Precision of Harmonic Amplitude Estimation on Jitter Corrupted Data

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Abstract – *In this paper we study the effect that jitter and phase noise have on the precision of amplitude estimation of the harmonics of a periodic signal. The analysis carried out is applicable to both Three-Parameter Sine Fitting as well as Coherent Discrete Fourier Transform procedures. We consider the case where jitter and phase noise on the signal or sampling clock is small when compared with the signal period.*

Index Terms – *Sine fitting, DFT, uncertainty, ADC, jitter, phase noise.*

I. Introduction

In analog to digital converter (ADC) testing [1][2] there are different methods that are used estimate a variety of ADC characteristics, from the transfer function [3][4] to random noise [5], harmonic distortion and jitter. Two of the recommended methods in the IEEE standard 1057 for digitizing waveform recorders [1] and the IEEE Standard 1241 for terminology and test methods for analog-to-digital converters [2] are the Three-Parameter Sine Fitting (section 4.6 in [1] and section 4.1.4 in [2]) and the Coherent Discrete Fourier Transform (section 4.5 in [1] and section 4.1.5 in [2]).

In particular these test methods allow the estimation of the harmonics of a digitized sine wave in order to compute the Total Harmonic Distortion (THD) or the Signal to Noise and Distortion ratio (SINAD) of ADCs and waveform digitizer, which are parameters that quantify the linearity of the devices.

These procedures have many other applications, from o particle size and velocity determination using laser anemometry [6] to impedance measurement [7]-[9].

As any measurement method, these too have uncertainty due to different non ideal effects which are present in the test setup and the devices themselves. It is thus important to be able to quantify the contribution of these sources of uncertainty to the uncertainty of the end result. the In [10] an asymptotic Cramér-Rao bound for the variance of three and four-parameter sine wave fitting parameters (amplitude, offset, initial phase and frequency) for a large number of samples is derived taking into account the presence of additive noise. In [11] the same bounds are evaluated when additive noise is present and data is quantized.

One of the sources of uncertainty that has not really been subject of much research is the presence of phase noise on the signals and jitter in the ADC [12]-[14]. These two uncertainty sources are equivalent and can be treated the same way. In this work we present the study carried out on the precision of the estimates of the fundamental and the harmonics of a periodic signal in the presence of jitter and phase noise. The expression derived can also be useful in determining the uncertainty of other measurements made from the amplitudes of the harmonics like THD, SINAD and the effective value of a signal.

II. Sine Wave Fitting

Consider M data points $x_1, x_2, \ldots, x_M$ given by

$$x_i = C + \sum_{n=1}^{\infty} A_n \cos(n\omega_x t_i + \alpha_n + \varphi) \,. \quad (1)$$

where φ is the initial phase and ω_s is the angular frequency ($2\pi f$). We consider the phase φ to be a random variable uniformly distributed in an interval of 2π.

This data is affected by white Gaussian phase noise, θ_i, with null mean and standard deviation σ_θ:

$$z_i = C + \sum_{n=1}^{\infty} A_n \cos(n\omega_x t_i + \alpha_n + \varphi + n\theta_i) \,, \quad (2)$$

We wish to estimate the sine wave that best fits, in a least square error sense, to these M points. The estimates of the sine wave are obtained, in a matrix form, with [1]

$$\begin{bmatrix} \widehat{A_I} \\ \widehat{A_Q} \\ \widehat{C} \end{bmatrix} = \left(D^T D\right)^{-1} D^T \begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_M \end{bmatrix} \text{ with } D = \begin{bmatrix} \cos(\omega_a t_1) & \sin(\omega_a t_1) & 1 \\ \cos(\omega_a t_2) & \sin(\omega_a t_2) & 1 \\ \ldots & \ldots & \ldots \\ \cos(\omega_a t_M) & \sin(\omega_a t_M) & 1 \end{bmatrix} \quad (3)$$

and

$$\hat{A} = \sqrt{\widehat{A_I}^2 + \widehat{A_Q}^2} \quad (4)$$

where ω_a is the angular frequency of the sinusoid we are trying to adjust to the data. Here we will consider that this sinusoid is an harmonic of the signal and thus we have $\omega_a = h\omega_x$, where h is the order of the harmonic.

Here we will assume that the number of samples (M) acquired covers exactly an integer number of periods (J) of the sine wave we are trying to fit to the data. This means that the sine wave frequency (f_a), sampling frequency (f_s) and

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number of samples satisfy

$$\frac{f_a}{f_s}=\frac{J}{M}\ ,\ J\in\mathbb{N}\ . \qquad (5)$$

Note that J and M should be mutually prime so that the M different samples acquired at M different time instants, correspond to M different sine wave phases. If not, you will have less that M different phases which will increase the uncertainty in the estimation of the sine wave parameters. In the case that J is a multiple of $M/2$, the sampling instants will correspond to only 2 different sine wave phases and matrix D^TD will be singular and hence not invertible (you can not estimate the 3 sine wave parameters with only two data points).

Note that the sampling instants are given by $t_i=i/f_s$. The assumption is reasonable because we can choose whatever values we want for those frequencies and the number of samples. In practice, however, due to instrument inaccuracies, the actual value of those frequencies may not be exactly the values chosen and which satisfy (5) but are close enough considering typical frequency errors smaller than 100 ppm. If a non integer number of periods is acquired a bias will affect the estimator. In this work, however, we will not consider this scenario.

