Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

DOCTORAL PROGRAM IN MATHEMATICS

QUALIFYING EXAM – GEOMETRY AND TOPOLOGY

MODEL – JANUARY 2003

solve 8 of the 12 proposed problems

present and justify all computations

write partial solutions, even when you cannot complete them

duration: 4 hours

(1) Let $A \subset \mathbb{R}^n$ be an open subset. Justify that any connected component of A is an open set, and that the set of connected components of A is countable.

(2) Justify that the collection of subsets of $\mathbb{R}$

$$\mathcal{T} = \{]a, +\infty[: -\infty \leq a \leq +\infty\}$$

is a topology in $\mathbb{R}$. Characterize the compact sets in that topology and show that the intersection of two compact sets may fail to be a compact set.

(3) Provide a proof or a counterexample for the following statement:
Two topological spaces with the same first singular homology group necessarily have the same fundamental group. (We identify isomorphic groups.)

(4) For any integers k and n with $1 \leq k \leq n$, let

$$S^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

be the n-dimensional sphere, and let $D^k \subset \mathbb{R}^{n+1}$ be the closed disk

$$D^k = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_k^2 \leq 1\,;\, x_{k+1} = \ldots = x_{n+1} = 0\} \subset \mathbb{R}^{n+1}\ .$$

Let $X_{n,k} = S^n \cup D^k$ be their union. Compute the homology groups $H_*(X_{n,k}; \mathbb{Z})$.

(5) Show that the $2n$-dimensional sphere S^{2n} cannot be equipped with a Lie group structure.

(6) Show that the orthogonal group $O(n)$ of real $n \times n$ matrices with the property that $AA^t = Id$ is a compact submanifold of dimension $n(n-1)/2$ of the set of all $n \times n$ real matrices $M(n) \simeq \mathbb{R}^{n^2}$.

(continues)

(7) Show that, for any manifold M (orientable or not), its tangent bundle TM is naturally an oriented manifold.

(8) For $p \in S^2$, we represent by $-p$ its antipode. Let $f : S^2 \to S^2$ be a continuous map such that $f(p) \neq f(-p)$ for all $p \in S^2$. Show that f must be surjective.

(9) In $\mathbb{R}^4$ with coordinates (x, y, z, w), let $\mathcal{D}$ be the distribution generated at each point by the values of the vector fields $u = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and $v = \frac{\partial}{\partial x} + y\frac{\partial}{\partial w}$. Determine the integral curve of u through the point $(2, 1, -1, 3)$. Show that $\mathcal{D}$ does not admit any 2-dimensional integral manifold.

(10) Write the vector field X which generates the flow in $\mathbb{R}^2$ given by rotation around the origin. Letting $\omega = dx \wedge dy$, compute the 1-form $\imath_X\omega$ defined by $\imath_X\omega(Y) = \omega(X, Y)$. Exhibit a function $\mu : \mathbb{R}^2 \to \mathbb{R}$ satisfying $d\mu = -\imath_X\omega$.

(11) Show that the Levi-Civita connection ∇ in $\mathbb{R}^n$ relative to the flat metric $g_0 = \sum_{i=1}^n dx_i \otimes dx_i$ is given by

$$(\nabla_X Y)_x = dY_x\,(X_x) \ ,$$

for any $x \in \mathbb{R}^n$, and any vector fields X and Y, regarded as maps $X, Y : \mathbb{R}^n \to \mathbb{R}^n$ (why can they be seen this way?). We use the notation $dY_x : \mathbb{R}^n \to \mathbb{R}^n$ for the differential of the map Y at the point x.

(12) Let $M \to \mathbb{R}^n$ be an embedding of a k-dimensional manifold into the n-dimensional euclidean space. Consider in M the riemannian metric induced by this embedding. Let $\gamma : (-1, 1) \to M$ be a curve in M and let $\widetilde{\gamma} : (-1, 1) \to \mathbb{R}^n$ be the composition of γ with the embedding. Suppose that $||\frac{d\widetilde{\gamma}}{dt}|| \equiv 1$. Show that γ is a geodesic if and only if $\frac{d^2\widetilde{\gamma}}{dt^2}$ is orthogonal to M at $\gamma(t)$ for all t.