Gaussian Jitter Induced Bias of Sine Wave Amplitude Estimation Using Three-Parameter Sine Fitting

F. Corrêa Alegria, *Member IEEE*, A. Cruz Serra, *Senior Member IEEE*

***Abstract*—In this paper we study the Gaussian jitter induced bias on the estimate of sine wave amplitude obtained with 3-parameter sine fitting of a set of data points. This is a source of uncertainty that is not usually considered because no analytical study exists on it. Nowadays it is becoming more and more important due to ever increasing sampling rates available in analog to digital converters which are used in innumerous application like high speed digital oscilloscopes.**

***Index Terms*—Sine wave fit, uncertainty, ADC, jitter, phase noise.**

I. Introduction

Sine fitting is a technique used in a never ending list of applications, from analog to digital converter testing [1]-[5] and impedance measurement [6] to particle size and velocity determination using laser anemometry [7].

Algorithms for sine wave fitting have been standardized in [4] and [5] and some work has been done on the uncertainty of the sine fitting parameters. In [8] and [9] an asymptotic Cramér-Rao bound for the variance of three and four-parameter sine wave fitting parameters (amplitude, offset, initial phase and frequency) for a large number of samples is derived taking into account the presence of additive noise. In [10] the same bounds are evaluated when additive noise is present and data is quantized. It considers both the case where quantization error can and cannot be considered an additional additive noise term normally distributed independent of the signal. In [11], the performance of the frequency estimator used in IEEE 1057 std. 4-parameter sine fitting algorithm is compared to the Cramér-Rao bound.

The presence of jitter in sampling systems as long been considered [12][13] and with the advent of high frequency digital oscilloscopes it has gained a fundamental importance. One of the concerns is the measurement of the amount of jitter present in the system has been subject of various works ([14][15][16][17]). Another is the minimization of the effect of jitter on measurements [18]. One of those measurements, the amplitude of a sine wave, which is the focus of this paper, has received little attention in the past [19][20].

There are different estimators used to determine the amplitude of a sine wave given a set of data points. In [19] the IEEE 1057 standard method [4], which minimizes the square of the residuals, was studied. It is pointed out that a bias in the estimated amplitude arises due to jitter in the sampling instant. It is also shown how to compute that bias in the asymptotic case (infinite number of samples). In [20] other estimators, based on least squares and on maximum likelihood approaches which minimize the difference between actual and the estimated jitter mean and variance, are studied. A bias of those estimators is also reported for the estimated sine wave amplitude although no expression is presented to calculate it.

Here we present such an expression for the bias of sine wave amplitude estimated using the IEEE 1057 standard method caused by jitter on the sampling instant when a finite number of samples are acquired. The bias that arises, and which is always negative, can be seen in the numerical simulation example presented in Fig. 1 were data points from a sine wave (solid thick line) were corrupted by normally distributed jitter with a standard deviation of 0.8 μs (dots). This is a high value for timing jitter considering what is usually encountered in practice, however it was chosen to better illustrate the effect it has on the amplitude bias of the estimated sine wave (thin solid line).

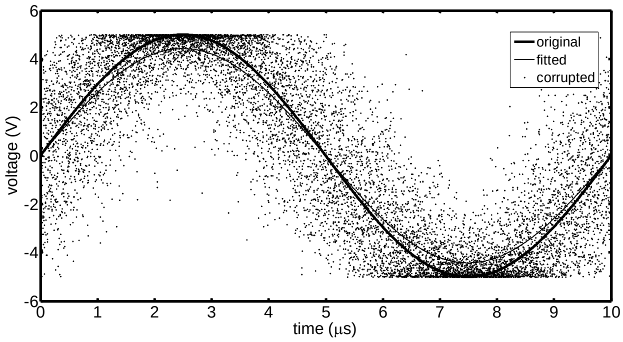

Fig. 1 – Example of simulated sine fitting of jitter corrupted data. The sine wave model has 5 V of amplitude and 100 kHz of frequency. The normally distributed jitter has 0.8 μs standard deviation. 10^4 data points acquired at a sampling frequency of 1 GHz are shown. The fitted sine wave has 4.425 V of amplitude due to jitter induced bias.

In Fig. 2 an even greater amount of jitter was simulated (1.6 μs). As can be seen the amplitude of the fitted sine wave is lower than in the case of Fig. 1. As the amount of jitter tends to infinity the contribution of the original sine

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F. A. C. Alegria is with Instituto de Telecomunicações and Instituto Superior Técnico, Technical University of Lisbon, Portugal (Ph: +351-218418376; Fax: +351-218417672; e-mail: falegria@lx.it.pt).

A. M. C. Serra is with Instituto de Telecomunicações and Instituto Superior Técnico, Technical University of Lisbon, Portugal (Ph: +351-218418490; Fax: +351-218417672; e-mail: acserra@ist.utl.pt).

wave to the voltage of each sample becomes negligible and we are left with random voltage noise with null mean. The sine wave that best fits this noise is a null amplitude one (in the case of an infinite number of samples).

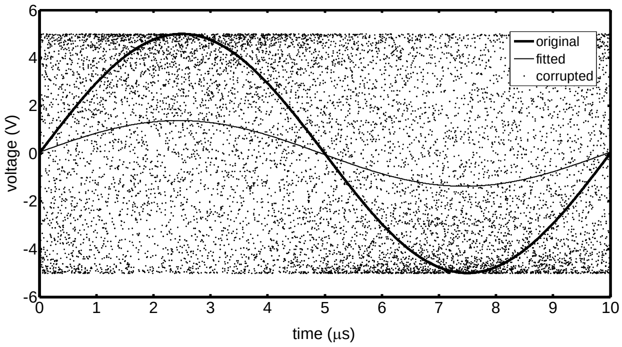

Fig. 2 – Example of simulated sine fitting of jitter corrupted data. The sine wave model has 5 V of amplitude and 100 kHz of frequency. The normally distributed jitter has 1.6 μs standard deviation. 10^4 data points acquired at a sampling frequency of 1 GHz are shown. The fitted sine wave has 4.425 V of amplitude due to jitter induced bias.