If the samples cover an integer number of sine wave periods, we have

$$\sum_{i=0}^{M-1}\cos(\omega_a t_i)=0\ ,\ \sum_{i=0}^{M-1}\sin(\omega_a t_i)=0 \text{ and } \sum_{i=0}^{M-1}\cos(\omega_a t_i)\sin(\omega_a t_i)=0 \qquad (6)$$

and

$$\sum_{i=0}^{M-1}\cos^2(\omega_a t_i)=\frac{M}{2} \text{ and } \sum_{i=0}^{M-1}\sin^2(\omega_a t_i)=\frac{M}{2}\ . \qquad (7)$$

Consequently matrix D^TD becomes diagonal and the sine wave parameters can thus be estimated with

$$\begin{bmatrix}\hat{A}_I\\ \hat{A}_Q\\ \hat{C}\end{bmatrix}=\begin{bmatrix}\frac{2}{M}\sum_{i=0}^{M-1}z_i\cos(\omega_a t_i)\\ \frac{2}{M}\sum_{i=0}^{M-1}z_i\sin(\omega_a t_i)\\ \frac{1}{M}\sum_{i=0}^{M-1}z_i\end{bmatrix}, \qquad (8)$$

and the square of the sine wave amplitude is given by

$$\hat{A}^2=\hat{A}_I^{\,2}+\hat{A}_Q^{\,2}=\frac{4}{M^2}\sum_{i,j}z_i z_j\cos\left[\omega_a\left(t_i-t_j\right)\right]. \qquad (9)$$

The harmonic amplitude is thus estimated with

$$\hat{A}_h=\frac{2}{M}\sqrt{\sum_{i,j}z_i z_j\cos\left[h\omega_x\left(t_i-t_j\right)\right]}\ . \qquad (10)$$

Note that this is the same expression one gets when using the Coherent Discrete Fourier Transform, apart from a $2/M$ factor.

III. Covariance of the Estimated Square Amplitude

In this section we will focus on the covariance of the square of the amplitude of two sine waves with frequency ω_g and ω_h. By definition the covariance is ([16], eq. 7-7)

$$Cov\left\{\widehat{A_g^2},\widehat{A_h^2}\right\}=E\left\{\widehat{A_g^2}\widehat{A_h^2}\right\}-E\left\{\widehat{A_g^2}\right\}E\left\{\widehat{A_h^2}\right\}. \qquad (11)$$

Using (9) we can write

$$Cov\left\{\widehat{A_g^2},\widehat{A_h^2}\right\}=\frac{16}{M^4}\sum_{i,j,k,l}Cov\{z_i z_j,z_k z_l\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right], \qquad (12)$$

where we have used

$$Cov\{ax,by\}=abCov\{x,y\}\ , \qquad (13)$$

being a and b constants and x and y random variables.

Note that the covariance of the square of the estimated amplitude does not depend on the sine wave offset and can thus be determined with

$$Cov\left\{\widehat{A_g^2},\widehat{A_h^2}\right\}=\frac{16}{M^4}\sum_{i,j,k,l}Cov\{w_i w_j,w_k w_l\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right], \qquad (14)$$

where

$$w_i=\sum_n A_n\cos\left(n\omega_x t_i+\alpha_n+\varphi+n\theta_i\right). \qquad (15)$$

Introducing now

$$g_{ni}=n\omega_x t_i+\alpha_n+n\theta_i\ , \qquad (16)$$

And making use of (15) we can write

$$Cov\{w_i w_j,w_k w_l\}= =Cov\left\{\sum_{n,q}A_n A_q\cos(g_{ni}+\varphi)\cos\left(g_{qj}+\varphi\right),\sum_{m,p}A_m A_p\cos(g_{mk}+\varphi)\cos\left(g_{pl}+\varphi\right)\right\}. \qquad (17)$$

Using ([20], p.346])

$$Cov\left\{\sum_i x_i,\sum_j x_j\right\}=\sum_{i,j}Cov\{x_i,y_i\}\ , \qquad (18)$$

with x_i and y_i random variables, we can write

$$Cov\{w_i w_j,w_k w_l\}= =\sum_{n,q,m,p}A_n A_q A_m A_p Cov\left\{\cos(g_{ni}+\varphi)\cos\left(g_{qj}+\varphi\right),\cos(g_{mk}+\varphi)\cos\left(g_{pl}+\varphi\right)\right\}. \qquad (19)$$

$$Cov\{w_i w_j,w_k w_l\}= =\frac{1}{4}\sum_{n,q,m,p}A_n A_q A_m A_p Cov\left\{\begin{matrix}\cos\left(g_{ni}+g_{qj}+2\varphi\right)+\cos\left(g_{ni}-g_{qj}\right),\\ \cos\left(g_{mk}+g_{pl}+2\varphi\right)+\cos\left(g_{mk}-g_{pl}\right)\end{matrix}\right\}. \qquad (20)$$

Considering (18) we can write (20) as

$$
\begin{aligned}
&Cov\{w_i w_j, w_k w_l\}=\\
&=\frac{1}{4}\sum_{n,q,m,p} A_n A_q A_m A_p Cov\{\cos(g_{ni}+g_{qj}+2\varphi),\cos(g_{mk}+g_{pl}+2\varphi)\}+\\
&+\frac{1}{4}\sum_{n,q,m,p} A_n A_q A_m A_p Cov\{\cos(g_{ni}+g_{qj}+2\varphi),\cos(g_{mk}-g_{pl})\}+\\
&+\frac{1}{4}\sum_{n,q,m,p} A_n A_q A_m A_p Cov\{\cos(g_{ni}-g_{qj}),\cos(g_{mk}+g_{pl}+2\varphi)\}+\\
&+\frac{1}{4}\sum_{n,q,m,p} A_n A_q A_m A_p Cov\{\cos(g_{ni}-g_{qj}),\cos(g_{mk}-g_{pl})\}
\end{aligned}
\quad . \tag{21}
$$

which leads to

$$
\begin{aligned}
&Cov\{w_i w_j, w_k w_l\}=\\
&=\frac{1}{4}\sum_{n,q,m,p} A_n A_q A_m A_p Cov\{\cos(g_{ni}-g_{qj}),\cos(g_{mk}-g_{pl})\}+\ .\\
&+\frac{1}{8}\sum_{n,q,m,p} A_n A_q A_m A_p E\{\cos(g_{ni}+g_{qj}-g_{mk}-g_{pl})\}
\end{aligned}
\tag{22}
$$

The derivation that follows and which will lead to an expression for the covariance between the estimated values of the squared amplitude of two harmonics with order g and h (equation (12)), is divided into 5 steps.

