Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
May 2005

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Let X be a topological space and $A \subset X$. Show that if $C \subset X$ is connected and C intersects A and $X \setminus A$ then C also intersects the boundary of A.

(2) Let X be a Hausdorff, locally compact topological space. Show that X is completely regular, i.e., show that if $F \subset X$ is closed and $x \in X \setminus F$ then there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

(3) Let $C_n = \{e^{\frac{2k\pi i}{n}} \mid k = 0, \ldots, n-1\}$ be the group of the n-roots of the unit and let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\}$ be the 3-sphere. For the action of C_n on S^3 determined by the formula

$$\alpha \cdot (z_1, z_2) = (\alpha z_1, \alpha z_2),$$

compute the fundamental group of the orbit space $L_n = S^3/C_n$ (with the quotient topology).

(4) Determine the homology groups of the topological subspace $A \subset \mathbb{R}^3$ given by:

$$A = \left\{(x, y, z) \in \mathbb{R}^3 \mid x\left((\sqrt{x^2+y^2} - 2)^2 + z^2 - 1\right) = 0\right\}.$$

(5) Let G be an abelian group and $k > 1$ an integer. Show that there exists a topological space X with homology group $H_k(X) = G$.

(6) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth map which is homogeneous of degree $k \neq 0$ (i.e., $f(tx) = t^k f(x)$ for all $t > 0$, $x \in \mathbb{R}^n$).

(a) Show that if $a \neq 0$ and $f^{-1}(a) \neq \varnothing$ then $f^{-1}(a)$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.
(b) Show that if $ab > 0$ then $f^{-1}(a)$ is diffeomorphic to $f^{-1}(b)$.

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(7) Let M be a simply connected manifold. Show that M is orientable.

(8) Let M be a differentiable manifold and $X, Y \in \mathfrak{X}(M)$ complete vector fields.

   (a) Show that if the flows of X and Y commute then $[X, Y] = 0$.

   (b) Show that if G is an abelian Lie group then its Lie algebra $\mathfrak{g}$ is abelian (i.e., $[v, w] = 0$ for all $v, w \in \mathfrak{g}$).

(9) Let M be a compact, orientable manifold with boundary. Show that there are no smooth maps $f : M \to \partial M$ such that $f|_{\partial M} = \text{id}$.

(10) Consider the sets

$$S^3 = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 2\right\};$$
$$T^2 = \left\{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = z^2 + w^2 = 1\right\}.$$

Check that $S^3 \setminus T^2$ has two connected components. If M is one of these connected components and

$$\omega = z dx \wedge dy \wedge dw - x dy \wedge dz \wedge dw,$$

compute $\int_M \omega$ for your choice of orientation of M.

(11) Prove that any Lie group has a unique connection ∇ with zero curvature such that the left-invariant vector fields are parallel. Compute the torsion tensor for this connection.

(12) Let (M, g) be a 2-dimensional Riemannian manifold, and $\Delta \subset M$ a *geodesic triangle*, i.e., an open set homeomorphic to a disk whose boundary is contained in the union of the images of three geodesics. Let α, β, γ be the internal angles of Δ, i.e., the angles between the geodesics at the intersection points in the boundary of Δ. Prove that

$$\alpha + \beta + \gamma = \pi + \int_\Delta K,$$

where K is the Gauss curvature of M.