In the limit when the number of samples goes to infinity, the expression presented here tends to the expression given in [19] as we will demonstrate.

II. Sine Wave Fitting

Consider M data points $x_1, x_2, \ldots, x_M$ given by

$$x_i = C + A\cos(\omega_x t_i + \varphi) \quad \text{with} \quad i = 1,\ldots,M \;. \tag{1}$$

where C is the offset, A is the amplitude, φ is the initial phase, ω_x is the angular frequency and t_i are the sampling instants. Generally the initial phase of the sine wave is not controlled and thus varies from acquisition to acquisition and from measurement to measurement. Statistically we can consider it to be a random variable uniformly distributed in an interval of length 2π.

In this work we consider only the presence of normally distributed jitter in the sampling instants and represent it by a null mean random variable δ_i with standard deviation σ_t. The actual voltage of the samples is thus given by

$$z_i = C + A\cos\left[\omega_x(t_i + \delta_i) + \varphi\right] . \tag{2}$$

To ease the derivations that follow, we will introduce the random variable $\theta_i = \omega_x\delta_i$. This variable will be a null mean random variable with standard deviation $\sigma_\theta = \omega_x\sigma_t$ since it is just a constant (ω_x) times a normally distributed random variable (δ_i) with standard deviation σ_t. Equation (2) can thus be written as

$$z_i = C + A\cos(\omega_x t_i + \theta_i + \varphi) . \tag{3}$$

We wish to estimate the sine wave that best fits, in a least square error sense, to these M points. The estimates of the sine wave are obtained, in a matrix form, with [1]

$$\begin{bmatrix} \widehat{A_I} \\ \widehat{A_Q} \\ \widehat{C} \end{bmatrix} = \left(D^T D\right)^{-1} D^T \begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_M \end{bmatrix} \text{with } D = \begin{bmatrix} \cos(\omega_a t_1) & \sin(\omega_a t_1) & 1 \\ \cos(\omega_a t_2) & \sin(\omega_a t_2) & 1 \\ \ldots & \ldots & \ldots \\ \cos(\omega_a t_M) & \sin(\omega_a t_M) & 1 \end{bmatrix} \tag{4}$$

and

$$\widehat{A} = \sqrt{\widehat{A_I}^2 + \widehat{A_Q}^2} \tag{5}$$

where ω_a is the angular frequency of the sinusoid we are trying to adjust to the data. Here we will assume that the frequency of the signal is exactly know and its value is used to fit the sine wave ($\omega_a = \omega_x$).

We will also assume that the number of samples (M) acquired covers exactly an integer number of periods (J) of the sine wave we are trying to fit to the data. This means that the sine wave frequency (f_a), sampling frequency (f_s) and number of samples satisfy

$$\frac{f_a}{f_s} = \frac{J}{M} \;,\; J \in \mathbb{N} \;. \tag{6}$$

Note that J and M should be mutually prime so that the M different samples acquired at M different time instants, correspond to M different sine wave phases. If not, you will have less that M different phases which will increase the uncertainty in the estimation of the sine wave parameters. In the case that J is a multiple of $M/2$, the sampling instants will correspond to only 2 different sine wave phases and matrix D^TD will be singular and hence not invertible (you can not estimate the 3 sine wave parameters with only two data points).

The assumption is reasonable because we can choose whatever values we want for those frequencies and the number of samples. In practice, however, due to instrument inaccuracies, the actual value of those frequencies may not be exactly the values chosen and which satisfy (6) but are close enough considering typical frequency errors smaller than 100 ppm. If a non integer number of periods is acquired a bias will affect the estimator. In this work, however, we will not consider this scenario.

If the samples cover exactly an integer number of sine wave periods, we have

$$\begin{array}{ll} \sum_{i=1}^{M}\cos(\omega_a t_i) = 0 & \sum_{i=1}^{M}\cos(\omega_a t_i)\sin(\omega_a t_i) = 0 \\ \sum_{i=1}^{M}\sin(\omega_a t_i) = 0 & \sum_{i=1}^{M}\sum_{j=1}^{M}\cos(\omega_a t_i - \omega_a t_j) = 0 \end{array} \tag{7}$$

and

$$\sum_{i=1}^{M}\cos^2(\omega_a t_i) = \frac{M}{2} \text{ and } \sum_{i=1}^{M}\sin^2(\omega_a t_i) = \frac{M}{2} . \tag{8}$$

Consequently, from (4) and (5),

$$\widehat{A}^2 = \widehat{A_I}^2 + \widehat{A_Q}^2 = \frac{4}{M^2}\sum_{i,j} z_i z_j \cos\left[\omega_a(t_i - t_j)\right] . \tag{9}$$

From now on, for the sake of compactness, we will eliminate the summation limits and assume that all indices go from 1 to M. The summation in (9) is thus a double summation on i and j which go from 1 to M.

III. Mean of Square Estimated Amplitude

The expected value of the square of the estimated sine wave amplitude is, from (9),

$$E\left\{\widehat{A}^2\right\} = \frac{4}{M^2}\sum_{i,j} E\left\{z_i z_j\right\}\cos\left[\omega_a(t_i - t_j)\right] . \tag{10}$$

Note that the expected value of a sum is equal to the sum of the expected values and that the expected value of a random variable times a constant is equal to that constant times the expected value of the random variable.