A. *Step 1*

The covariance in the first term of the second member can be written as ([16], eq. 7-7)

$$
\begin{aligned}
&Cov\{\cos(g_{ni}-g_{qj}),\cos(g_{mk}-g_{pl})\}=\\
&E\{\cos(g_{ni}-g_{qj})\cos(g_{mk}-g_{pl})\}-E\{\cos(g_{ni}-g_{qj})\}E\{\cos(g_{mk}-g_{pl})\}
\end{aligned}
\ . \tag{23}
$$

Making use of

$$
\cos(a)\cos(b)=\frac{1}{2}\cos(a+b)+\frac{1}{2}\cos(a-b)\ , \tag{24}
$$

we can write (23) as

$$
\begin{aligned}
&Cov\{\cos(g_{ni}-g_{qj}),\cos(g_{mk}-g_{pl})\}=\\
&\frac{1}{2}E\{\cos(g_{ni}-g_{qj}+g_{mk}-g_{pl})\}+\frac{1}{2}E\{\cos(g_{ni}-g_{qj}-g_{mk}+g_{pl})\}-\ ,\\
&-E\{\cos(g_{ni}-g_{qj})\}E\{\cos(g_{mk}-g_{pl})\}
\end{aligned}
\tag{25}
$$

The first and second terms of (25) end up, after inserting into (22) and then into (14), having the same value since we can swap the indices k with l and m with p. We will thus change (25) to

$$
\begin{aligned}
&Cov\{\cos(g_{ni}-g_{qj}),\cos(g_{mk}-g_{pl})\}=\\
&E\{\cos(g_{ni}-g_{qj}+g_{mk}-g_{pl})\}-E\{\cos(g_{ni}-g_{qj})\}E\{\cos(g_{mk}-g_{pl})\}
\end{aligned}
\ , \tag{26}
$$

Note that (26) is not the same as (25) but using (26) instead of (25) gives the same result for (14) which is what we want to determine.

Using (16) we can write

$$
E\{\cos(g_{ni}-g_{qj}-g_{mk}+g_{pl})\}=E\left\{\cos\left[\begin{aligned}&(n\omega_x t_i-q\omega_x t_j-m\omega_x t_k+p\omega_x t_l)+\\&+(\alpha_n-\alpha_q-\alpha_m+\alpha_p)+\\&+(n\theta_i-q\theta_j-m\theta_k+p\theta_l)\end{aligned}\right]\right\}.\tag{27}
$$

The expected value of the cosine of a random variable x, normally distributed with null mean and standard deviation σ_x, is

$$
E\{\cos(a+bx)\}=\cos(a)e^{-\frac{1}{2}b^2\sigma_x^2}\ . \tag{28}
$$

Using this we can write (27) as

$$
E\{\cos(g_{ni}-g_{qj}+g_{mk}-g_{pl})\}=
$$

$$
=\begin{cases}
Y_0\cos\left[(n-q+m-p)\omega_x t_i+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\bullet\bullet\bullet\\
Y_1\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_l+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\bullet\bullet\times\\
Y_2\cos\left[(n-q-p)\omega_x t_i+m\omega_x t_k+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\bullet\times\bullet\\
Y_3\cos\left[(n+m-p)\omega_x t_i-q\omega_x t_j+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\times\bullet\bullet\\
Y_4\cos\left[(-q+m-p)\omega_x t_j+n\omega_x t_i+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \times\bullet\bullet\bullet\\
Y_5\cos\left[(n-q)\omega_x t_i+(m-p)\omega_x t_k+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\bullet\times\times\\
Y_6\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\times\bullet\times\\
Y_7\cos\left[(n-p)\omega_x t_i-(q-m)\omega_x t_j+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\times\times\bullet\\
Y_8\cos\left[(n-q)\omega_x t_i+m\omega_x t_k-p\omega_x t_l+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\bullet\times\circ\\
Y_9\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_l+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\times\bullet\circ\\
Y_{10}\cos\left[(n-p)\omega_x t_i-q\omega_x t_j+m\omega_x t_k+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\times\circ\bullet\\
Y_{11}\cos\left[n\omega_x t_i+(-q+m)\omega_x t_j-p\omega_x t_l+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \times\bullet\bullet\circ\\
Y_{12}\cos\left[n\omega_x t_i+(-q-p)\omega_x t_j+m\omega_x t_k+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \times\bullet\circ\bullet\\
Y_{13}\cos\left[n\omega_x t_i-q\omega_x t_j+(m-p)\omega_x t_l+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \times\circ\bullet\bullet\\
Y_{14}\cos\left[(n\omega_x t_i-q\omega_x t_j+m\omega_x t_k-p\omega_x t_l)+(\alpha_n-\alpha_q+\alpha_m-\alpha_p)\right] & \bullet\times\circ *
\end{cases}
. \tag{29}
$$

