the degree of g is $\frac{1}{2}\chi(M)$.