Using (2) we can write

$$E\{z_i z_j\} = E\left\{\left[C + A\cos(\omega_x t_i + \theta_i + \varphi)\right]\cdot\left[C + A\cos(\omega_x t_j + \theta_j + \varphi)\right]\right\} . \quad (11)$$

which can be written as

$$\begin{aligned} E\{z_i z_j\} &= C^2 + A^2 E\{\cos(\omega_x t_i + \varphi + \theta_i)\cos(\omega_x t_j + \varphi + \theta_j)\} + \\ &+ CAE\{\cos(\omega_x t_i + \varphi + \theta_i)\} + CAE\{\cos(\omega_x t_j + \varphi + \theta_j)\} = \\ &= C^2 + \frac{1}{2}A^2 E\{\cos(\omega_x t_i + \omega_x t_j + 2\varphi + \theta_i + \theta_j)\} + \\ &+ \frac{1}{2}A^2 E\{\cos(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j)\} + \\ &+ CAE\{\cos(\omega_x t_i + \varphi + \theta_i)\} + CAE\{\cos(\omega_x t_j + \varphi + \theta_j)\} . \end{aligned} \quad (12)$$

Considering that φ is a uniformly distributed random variable between 0 and 2π we have

$$E\{z_i z_j\} = C^2 + \frac{1}{2}A^2 E\{\cos(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j)\} , \quad (13)$$

since

$$E\{\cos(a + \varphi)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(a + \varphi)d\varphi = 0 . \quad (14)$$

To compute the expected value in (13) we have to consider two cases – equal or different values of indices i and j. If they are equal then θ_i and θ_j cancel each other and we cease to have any random variables in the equation. The expected value is thus

$$E\{z_i z_j\}_{i=j} = C^2 + \frac{1}{2}A^2 . \quad (15)$$

On the other hand, if the indices are different, we have, considering that θ_i and θ_j are normally distributed random variables with standard deviation σ_θ,

$$E\{z_i z_j\}_{i \neq j} = C^2 + \frac{1}{2}A^2 \cos(\omega_x t_i - \omega_x t_j) e^{-\sigma_\theta^2} , \quad (16)$$

since

$$E\{\cos(a + \theta)\} = \int_{-\infty}^{\infty}\cos(a + \theta)\frac{1}{\sqrt{2\pi}\sigma_\theta} e^{-\frac{\theta^2}{2\sigma_\theta^2}} d\varphi = \cos(a) e^{-\sigma_\theta^2} . \quad (17)$$

Having determined $E\{z_i z_j\}$ we are now ready to address the determination of $E\{\hat{A}^2\}$ given by (10). Notice however that the expression to use for the argument of the double summation is different whether indices i and j are equal or not, namely (15) and (16) respectively. In order to proceed with the derivation we need to have complete summations, that is, summations whose indices span all possible values, and that have in its argument a single expression for all cases of the indices. This can be achieved by splitting the summation in (10) into three summations as illustrated in Fig. 3.

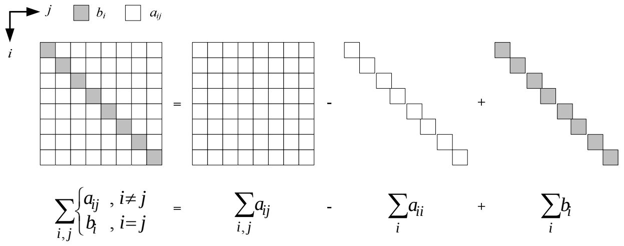

Fig. 3 – Illustration of a double summation split into three other summations (one double and two simple).

This summation can be divided into two terms: the first one a double summation on i and j for $i \neq j$ using (16) in its argument; and the second one, a simple summation on i where $j = i$ using (15) in its argument. The new double summation, however can be written as a double summation for all values of i and j (using (16)) minus a simple summation with $j = i$ using (16) in its argument:

$$E\{\hat{A}^2\} = \frac{2}{M^2}\sum_{i,j} A^2 \cos(\omega_x t_i - \omega_x t_j)\cos[\omega_a(t_i - t_j)] e^{-\sigma_\theta^2} - \frac{2}{M}A^2 e^{-\sigma_\theta^2} + \frac{2}{M}A^2 . \quad (18)$$

Note that, taking into account (7), that the terms in C^2 become null.

Considering that we know the signal frequency and use it for the sine wave we are trying to fit to the data, ($\omega_a = \omega_x$), we have

$$E\{\hat{A}^2\} = A^2 e^{-\sigma_\theta^2} + \frac{1}{M^2}A^2 e^{-\sigma_\theta^2}\sum_{i,j}\cos(2\omega_x t_i - 2\omega_x t_j) + \frac{2}{M}A^2\left(1 - e^{-\sigma_\theta^2}\right) , \quad (19)$$

where we used a trigonometric relation to transform the product of two cosine function into the sum of two cosine function. Since we are considering that the sine wave fit to the data covers an integer number of periods, the summation in i and j is 0, leading to

$$\mu_{\hat{A}^2} = E\{\hat{A}^2\} = A^2 e^{-\sigma_\theta^2} + \frac{2}{M}A^2\left(1 - e^{-\sigma_\theta^2}\right) . \quad (20)$$

IV. VARIANCE OF ESTIMATED SQUARE AMPLITUDE

The variance of a random variable can be expressed as the difference between the second moment and the square of the mean [21]. In the case of the variance of $\hat{A}^2$ this leads to

$$\sigma^2_{\hat{A}^2} = E\{\hat{A}^4\} - E^2\{\hat{A}^2\} . \quad (21)$$

Using (9) we can write

$$\hat{A}^4 = \frac{16}{M^4}\sum_{i,j,k,l} z_i z_j z_k z_l \cos[\omega_a(t_i - t_j)]\cos[\omega_a(t_k - t_l)] . \quad (22)$$