where

$$
\begin{array}{lll}
Y_0=e^{\frac{-1}{2}(n-q+m-p)^2\sigma_\theta^2} & Y_5=e^{\frac{-1}{2}\left[(n-q)^2+(m-p)^2\right]\sigma_\theta^2} & Y_{10}=e^{\frac{-1}{2}\left[(n-p)^2+q^2+m^2\right]\sigma_\theta^2}\\
Y_1=e^{\frac{-1}{2}\left[(n-q+m)^2+p^2\right]\sigma_\theta^2} & Y_6=e^{\frac{-1}{2}\left[(n+m)^2+(q+p)^2\right]\sigma_\theta^2} & Y_{11}=e^{\frac{-1}{2}\left[n^2+(-q+m)^2+p^2\right]\sigma_\theta^2}\\
Y_2=e^{\frac{-1}{2}\left[(n-q-p)^2+m^2\right]\sigma_\theta^2} & Y_7=e^{\frac{-1}{2}\left[(n-p)^2+(q-m)^2\right]\sigma_\theta^2} & Y_{12}=e^{\frac{-1}{2}\left[n^2+(-q-p)^2+m^2\right]\sigma_\theta^2}\\
Y_3=e^{\frac{-1}{2}\left[(n+m-p)^2+q^2\right]\sigma_\theta^2} & Y_8=e^{\frac{-1}{2}\left[(n-q)^2+m^2+p^2\right]\sigma_\theta^2} & Y_{13}=e^{\frac{-1}{2}\left[n^2+q^2+(m-p)^2\right]\sigma_\theta^2}\\
Y_4=e^{\frac{-1}{2}\left[(-q+m-p)^2+n^2\right]\sigma_\theta^2} & Y_9=e^{\frac{-1}{2}\left[(n+m)^2+q^2+p^2\right]\sigma_\theta^2} & Y_{14}=e^{\frac{-1}{2}\left[n^2+q^2+m^2+p^2\right]\sigma_\theta^2}
\end{array}
\ . \tag{30}
$$

On the other hand we have

$$
E\{\cos(g_{ni}-g_{qj})\}=\begin{cases}
e^{-\frac{(n-q)^2}{2}\sigma_\theta^2}\cos\left[(n-q)\omega_x t_i+(\alpha_n-\alpha_q)\right] & i=j\\
e^{-\frac{n^2+q^2}{2}\sigma_\theta^2}\cos\left[n\omega_x t_i-q\omega_x t_j+(\alpha_n-\alpha_q)\right] & i\neq j
\end{cases}
. \tag{31}
$$

and

$$
E\{\cos(g_{mk}-g_{pl})\}=\begin{cases}
e^{-\frac{(m-p)^2}{2}\sigma_\theta^2}\cos\left[(m-p)\omega_x t_k+(\alpha_m-\alpha_p)\right] & k=l\\
e^{-\frac{m^2+p^2}{2}\sigma_\theta^2}\cos\left[m\omega_x t_k-p\omega_x t_l+(\alpha_m-\alpha_p)\right] & k\neq l
\end{cases}
. \tag{32}
$$

The second term in the second member of (26) is thus

$$E\left\{\cos\left(g_{ni}-g_{qj}\right)\right\}E\left\{\cos\left(g_{mk}-g_{pl}\right)\right\}=$$
$$=\begin{cases}
Y_5\cos\left[(n-q)\omega_x t_i+\left(\alpha_n-\alpha_q\right)\right]\cos\left[(m-p)\omega_x t_i+\left(\alpha_m-\alpha_p\right)\right] & \bullet\bullet\bullet\bullet \\
Y_8\cos\left[(n-q)\omega_x t_i+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_i-p\omega_x t_l+\left(\alpha_m-\alpha_p\right)\right] & \bullet\bullet\bullet\times \\
Y_8\cos\left[(n-q)\omega_x t_i+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_k-p\omega_x t_i+\left(\alpha_m-\alpha_p\right)\right] & \bullet\bullet\times\bullet \\
Y_{13}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[(m-p)\omega_x t_i+\left(\alpha_m-\alpha_p\right)\right] & \bullet\times\bullet\bullet \\
Y_{13}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[(m-p)\omega_x t_j+\left(\alpha_m-\alpha_p\right)\right] & \times\bullet\bullet\bullet \\
Y_5\cos\left[(n-q)\omega_x t_i+\left(\alpha_n-\alpha_q\right)\right]\cos\left[(m-p)\omega_x t_k+\left(\alpha_m-\alpha_p\right)\right] & \bullet\bullet\times\times \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_i-p\omega_x t_j+\left(\alpha_m-\alpha_p\right)\right] & \bullet\times\bullet\times \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_j-p\omega_x t_i+\left(\alpha_m-\alpha_p\right)\right] & \bullet\times\times\bullet \\
Y_8\cos\left[(n-q)\omega_x t_i+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_k-p\omega_x t_l+\left(\alpha_m-\alpha_p\right)\right] & \bullet\bullet\times\circ \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_i-p\omega_x t_l+\left(\alpha_m-\alpha_p\right)\right] & \bullet\times\bullet\circ \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_k-p\omega_x t_i+\left(\alpha_m-\alpha_p\right)\right] & \bullet\times\circ\bullet \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_j-p\omega_x t_l+\left(\alpha_m-\alpha_p\right)\right] & \times\bullet\bullet\circ \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_k-p\omega_x t_j+\left(\alpha_m-\alpha_p\right)\right] & \times\bullet\circ\bullet \\
Y_{13}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[(m-p)\omega_x t_k+\left(\alpha_m-\alpha_p\right)\right] & \times\circ\bullet\bullet \\
Y_{14}\cos\left[n\omega_x t_i-q\omega_x t_j+\left(\alpha_n-\alpha_q\right)\right]\cos\left[m\omega_x t_k-p\omega_x t_l+\left(\alpha_m-\alpha_p\right)\right] & \bullet\times\circ *
\end{cases} . \qquad (33)$$

Using the same reasoning presented when justifying the replacement of (25) by (27) we are lead to conclude that each product of cosines found in (33) can be replaced by one cosine with an argument that is the sum of the arguments of the two cosines.

