Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

DOCTORAL PROGRAM IN MATHEMATICS

QUALIFYING EXAM – GEOMETRY AND TOPOLOGY

30 SEPTEMBER 2003

solve 8 of the 12 proposed problems

present and justify all computations

write partial solutions, even when you cannot complete them

duration: 4 hours

(1) Let E be a topological space and $\Delta = \{(a,a) : a \in E\}$ the diagonal of E^2. Prove that Δ is closed in E^2 if and only if E is Hausdorff.

(2) Let X and Y be metric spaces with distance functions $d_X : X \times X \to \mathbb{R}$ and $d_Y : Y \times Y \to \mathbb{R}$. Let $f : X \to Y$ be a map such that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \ , \quad \forall x_1, x_2 \in X \ .$$

Show that f is a homeomorphism onto its image.

(3) Let X be a n-dimensional compact manifold, and let $Y \subset X$ be a closed m-dimensional submanifold. Show that the Euler characteristic $\chi(X \setminus Y)$ of the complement of Y in X is given by

$$\chi(X \setminus Y) = \chi(X) + (-1)^{n-m-1}\chi(Y) \ .$$

(4) Prove Brouwer's fixed point theorem: any continuous map of the closed n-dimensional disk $D^n \subset \mathbb{R}^n$ to itself must have a fixed point.

(5) Prove that the set of all 2×2 matrices with rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4$.

(6) Show that the real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd. (Hint: show that the antipodal map on the sphere S^n preserves orientations if and only if n is odd.)

(continues)

(7) A symplectic form on a $2n$-dimensional manifold is a closed 2-form ω whose n-th exterior power ω^n is a volume form. Show that no sphere S^{2n} with dimension $2n > 2$ admits symplectic forms.

(8) Let M be a n-dimensional compact manifold, and let $f : M \to \mathbb{R}^n$ be a smooth map. Show that f cannot be everywhere regular.

(9) Let M be a connected manifold, and let $\pi : M \times N \to N$ be the natural projection. Prove that a p-form β on $M \times N$ is given by $\pi^*\alpha$ for some p-form α on N if and only if $\imath_X \beta = 0$ and $\mathcal{L}_X \beta = 0$ for all vector fields X on $M \times N$ satisfying $d\pi(X) = 0$ everywhere.

(10) Let

$$\alpha = \frac{1}{2\pi} \cdot \frac{x\,dy - y\,dx}{x^2 + y^2} .$$

Prove that α is a closed 1-form on $\mathbb{R}^2 \setminus \{0\}$. Compute the integral of α over the unit circle S^1. How does the result show that α is not exact? And how does it show that $i^*\alpha$ is not exact, where $i : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is the usual embedding?

(11) Let N be a n-dimensional Riemannian manifold with Levi-Civita connection ∇. Let M be a hypersurface (i.e., a $(n-1)$-dimensional embedded submanifold) in N with unit normal vector field Y, and let X be any vector field tangent to M. Prove that the covariant derivative $\nabla_X Y$ is tangent to M.

(12) Let ∇ and $\widetilde{\nabla}$ be connections on a differentiable manifold M, and define

$$B(X,Y) = \nabla_X Y - \widetilde{\nabla}_X Y .$$

Show that B is tensorial, i.e., $B(X,Y)(p)$ depends only on the values X_p and Y_p of the vector fields at point p.
Show that ∇ and $\widetilde{\nabla}$ have the same geodesics if and only if $B(X,X) = 0$ for all vector fields X.