The expected value of the forth power of the estimated amplitude is thus

$$E\{\hat{A}^4\} = \frac{16}{M^4}\sum_{i,j,k,l} E\{z_i z_j z_k z_l\}\cos[\omega_a(t_i - t_j)]\cos[\omega_a(t_k - t_l)] . \quad (23)$$

Inserting (10) and (23) into (21) and making use of ([21], eq. 7-7)

$$Cov\{x,y\} = E\{xy\} - E\{x\}E\{y\} , \quad (24)$$

we have for the variance of the square estimated amplitude

$$\sigma^2_{\hat{A}^2} = \frac{16}{M^4}\sum_{i,j,k,l} Cov\left\{z_i z_j, z_k z_l\right\}\cos\left[\omega_a\left(t_i-t_j\right)\right]\cos\left[\omega_a\left(t_k-t_l\right)\right] . \quad (25)$$

The actual voltage of a sample, z, can be expressed as

$$z_i = C + w_i , \quad (26)$$

where

$$w_i = A\cos(\omega_x t_i + \varphi + \theta_i) , \quad (27)$$

As seen in Appendix A, the variance of the estimated square value of amplitude does not depend on the stimulus signal offset. As such (25) can be written as

$$\sigma^2_{\hat{A}^2} = \frac{16}{M^4}\sum_{i,j,k,l} Cov\left\{w_i w_j, w_k w_l\right\}\cos\left[\omega_a\left(t_i-t_j\right)\right]\cos\left[\omega_a\left(t_k-t_l\right)\right] . \quad (28)$$

Since the argument of the two cosine function are not random variables we can place them inside the covariance:

$$\sigma^2_{\hat{A}^2} = \frac{16}{M^4}\sum_{i,j,k,l} Cov\left\{w_i w_j \cos\left[\omega_a\left(t_i-t_j\right)\right], w_k w_l \cos\left[\omega_a\left(t_k-t_l\right)\right]\right\} . \quad (29)$$

Inserting (27) into (29) leads to

$$\sigma^2_{\hat{A}^2} = \frac{16A^4}{M^4}\sum_{i,j,k,l} Cov\left\{\begin{array}{l}\cos\left(\omega_x t_i+\theta_i+\varphi\right)\cos\left(\omega_x t_j+\theta_j+\varphi\right)\cos\left[\omega_a\left(t_i-t_j\right)\right],\\ \cos\left(\omega_x t_k+\theta_k+\varphi\right)\cos\left(\omega_x t_l+\theta_l+\varphi\right)\cos\left[\omega_a\left(t_k-t_l\right)\right]\end{array}\right\} \quad (30)$$

Again using

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b) , \quad (31)$$

we can write

$$\sigma^2_{\hat{A}^2} = \frac{4A^4}{M^4}\sum_{i,j,k,l} Cov\left\{\begin{array}{l}\cos\left(\omega_x t_i-\omega_x t_j+\theta_i-\theta_j\right)\cos\left[\omega_a\left(t_i-t_j\right)\right]+\\ +\cos\left(\omega_x t_i+\omega_x t_j+\theta_i+\theta_j+2\varphi\right)\cos\left[\omega_a\left(t_i-t_j\right)\right],\\ \cos\left(\omega_x t_k-\omega_x t_l+\theta_k-\theta_l\right)\cos\left[\omega_a\left(t_k-t_l\right)\right]+\\ +\cos\left(\omega_x t_k+\omega_x t_l+\theta_k+\theta_l+2\varphi\right)\cos\left[\omega_a\left(t_k-t_l\right)\right]\end{array}\right\} . \quad (32)$$

Now using

$$Cov\{a+b, c+d\} = Cov\{a,c\} + Cov\{b,d\} , \quad (33)$$

we can write (32) as

$$\begin{array}{l}\sigma^2_{\hat{A}^2} = \frac{4A^4}{M^4}\sum_{i,j,k,l} Cov\left\{\begin{array}{l}\cos\left(\omega_x t_i-\omega_x t_j+\theta_i-\theta_j\right)\cos\left[\omega_a\left(t_i-t_j\right)\right],\\ \cos\left(\omega_x t_k-\omega_x t_l+\theta_k-\theta_l\right)\cos\left[\omega_a\left(t_k-t_l\right)\right]\end{array}\right\}+\\ +\frac{4A^4}{M^4}\sum_{i,j,k,l} Cov\left\{\begin{array}{l}\cos\left(\omega_x t_i+\omega_x t_j+\theta_i+\theta_j+2\varphi\right)\cos\left[\omega_a\left(t_i-t_j\right)\right],\\ \cos\left(\omega_x t_k+\omega_x t_l+\theta_k+\theta_l+2\varphi\right)\cos\left[\omega_a\left(t_k-t_l\right)\right]\end{array}\right\}\end{array} . \quad (34)$$

Using (31) leads to

$$\begin{array}{l}\sigma^2_{\hat{A}^2} = \frac{A^4}{M^4}\sum_{i,j,k,l} Cov\left\{\begin{array}{l}\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right)+\cos\left(\theta_i-\theta_j\right),\\ \cos\left(2\omega_x t_k-2\omega_x t_l+\theta_k-\theta_l\right)+\cos\left(\theta_k-\theta_l\right)\end{array}\right\}+\\ +\frac{A^4}{M^4}\sum_{i,j,k,l} Cov\left\{\begin{array}{l}\cos\left(2\omega_x t_i+\theta_i+\theta_j+2\varphi\right)+\cos\left(2\omega_x t_j+\theta_i+\theta_j+2\varphi\right),\\ \cos\left(2\omega_x t_k+\theta_k+\theta_l+2\varphi\right)+\cos\left(2\omega_x t_l+\theta_k+\theta_l+2\varphi\right)\end{array}\right\}\end{array} . \quad (35)$$