Using (29) and (33) we can write (26) as

$$Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}=$$
$$=\begin{cases}
\left(Y_0-Y_5\right)\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right] & \bullet\bullet\bullet\bullet \\
\left(Y_1-Y_8\right)\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_l+\beta_{nqmp}\right] & \bullet\bullet\bullet\times \\
\left(Y_2-Y_8\right)\cos\left[(n-q-p)\omega_x t_i+m\omega_x t_k+\beta_{nqmp}\right] & \bullet\bullet\times\bullet \\
\left(Y_3-Y_{13}\right)\cos\left[(n+m-p)\omega_x t_i-q\omega_x t_j+\beta_{nqmp}\right] & \bullet\times\bullet\bullet \\
\left(Y_4-Y_{13}\right)\cos\left[(-q+m-p)\omega_x t_j+n\omega_x t_i+\beta_{nqmp}\right] & \times\bullet\bullet\bullet \\
0 & \bullet\bullet\times\times \\
\left(Y_6-Y_{14}\right)\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+\beta_{nqmp}\right] & \bullet\times\bullet\times \\
\left(Y_7-Y_{14}\right)\cos\left[(n-p)\omega_x t_i-(q-m)\omega_x t_j+\beta_{nqmp}\right] & \bullet\times\times\bullet \\
0 & \bullet\bullet\times\circ \\
\left(Y_9-Y_{14}\right)\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_l+\beta_{nqmp}\right] & \bullet\times\bullet\circ \\
\left(Y_{10}-Y_{14}\right)\cos\left[(n-p)\omega_x t_i-q\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right] & \bullet\times\circ\bullet \\
\left(Y_{11}-Y_{14}\right)\cos\left[n\omega_x t_i+(-q+m)\omega_x t_j-p\omega_x t_l+\beta_{nqmp}\right] & \times\bullet\bullet\circ \\
\left(Y_{12}-Y_{14}\right)\cos\left[n\omega_x t_i+(-q-p)\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right] & \times\bullet\circ\bullet \\
0 & \times\circ\bullet\bullet \\
0 & \bullet\times\circ *
\end{cases} . \qquad (34)$$

where

$$\beta_{nqmp}=\alpha_n-\alpha_q+\alpha_m-\alpha_p \ . \qquad (35)$$

B. Step 2

Now we are going to compute the summation of (34) times two cosine functions:

$$\sum_{i,j,k,l} Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]=$$
$$=\left(Y_0-Y_5\right)\sum_i\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+$$
$$+2\left(Y_1-Y_8\right)\sum_{i\neq j}\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+$$
$$+2\left(Y_3-Y_{13}\right)\sum_{i\neq j}\cos\left[(n+m-p)\omega_x t_i-q\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]+$$
$$+\left(Y_6-Y_{14}\right)\sum_{i\neq j}\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+$$
$$+\left(Y_7-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-p)\omega_x t_i-(q-m)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+$$
$$+2\left(Y_9-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]+$$
$$+2\left(Y_{10}-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n-p)\omega_x t_i-q\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right] \qquad (36)$$

Note that since the terms in (34) go into a summation on *n*, *q*, *m* and *p* (equation (22)), we can swap some of the indices without changing the final result. This was used to join the terms in the 2nd member of (34), namely the 2nd and 3rd terms in (34) into the 2nd term in (36), the 4th and 5th terms in (34) into the 3rd term in (36), the 9th and 12th terms in (34) into the 6th term in (36) and the 10th and 11th terms in (34) into the 7th term in (36).

The double and triple summations in (36) are not complete, that is, their indices do not span the entire range from 1 to *M* because the different summation indices cannot be equal. Note that this is so because the expression where they came from, in (34) are only valid for certain arrangements of *i*, *j*, *k* and *l*. Those summation can however be transformed into complete summation using

$$\sum_{i\neq j}a_{ij}=\sum_{i,j}a_{ij}-\sum_i a_{ii} \ , \qquad (37)$$

For the double summations and

$$\sum_{i\neq j\neq k}a_{ijk}=\sum_{i,j,k}a_{ijk}-\sum_{i\neq j}a_{iij}-\sum_{i\neq j}a_{ijj}-\sum_{i\neq j}a_{iji}-\sum_i a_{iii} \ , \qquad (38)$$

for the triple summation. Using (37) we can write (38) as

$$\sum_{i\neq j\neq k}a_{ijk}=\sum_{i,j,k}a_{ijk}-\sum_{i,j}a_{iij}-\sum_{i,j}a_{ijj}-\sum_{i,j}a_{iji}+2\sum_i a_{iii} \ . \qquad (39)$$

Making use of (37) and (39) allows us to write (36) as

$$\begin{aligned}
&\sum_{i,j,k,l} Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]=\\
&=\left(Y_0-2Y_1-2Y_3-Y_5-Y_6-Y_7+2Y_8+4Y_9+4Y_{10}+2Y_{13}-6Y_{14}\right)\sum_i\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+2\left(Y_1-Y_8-Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+2\left(Y_3-Y_9-Y_{10}-Y_{13}+2Y_{14}\right)\sum_{i,j}\cos\left[(n+m-p)\omega_x t_i-q\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]+\\
&+\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p)\omega_x t_i-(q-m)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+2\left(Y_{14}-Y_{10}\right)\sum_{i,j}\cos\left[(n-q-p)\omega_x t_i+m\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+2\left(Y_9-Y_{14}\right)\sum_{i,j,k}\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]+\\
&+2\left(Y_{10}-Y_{14}\right)\sum_{i,j,k}\cos\left[(n-p)\omega_x t_i-q\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]
\end{aligned} \tag{40}$$

Since (40) is going to be used inside a quadruple summation in *n*, *q*, *m* and *p*, in (22), we can swap *n* with *q* and *m* with *p* in the 6th term of the second member of (40) to see that the summation is the same as the one in the 2nd term. This index swap transforms Y_{10} into Y_{11} as can be seen in (30). Expression (40) then becomes

$$\begin{aligned}
&\sum_{i,j,k,l} Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]=\\
&=\left(Y_0-2Y_1-2Y_3-Y_5-Y_6-Y_7+2Y_8+4Y_9+4Y_{10}+2Y_{13}-6Y_{14}\right)\sum_i\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+2\left(Y_1-Y_8-Y_9-Y_{11}+2Y_{14}\right)\sum_{i,j}\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+2\left(Y_3-Y_9-Y_{10}-Y_{13}+2Y_{14}\right)\sum_{i,j}\cos\left[(n+m-p)\omega_x t_i-q\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]+\\
&+\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p)\omega_x t_i-(q-m)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+2\left(Y_9-Y_{14}\right)\sum_{i,j,k}\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]+\\
&+2\left(Y_{10}-Y_{14}\right)\sum_{i,j,k}\cos\left[(n-p)\omega_x t_i-q\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]
\end{aligned} \tag{41}$$

