Instituto Superior Técnico
Departamento de Matemática

PhD Program in Mathematics
Qualifying Exam - Geometry and Topology
March 2004

solve 8 of the 12 problems
justify all your answers
partial solutions will be considered
duration: 4 hours

(1) Let X be a topological space, $U \subset X$ an open set and $K \subset X$ a compact set. Show that if $C \subset X$ is a closed, connected set, which is Hausdorff for the subspace topology, and $C \cap U = C \cap K \neq \emptyset$, then $C \subset U$.

(2) Let X be a compact metric space and $F \subset X$ a closed subset. Let $p : X \to Y$ a local homeomorphism such that the restriction $p|_F$ is injective. Show that there exists a neighborhood U of F such that $p|_U$ is injective.

(3) Describe the universal cover of the Möbius strip (without boundary), as well as the action of the group of deck transformations.

(4) Let M and N be connected d-dimensional manifolds. Let $M\#N$ be the connected sum of M and N. Check that the Euler characteristics of M, N and $M\#N$ are related by:

$$\chi(M\#N) = \chi(M) + \chi(N) - \chi(\mathbb{S}^d).$$

Recall that $M\#N$ is the manifold obtained by gluing M and N along the boundary of open sets diffeomorphic to the ball $\{x \in \mathbb{R}^d : \|x\| < 1\}$.

(5) Compute, for each $n \in \mathbb{N}$, the homology groups $H_*(\mathbb{T}^n; \mathbb{Z})$ of the n-dimensional torus $\mathbb{T}^n$.

(6) Let V be an n-dimensional vector space and consider the set $Gr(n,k)$ formed by the k-dimensional subspaces of V. Show that $Gr(n,k)$ is a $k(n-k)$-dimensional manifold.

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(7) Let M and N be manifolds. If $M \times N$ is orientable, what can you say about the orientability of M and N?

(8) Let $p : M \to N$ be a differentiable map between two compact, connected, orientable manifolds with the same dimension. Show that if p has nonzero degree then p is surjective.

(9) Let $a_1(x), \dots, a_n(x)$ be smooth functions in $\mathbb{R}^n$ and let (x, z) be coordinates in $\mathbb{R}^{n+1}$, with $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Consider the n-dimensional distribution in $\mathbb{R}^{n+1}$ generated by the vector fields

$$X_i = \frac{\partial}{\partial x_i} + a_i(x)\frac{\partial}{\partial z} \,, \qquad i = 1, \dots, n \,.$$

Give conditions for this distribution to be involutive.

(10) Let M be a n-dimensional manifold and $f \in C^\infty(M)$ a function with a critical point $p \in M$. Given two vectors $X, Y \in T_pM$, let $\widetilde{X}$ and $\widetilde{Y}$ be vector fields defined on a neighborhood of p which extend X and Y. Show that the map (called the *Hessian* of f at p)

$$\begin{array}{rccc} H : & T_pM \times T_pM & \longrightarrow & \mathbb{R} \\ & (X, Y) & \mapsto & \left(L_{\widetilde{X}}(L_{\widetilde{Y}} f)\right)(p) \end{array}$$

is well defined (i.e. is independent of the choice of extensions of X and Y), and is bilinear and symmetric. Let $(\mathcal{U}, x_1, \dots, x_n)$ be a coordinate chart for M centered at p. Show that the matrix that represents H in the basis of T_pM induced by these coordinates is the matrix of the second order partial derivatives

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right) \,.$$

(11) Let ∇ be a connection on a differentiable manifold M. Show that there exists a connection $\widetilde{\nabla}$ on M with zero torsion which has the same geodesics as ∇.

(12) Let (M, g) be a Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function. The *gradient* of f is the vector field $\nabla f \in \Gamma(TM)$ defined by the formula

$$g(\nabla f, X) = df(X) \,, \ \forall X \in \Gamma(TM) \,.$$

Show that if $\|\nabla f\| \equiv 1$ then the integral curves of ∇f are geodesics.