Note that the index of the summations can be exchanged. For example i can become j and j can become i without changing the result of the summation. Using this and (52) allows us to write (35) as

$$\begin{array}{l}\frac{M^4}{A^4}\sigma^2_{\hat{A}^2} = \sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right), \cos\left(2\omega_x t_k-2\omega_x t_l+\theta_k-\theta_l\right)\right\}+\\ +2\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right), \cos\left(\theta_k-\theta_l\right)\right\}+\\ +\sum_{i,j,k,l} Cov\left\{\cos\left(\theta_i-\theta_j\right), \cos\left(\theta_k-\theta_l\right)\right\}+\\ +4\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i+\theta_i+\theta_j+2\varphi\right), \cos\left(2\omega_x t_k+\theta_k+\theta_l+2\varphi\right)\right\}\end{array} . \quad (36)$$

Note that, being φ an uniformly distributed random variable between $-\pi$ and π, $E\{\cos(\alpha+\varphi)\}=0$. Using this we can simplify the 4th term of the second member of (36) since

$$\begin{array}{l}Cov\{\cos(\alpha+2\varphi), \cos(\beta+2\varphi)\} =\\ = E\{\cos(\alpha+2\varphi)\cos(\beta+2\varphi)\} - E\{\cos(\alpha+2\varphi)\}E\{\cos(\beta+2\varphi)\} =\\ = \frac{1}{2}E\{\cos(\alpha-\beta)\} + \frac{1}{2}E\{\cos(\alpha+\beta+4\varphi)\} - E\{\cos(\alpha+2\varphi)\}E\{\cos(\beta+2\varphi)\} =\\ = \frac{1}{2}E\{\cos(\alpha-\beta)\}\end{array} . \quad (37)$$

We have then

$$\begin{array}{l}\frac{M^4}{A^4}\sigma^2_{\hat{A}^2} = \sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right), \cos\left(2\omega_x t_k-2\omega_x t_l+\theta_k-\theta_l\right)\right\}+\\ +2\sum_{i,j,k,l} Cov\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right), \cos\left(\theta_k-\theta_l\right)\right\}+\\ +\sum_{i,j,k,l} Cov\left\{\cos\left(\theta_i-\theta_j\right), \cos\left(\theta_k-\theta_l\right)\right\}+\\ +2\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x\left(t_i-t_k\right)+\theta_i+\theta_j-\theta_k-\theta_l\right)\right\}\end{array} . \quad (38)$$

Substituting the covariance by expected values $(Cov\{a,b\}=E\{ab\}-E\{a\}E\{b\})$, (38) can be written as

$$\begin{array}{l}\frac{M^4}{A^4}\sigma^2_{\hat{A}^2} = \sum_{i,j,k,l} E\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right)\cos\left(2\omega_x t_k-2\omega_x t_l+\theta_k-\theta_l\right)\right\}-\\ -\left[\sum_{i,j} E\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right)\right\}\right]^2+\\ +2\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right)\cos\left(\theta_k-\theta_l\right)\right\}-\\ -2\sum_{i,j} E\left\{\cos\left(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j\right)\right\}\sum_{k,l} E\left\{\cos\left(\theta_k-\theta_l\right)\right\}+\\ +\sum_{i,j,k,l} E\left\{\cos\left(\theta_i-\theta_j\right)\cos\left(\theta_k-\theta_l\right)\right\} - \left[\sum_{i,j} E\left\{\cos\left(\theta_i-\theta_j\right)\right\}\right]^2+\\ +2\sum_{i,j,k,l} E\left\{\cos\left(2\omega_x\left(t_i-t_k\right)+\theta_i+\theta_j-\theta_k-\theta_l\right)\right\}\end{array} . \quad (39)$$

Note that the product of cosine function may be written as the sum of cosine functions. For instance, looking at the 5th term in the second member of (39), we have

$$\begin{array}{l}\sum_{i,j,k,l} E\left\{\cos\left(\theta_i-\theta_j\right)\cos\left(\theta_k-\theta_l\right)\right\} =\\ = \sum_{i,j,k,l} E\left\{\frac{1}{2}\cos\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right\} + \sum_{i,j,k,l} E\left\{\frac{1}{2}\cos\left(\theta_i-\theta_j-\theta_k+\theta_l\right)\right\}\end{array} . \quad (40)$$

Since the cosine functions are inside a summation, we can swap index k with index l in the last term of (40) without altering the summation. Doing this, results in the two terms in the second member of (40) to being exactly the same. We thus have

$$\sum_{i,j,k,l} E\left\{\cos(\theta_i-\theta_j)\cos(\theta_k-\theta_l)\right\} = \sum_{i,j,k,l} E\left\{\cos(\theta_i-\theta_j+\theta_k-\theta_l)\right\} . \quad (41)$$

Applying this reasoning also to the 1st and 3rd terms of the second member of (39) leads to

$$\begin{aligned}\frac{M^4}{A^4}\sigma^2_{\hat{A}^2} = &\sum_{i,j,k,l} E\left\{\cos\left[2\omega_x(t_i-t_j+t_k-t_l)+\theta_i-\theta_j+\theta_k-\theta_l\right]\right\} - \\ &-\left[\sum_{i,j}E\left\{\cos\left[2\omega_x(t_i-t_j)+\theta_i-\theta_j\right]\right\}\right]^2 + \\ &+4\sum_{i,j,k,l}E\left\{\cos\left[2\omega_x(t_i-t_j)+\theta_i-\theta_j+\theta_k-\theta_l\right]\right\} - \\ &-2\sum_{i,j}E\left\{\cos\left[2\omega_x(t_i-t_j)+\theta_i-\theta_j\right]\right\}\sum_{i,j}E\left\{\cos(\theta_i-\theta_j)\right\} + \\ &+\sum_{i,j,k,l}E\left\{\cos(\theta_i-\theta_j+\theta_k-\theta_l)\right\} - \left[\sum_{i,j}E\left\{\cos(\theta_i-\theta_j)\right\}\right]^2\end{aligned} . \quad (42)$$