C. Step 3

We are now going to apply (24) twice to each of the terms in (41). We are also going to use the fact that we are studding the case where an integer number of sine wave periods are fitted to the data which leads to the summation like $\sum_i\cos\left(a\omega_x t_i+b\right)$ are null except if *a* is null. In that case the summation is $M\cos(b)$. So, when applying (24) to (41) we will encounter terms like

$$\sum_{i,j}\cos\left[(n-q+m+h)\omega_x t_i-(p+h)\omega_x t_j+\beta_{nqmp}\right] \tag{42}$$

which is null since *p* and *h* are always positive integers equal to or greater than 1. Hence *p*+*h* can never be 0 and the summation in (42) is null. Those terms are discarded and we are left with the following terms:

$$\begin{aligned}
&\sum_{i,j,k,l} Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]=\\
&=\left(Y_0-2Y_1-2Y_3-Y_5-Y_6-Y_7+2Y_8+4Y_9+4Y_{10}+2Y_{13}-6Y_{14}\right)\sum_i\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+\left(Y_1-Y_8-Y_9-Y_{11}+2Y_{14}\right)\sum_{i,j}\cos\left[(n-q+m-h)\omega_x t_i-(p-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_3-Y_9-Y_{10}-Y_{13}+2Y_{14}\right)\sum_{i,j}\cos\left[(n+m-p-g)\omega_x t_i-(q-g)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m-g+h)\omega_x t_i-(q+p-g+h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m+g-h)\omega_x t_i-(q+p+g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m-g-h)\omega_x t_i-(q+p-g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p+g+h)\omega_x t_i-(q-m+g+h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p-g+h)\omega_x t_i-(q-m-g+h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p+g-h)\omega_x t_i-(q-m+g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{4}\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p-g-h)\omega_x t_i-(q-m-g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\frac{1}{2}\left(Y_9-Y_{14}\right)\sum_{i,j,k}\cos\left[(n+m-g-h)\omega_x t_i-(q-g)\omega_x t_j-(p-h)\omega_x t_k+\beta_{nqmp}\right]+\\
&+\frac{1}{2}\left(Y_{10}-Y_{14}\right)\sum_{i,j,k}\cos\left[(n-p-g+h)\omega_x t_i-(q-g)\omega_x t_j+(m-h)\omega_x t_k+\beta_{nqmp}\right]
\end{aligned} \tag{43}$$

D. Step 4

After inserting (22) into (14) we are left with

$$\begin{aligned}
Cov\left\{\widehat{A_g^2},\widehat{A_h^2}\right\}=&\frac{4}{M^4}\sum_{n,q,m,p}A_nA_qA_mA_p\sum_{i,j,k,l}\left[\begin{array}{l}Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}\times\\ \times\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]\end{array}\right]+\\
&+\frac{2}{M^4}\sum_{n,q,m,p}A_nA_qA_mA_p\sum_{i,j,k,l}\left[\begin{array}{l}E\left\{\cos\left(g_{ni}+g_{qj}-g_{mk}-g_{pl}\right)\right\}\times\\ \times\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]\end{array}\right]
\end{aligned}. \tag{44}$$

We can swap the indices j with k and q with m in the second term that the result does not change:

$$\begin{aligned}
&\sum_{n,q,m,p}A_nA_qA_mA_p\sum_{i,j,k,l}\left[\begin{array}{l}E\left\{\cos\left(g_{ni}+g_{qj}-g_{mk}-g_{pl}\right)\right\}\times\\ \times\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]\end{array}\right]=\\
&=\sum_{n,q,m,p}A_nA_qA_mA_p\sum_{i,j,k,l}\left[\begin{array}{l}E\left\{\cos\left(g_{ni}-g_{qj}+g_{mk}-g_{pl}\right)\right\}\times\\ \times\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_l\right)\right]\end{array}\right]
\end{aligned}. \tag{45}$$

Note that the expectation found in the second member is the same that was computed in (29). The next step is thus to insert (29) into (45) and compute the quadruple summation in *i*, *j*, *k* and *l*. To that effect note that the summation of the last term in (29) can be written as a complete summation for any value of the indices minus the cases where two or more of the four indices are the same. Those situations are exactly the ones covered by all the other terms in (29). We can thus write

$$\begin{aligned}
&\sum_{i,j,k,l} E\left\{\cos\left(g_{ni}-g_{qj}+g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_l\right)\right]=\\
&=\left(Y_0-Y_{14}\right)\sum_{i}\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+\left(Y_1-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_2-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q-p)\omega_x t_i+m\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]+\\
&+\left(Y_3-Y_{14}\right)\sum_{i\neq j}\cos\left[(n+m-p)\omega_x t_i-q\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_4-Y_{14}\right)\sum_{i\neq j}\cos\left[n\omega_x t_i-(q-m+p)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]+\\
&+\left(Y_5-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q)\omega_x t_i+(m-p)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_6-Y_{14}\right)\sum_{i\neq j}\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_7-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-p)\omega_x t_i-(q-m)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_8-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n-q)\omega_x t_i+m\omega_x t_k-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]+\\
&+\left(Y_9-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_j-t_k\right)\right]+\\
&+\left(Y_{10}-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n-p)\omega_x t_i-q\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_l\right)\right]+\\
&+\left(Y_{11}-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[n\omega_x t_i+(-q+m)\omega_x t_j-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_j-t_k\right)\right]+\\
&+\left(Y_{12}-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[n\omega_x t_i+(-q-p)\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_k\right)\right]+\\
&+\left(Y_{13}-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[n\omega_x t_i-q\omega_x t_j+(m-p)\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_k\right)\right]
\end{aligned} \quad . \tag{46}$$