The double summation in the 1st and 4th terms of the second member is equal to

$$\begin{aligned}&\sum_{i,j}E\left\{\cos(2\omega_x t_i-2\omega_x t_j+\theta_i-\theta_j)\right\} = \\ &=\sum_{i,j}\begin{cases}1 & , i=j \\ e^{-\sigma_\theta^2}\cos(2\omega_x t_i-2\omega_x t_j) & , i\neq j\end{cases} = \\ &=\sum_{i,j}e^{-\sigma_\theta^2}\cos(2\omega_x t_i-2\omega_x t_j)-\sum_{i=j}e^{-\sigma_\theta^2}\cos(2\omega_x t_i-2\omega_x t_j)+\sum_i 1 = \\ &=0-Me^{-\sigma_\theta^2}+M = \\ &=M\left(1-e^{-\sigma_\theta^2}\right)\end{aligned} . \quad (43)$$

The double summation in the 4th and 6th terms is

$$\begin{aligned}&\sum_{i,j}E\left\{\cos(\theta_i-\theta_j)\right\}=\sum_{i,j}\begin{cases}1 & , i=j \\ e^{-\sigma_\theta^2} & , i\neq j\end{cases} = \\ &=\sum_{i,j}e^{-\sigma_\theta^2}-\sum_i e^{-\sigma_\theta^2}+\sum_i 1 = \\ &=M^2e^{-\sigma_\theta^2}-Me^{-\sigma_\theta^2}+M\end{aligned} . \quad (44)$$

The other summation are computed in Appendix B, C and D. Inserting (43), (44), (73), (76) and (79) into (42) leads to

$$\begin{aligned}\sigma^2_{\hat{A}^2} = &\frac{A^4}{M}\left(4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+2e^{-3\sigma_\theta^2}\right) + \\ &+\frac{A^4}{M^2}\left(4-20e^{-\sigma_\theta^2}+29e^{-2\sigma_\theta^2}-14e^{-3\sigma_\theta^2}+e^{-4\sigma_\theta^2}\right) + \\ &+\frac{A^4}{M^3}\left(-6+24e^{-\sigma_\theta^2}-36e^{-2\sigma_\theta^2}+24e^{-3\sigma_\theta^2}-6e^{-4\sigma_\theta^2}\right)\end{aligned} . \quad (45)$$

V. Bias of the Estimated Sine Wave Amplitude

We are going to use the Taylor series to approximate the non linear relation between square amplitude and amplitude by a polynomial. This allows us to approximately determine the expected value of the estimated amplitude from the expected value of the square amplitude, given by (20), and the variance of the square amplitude, given by (45) as done in [21]:

$$\mu_{\hat{A}} \simeq \sqrt{\mu_{\hat{A}^2}} - \frac{\sigma^2_{\hat{A}^2}}{8\sqrt{\mu^3_{\hat{A}^2}}} . \quad (46)$$

We define now the relative error of the estimation as

$$\varepsilon_A = \frac{\mu_{\hat{A}}-A}{A} . \quad (47)$$

Inserting (20), (45) into (46) and (46) into (47), leads to

$$\boxed{\varepsilon_A=\sqrt{e^{-\sigma_\theta^2}+\frac{2}{M}\left(1-e^{-\sigma_\theta^2}\right)}-\frac{\left[\begin{aligned}&\frac{1}{M}\left(4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+2e^{-3\sigma_\theta^2}\right)+\\&+\frac{1}{M^2}\left(4-20e^{-\sigma_\theta^2}+29e^{-2\sigma_\theta^2}-14e^{-3\sigma_\theta^2}+e^{-4\sigma_\theta^2}\right)+\\&+\frac{1}{M^3}\left(-6+24e^{-\sigma_\theta^2}-36e^{-2\sigma_\theta^2}+24e^{-3\sigma_\theta^2}-6e^{-4\sigma_\theta^2}\right)\end{aligned}\right]}{8\sqrt{\left(e^{-\sigma_\theta^2}+\frac{2}{M}\left(1-e^{-\sigma_\theta^2}\right)\right)^3}}-1} , \quad (48)$$

which is the relative bias of the sine wave amplitude estimation using the IEEE 1057 sine fitting algorithm in the presence of jitter. Note that the relative error does not depend on the sine wave amplitude, but only on the number of samples and the phase noise (or jitter) standard deviation).

In order to validate the approximation made in (46) and to check the correctness of the derivations carried out, we did a Monte Carlo analysis of the estimator bias by simulating on a computer a set of data points from a sine wave with sampling instants corrupted by jitter, applying the sine fitting to estimate the amplitude and repeated the procedure 10^4 times to compute the expected value of the estimated amplitude. In Fig. 4 (markers) the relative error obtained is depicted as a function of the phase noise standard deviation for 10 and for 1000 samples (M).

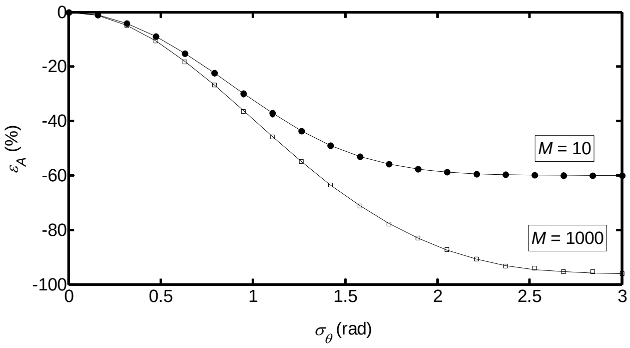

Fig. 4 – Relative error of the estimated sine wave amplitude as a function of phase noise standard deviation (markers). A 2 V sine wave with $f_s/f_x = M$ was used and 10^4 repetitions were carried out. The confidence intervals for a confidence level of 99.9 % are two small to be represented. The solid lines represent the theoretical value given (48).

It can be seen that the relative error of the expected value of the estimated amplitude obtained through numerical simulation, is in accordance with the theoretical value given by (48). In Fig. 5 the deviation of estimated amplitude relative error and theoretical value as a function of phase noise standard deviation is shown with confidence intervals corresponding to 99.9 % confidence level for a normal distribution. All confidence intervals are around 0 (null deviation from numerical simulation and theoretical values) which shows that the approximation made in (46) is valid for the conditions simulated.

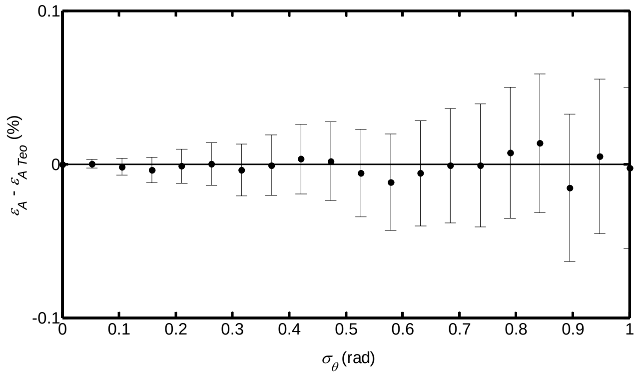

Fig. 5 – Deviation of estimated amplitude relative error and theoretical value given by (46) as a function of phase noise standard deviation for a 2 V sine wave with $f_s/f_x = M = 100$. 2×10^5 repetitions were carried out to determine the mean estimated amplitude. The vertical bars represent the confidence interval for a 99.9 % confidence level assuming a normal distribution.

In Fig. 6, the dependence of the relative estimation error on the number of samples can more easily be observed. Again we see that as the number of samples increase, the relative error gets bigger.

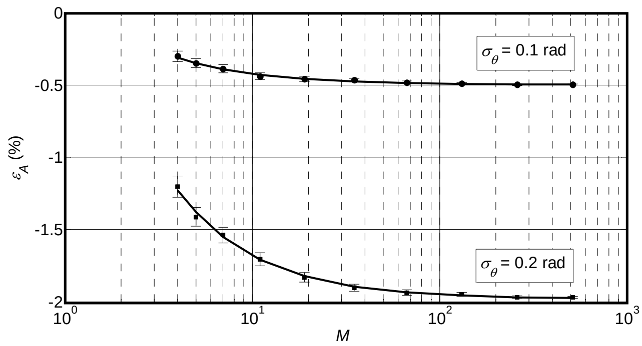

Fig. 6 – Estimated amplitude relative error and theoretical value given by (48) as a function of the number of samples for a 2 V sine wave with $f_s/f_x = M$ and a phase noise standard deviation of 0.1 rad (top) and 0.2 rad (bottom). 10^5 repetitions were carried out to determine the mean estimated amplitude. The vertical bars represent the confidence interval for a 99.9 % confidence level assuming a normal distribution.

From (48) we can compute the limit when the number of samples goes to infinity:

$$\lim_{M\to\infty} \varepsilon_A = e^{-\frac{1}{2}\sigma_\theta^2} - 1 \,. \qquad (49)$$

This, which is the result obtained in [19], shows that the estimator is asymptotically biased in the presence of jitter since the relative error does not go to 0 when the number of samples tends to infinity.

The theoretical results presented here we also subject of experimental validation. Preliminary results show a good agreement between theory and practice. We will dedicate a future publication to the presentation of the full experimental results as well as a detailed description of the test setup created to inject jitter into the ADC in a controllable manner including its calibration.

VI. Conclusions

The expression derived here for the bias of the fitted sine wave amplitude obtained with the 3-parameter sine algorithm, given in (48), shows that the estimator is biased when the acquired samples are affected by jitter which can be due to the analog converter itself or to phase noise in the sampling clock. The existence of this bias was previously mentioned in [19] but only the case of an infinite number of samples was considered. Here we presented an expression that allows the computation of the estimator relative bias given the number of acquired samples and the standard deviation of the jitter or phase noise.

Expression (48) can be used to correct the bias of the estimator if the amount of jitter present is known which can be accomplished using, for instance, the methods recommended in [4].

We limited here our study to the effect of jitter on the estimation of the sine wave amplitude, however we proceed doing work on the effect of jitter on other estimator related to the sine fitting, namely the sine wave offset, initial phase and frequency as well as other parameters derived from them like the module and argument of impedances determined with the help of sine fitting, or signal to noise and distortion ratio (SINAD) of analog to digital converters.

The influence of other non ideal factors, like harmonic distortion, additive noise and frequency error, on the bias and on the variance of the estimators has also to be studied in the future to achieve a full understanding of the performance of sine fitting algorithms in real conditions.

Appendix A

Here we show that the variance of the estimated square amplitude does, given by (25),

$$\sigma^2_{\hat{A}^2} = \frac{16}{M^4} \sum_{i,j,k,l} Cov\{z_i z_j, z_k z_l\} \cos\left[\omega_a (t_i - t_j)\right] \cos\left[\omega_a (t_k - t_l)\right] \qquad (50)$$

does not depend on the stimulus signal offset, C.