Note that since the terms in (46) go into a summation on *n*, *q*, *m* and *p* (equation (22)), we can swap some of the indices without changing the final result. This allows us to write

$$\begin{aligned}
&\sum_{i,j,k,l} E\left\{\cos\left(g_{ni}-g_{qj}+g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_l\right)\right]=\\
&=\left(Y_0-Y_{14}\right)\sum_{i}\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+2\left(Y_1-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q+m)\omega_x t_i-p\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+2\left(Y_2-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q-p)\omega_x t_i+m\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]+\\
&+2\left(Y_5-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q)\omega_x t_i+(m-p)\omega_x t_j+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_j\right)\right]+\\
&+\left(Y_6-Y_{14}\right)\sum_{i\neq j}\cos\left[(n+m)\omega_x t_i-(q+p)\omega_x t_j+\beta_{nqmp}\right]+\\
&+4\left(Y_8-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n-q)\omega_x t_i+m\omega_x t_k-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_i-t_k\right)\right]+\\
&+\left(Y_9-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n+m)\omega_x t_i-q\omega_x t_j-p\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_h\left(t_j-t_k\right)\right]+\\
&+\left(Y_{12}-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[n\omega_x t_i+(-q-p)\omega_x t_j+m\omega_x t_k+\beta_{nqmp}\right]\cos\left[\omega_g\left(t_i-t_k\right)\right]
\end{aligned} \quad . \tag{47}$$

Again using (37) and (39) we can write

$$\begin{aligned}
&\sum_{i,j,k,l} E\left\{\cos\left(g_{ni}-g_{qj}+g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_l\right)\right]=\\
&=\left(Y_0-2Y_1-2Y_2-2Y_5-Y_6+8Y_8+2Y_9+2Y_{12}-6Y_{14}\right)\sum_{i}\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+\left(Y_1-2Y_8-Y_9+2Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q+m-h)\omega_x t_i-(p-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_2-2Y_8-Y_{12}+2Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q-p+g)\omega_x t_i+(m-g)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_5-2Y_8-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q+g+h)\omega_x t_i+(m-p-g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_5-2Y_8-Y_{14}\right)\sum_{i\neq j}\cos\left[(n-q-g+h)\omega_x t_i+(m-p+g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_8-Y_{14}\right)\sum_{i\neq j\neq k}\cos\left[(n-q+g-h)\omega_x t_i+(m-g)\omega_x t_k-(p-h)\omega_x t_k+\beta_{nqmp}\right]
\end{aligned} \quad . \tag{48}$$

E. Step 5

Adding 4 times (43) plus 2 times (48) leads to

$$\begin{aligned}
&4\sum_{i,j,k,l} Cov\left\{\cos\left(g_{ni}-g_{qj}\right),\cos\left(g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_j\right)\right]\cos\left[\omega_h\left(t_k-t_l\right)\right]+\\
&+2\sum_{i,j,k,l} E\left\{\cos\left(g_{ni}-g_{qj}+g_{mk}-g_{pl}\right)\right\}\cos\left[\omega_g\left(t_i-t_k\right)\right]\cos\left[\omega_h\left(t_j-t_l\right)\right]=\\
&=\begin{pmatrix}6Y_0-12Y_1-4Y_2-8Y_3-8Y_5-6Y_6-4Y_7+\\+24Y_8+20Y_9+16Y_{10}+14Y_{12}+8Y_{13}-36Y_{14}\end{pmatrix}\sum_{i}\cos\left[(n-q+m-p)\omega_x t_i+\beta_{nqmp}\right]+\\
&+\left(6Y_1-8Y_8-6Y_9-4Y_{11}+12Y_{14}\right)\sum_{i,j}\cos\left[(n-q+m-h)\omega_x t_i-(p-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(6Y_3-6Y_9-4Y_{10}-8Y_{13}+12Y_{14}\right)\sum_{i,j}\cos\left[(n+m-p-g)\omega_x t_i-(q-g)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m-g+h)\omega_x t_i-(q+p-g+h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m+g-h)\omega_x t_i-(q+p+g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_6-2Y_9+Y_{14}\right)\sum_{i,j}\cos\left[(n+m-g-h)\omega_x t_i-(q+p-g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(3Y_7-6Y_{10}+3Y_{14}\right)\sum_{i,j}\cos\left[(n-p+g+h)\omega_x t_i-(q-m+g+h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p-g+h)\omega_x t_i-(q-m-g+h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p+g-h)\omega_x t_i-(q-m+g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(Y_7-2Y_{10}+Y_{14}\right)\sum_{i,j}\cos\left[(n-p-g-h)\omega_x t_i-(q-m-g-h)\omega_x t_j+\beta_{nqmp}\right]+\\
&+\left(2Y_9-2Y_{14}\right)\sum_{i,j,k}\cos\left[(n+m-g-h)\omega_x t_i-(q-g)\omega_x t_j-(p-h)\omega_x t_k+\beta_{nqmp}\right]+\\
&+\left(4Y_{10}-4Y_{14}\right)\sum_{i,j,k}\cos\left[(n-p-g+h)\omega_x t_i-(q-g)\omega_x t_j+(m-h)\omega_x t_k+\beta_{nqmp}\right]
\end{aligned} \tag{49}$$

Considering, as stated earlier, that the summations are only different from 0 if the factors that multiply the time instants are null, we can write, inserting (49) into (22) and into (14),

$$Cov\left\{\widehat{A_g^2},\widehat{A_h^2}\right\}=\frac{1}{M^4}\sum_{n,q,m,p}A_nA_qA_mA_p\cos\left(\beta_{nqmp}\right)\Gamma_{nqmp}$$

with

$$\Gamma_{nqmp}=\begin{cases}M\left(6-24Y_1-12Y_5-6Y_6+48Y_8+24Y_9-36Y_{14}\right)+ & \text{, if } n-q+m-p=0\\ +M^2\left(6Y_1-8Y_8-6Y_9-4Y_{11}+12Y_{14}\right)+ & \text{, if } n-q+m=h\wedge p=h\\ +M^2\left(6Y_3-6Y_9-4Y_{10}-8Y_{13}+12Y_{14}\right)+ & \text{, if } n+m-p=g\wedge q=g\\ +M^2\left(Y_6-2Y_9+Y_{14}\right)+ & \text{, if } n+m=g-h\wedge q+p=g-h\\ +M^2\left(Y_6-2Y_9+Y_{14}\right)+ & \text{, if } n+m=h-g\wedge q+p=h-g\\ +M^2\left(Y_6-2Y_9+Y_{14}\right)+ & \text{, if } n+m=g+h\wedge q+p=g+h\\ +M^2\left(3Y_7-6Y_{10}+3Y_{14}\right)+ & \text{, if } p-n=g+h\wedge m-q=g+h\\ +M^2\left(3Y_7-6Y_{10}+3Y_{14}\right)+ & \text{, if } n-p=g-h\wedge q-m=g-h\\ +M^2\left(Y_7-2Y_{10}+Y_{14}\right)+ & \text{, if } p-n=g-h\wedge m-q=g-h\\ +M^2\left(Y_7-2Y_{10}+Y_{14}\right)+ & \text{, if } n-p=g+h\wedge q-m=g+h\\ +M^3\left(2Y_9-2Y_{14}\right)+ & \text{, if } n+m=g+h\wedge q=g\wedge p=h\\ +M^3\left(4Y_{10}-4Y_{14}\right) & \text{, if } n-p=g-h\wedge q=g\wedge m=h\end{cases} \quad (50)$$

To obtain an approximate expression for the standard deviation of the estimated harmonic amplitude we will use ([16], eq. 5-56)

$$\sigma_{\widehat{A_h}}\approx\frac{\sigma_{\widehat{A_h^2}}}{2\sqrt{\mu_{\widehat{A_h^2}}}}\,. \quad (51)$$

Furthermore, in order to obtain a more compact expression, we will consider the case of low amount of jitter/phase noise. This allows us to use

$$e^{-a\sigma_\theta^2}\approx1-a\sigma_\theta^2\,, \quad (52)$$

and

$$\mu_{\widehat{A_h^2}}\approx A_h^2\,. \quad (53)$$

Using (51), (52), (53) and making $g = h$ in(50), we are lead to

$$\boxed{\sigma_{\widehat{A_h}}\approx\frac{\sigma_\theta}{\sqrt{M}}\sqrt{\sum_n n^2A_h^2+\sum_n\left(\frac{1}{2}n^2-h^2\right)A_nA_{2h-n}\cos\left(\alpha_n+\alpha_{2h-n}-2\alpha_h\right)}}\,. \quad (54)$$

Note that for a modest number of samples we can discard the terms in M and M^2 in (50).

IV. Monte Carlo Analysis

To validate expression (54) we did a Monte Carlo Analysis on a numerically simulated sine wave with two harmonics. The results and the analytical expression are in agreement as seen in Fig. 1.

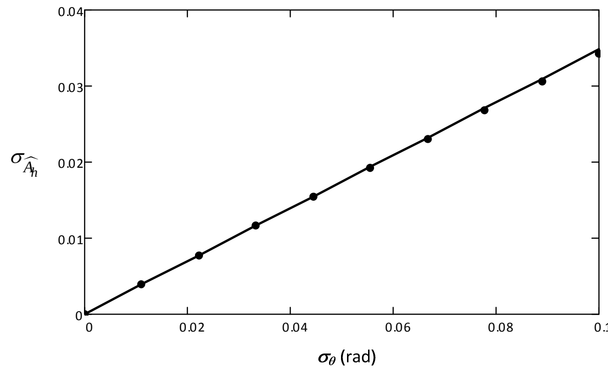

Fig. 1 – Standard deviation of the estimated 2nd harmonic amplitude as a function of phase noise standard deviation obtained from numerical simulation and Monte Carlo Analysis (circles) the solid line represents the values given by (54). The numerically simulated signal had one 2 V fundamental, one 0.6 V 2nd harmonic and one 1 V 3rd harmonic. The number of samples is 100 and the number of repetitions made in the Monte Carlo Analysis is 2×10^5.

In Fig. 2 we see that the approximations made are valid for values of phase noise standard deviation lower than 0.05 rad.

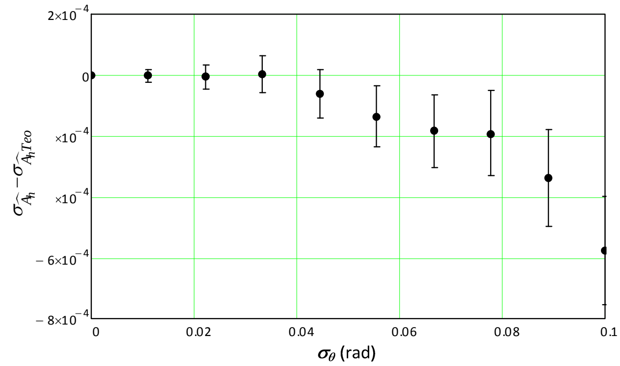

Fig. 2 – Difference from standard deviation of the estimated 2nd harmonic amplitude as a function of phase noise standard deviation obtained from numerical simulation and Monte Carlo Analysis (circles) and the analytical value given by (54). The numerically simulated signal had one 2 V fundamental, one 0.6 V 2nd harmonic and one 1 V 3rd harmonic. The number of samples is 100 and the number of repetitions made in the Monte Carlo Analysis is 2×10^5. The vertical bars represent the confidence interval for a 99.9% confidence level.

V. Conclusions

In this paper we derived an expression for the standard deviation of the estimated amplitude of the harmonics of a periodic signal obtained with the Three-Parameter Sine Fitting or the DFT methods affected by phase noise and jitter.

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