We can write the covariance of $z_i z_j$ with $z_k z_l$ as

$$\begin{aligned} Cov\{z_i z_j, z_k z_l\} &= Cov\{(C+w_i)(C+w_j),(C+w_k)(C+w_l)\} = \\ &= Cov\{C^2 + C(w_i + w_j) + w_i w_j, C^2 + C(w_k + w_l) + w_k w_l\} \end{aligned} \qquad (51)$$

where we have made use of (26). The covariance of the sum of random variables can be expressed as the sum of the covariance between the different summation terms:

$$Cov\left\{\sum_i x_i, \sum_j y_j\right\} = \sum_i \sum_j Cov\{x_i, y_j\} \,. \qquad (52)$$

This allows (51) to be written as

$$\begin{aligned} Cov\{z_i z_j, z_k z_l\} &= Cov\{C^2, C^2\} + Cov\{C^2, C(w_k + w_l)\} + \\ &+ Cov\{C^2, w_k w_l\} + Cov\{C(w_i + w_j), C^2\} + \\ &+ Cov\{C(w_i + w_j), C(w_k + w_l)\} + Cov\{C(w_i + w_j), w_k w_l\} + \\ &+ Cov\{w_i w_j, C^2\} + Cov\{w_i w_j, C(w_k + w_l)\} + Cov\{w_i w_j, w_k w_l\} \end{aligned} \,. \qquad (53)$$

Using the covariance property

$$Cov\{a, x\} = E\{ax\} - E\{a\}E\{x\} = aE\{x\} - aE\{x\} = 0 \,, \qquad (54)$$

where a is a constant and x is a random variable, we can write (51) as

$$\begin{aligned} Cov\{z_i z_j, z_k z_l\} &= Cov\{C(w_i + w_j), C(w_k + w_l)\} + \\ &+ Cov\{C(w_i + w_j), w_k w_l\} + \\ &+ Cov\{w_i w_j, C(w_k + w_l)\} + \\ &+ Cov\{w_i w_j, w_k w_l\} \end{aligned} \,. \qquad (55)$$

Using another covariance property, specifically

$$\begin{aligned}Cov\{ax,by\}&=E\{abxy\}-E\{ax\}E\{by\}=\\&=abE\{xy\}-abE\{x\}E\{y\}=\\&=abCov\{x,y\}\end{aligned}, \qquad (56)$$

where a and b are constants and x and y are random variables, we have

$$\begin{aligned}Cov\{z_i z_j,z_k z_l\}&=C^2 Cov\{w_i+w_j,w_k+w_l\}+C\cdot Cov\{w_i+w_j,w_k w_l\}+\\&+C\cdot Cov\{w_i w_j,w_k+w_l\}+Cov\{w_i w_j,w_k w_l\}\end{aligned}. \qquad (57)$$

Using (52) leads to

$$\begin{aligned}&Cov\{z_i z_j,z_k z_l\}=Cov\{w_i w_j,w_k w_l\}+\\&+C^2\left[Cov\{w_i,w_k\}+Cov\{w_i,w_l\}+Cov\{w_j,w_k\}+Cov\{w_j,w_l\}\right]+\\&+C\left[Cov\{w_i,w_k w_l\}+Cov\{w_j,w_k w_l\}+Cov\{w_i w_j,w_k\}\cdot Cov\{w_i w_j,w_l\}\right]\end{aligned}. \qquad (58)$$

Making use of the definition of covariance we have

$$Cov\{w_i,w_j w_k\}=E\{w_i w_j w_k\}-E\{w_i\}E\{w_j w_k\}. \qquad (59)$$

The expected value of w_i is

$$E\{w_i\}=E\{A\cos(\omega_x t_i+\varphi+\theta_i)\}=0. \qquad (60)$$

considering φ a uniform random variable distributed between 0 and 2π. The same can be said of the product $w_i w_j w_k$:

$$E\{w_i w_j w_k\}=E\{A^3\cos(\omega_x t_i+\varphi+\theta_i)\cos(\omega_x t_j+\varphi+\theta_j)\cos(\omega_x t_k+\varphi+\theta_k)\}=0. \qquad (61)$$

Inserting (60) and (61) into (59) leads to

$$Cov\{w_i,w_j w_k\}=0. \qquad (62)$$

Equation (58) than becomes

$$\begin{aligned}&Cov\{z_i z_j,z_k z_l\}=Cov\{w_i w_j,w_k w_l\}+\\&+C^2\left[Cov\{w_i,w_k\}+Cov\{w_i,w_l\}+Cov\{w_j,w_k\}+Cov\{w_j,w_l\}\right]\end{aligned}. \qquad (63)$$

We can insert this into (50) and write

$$\begin{aligned}\sigma^2_{\hat{A}^2}=&\frac{16}{M^4}\sum_{i,j,k,l}Cov\{w_i w_j,w_k w_l\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]+\\&+\frac{16}{M^4}C^2\sum_{i,j,k,l}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]\\&+\frac{16}{M^4}C^2\sum_{i,j,k,l}Cov\{w_i,w_l\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]\\&+\frac{16}{M^4}C^2\sum_{i,j,k,l}Cov\{w_j,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]\\&+\frac{16}{M^4}C^2\sum_{i,j,k,l}Cov\{w_j,w_l\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]\end{aligned}. \qquad (64)$$

By exchanging the indexed of the summations we realize that all summations that are multiplied by C^2 are the same. We thus have

$$\begin{aligned}\sigma^2_{\hat{A}^2}=&\frac{16}{M^4}\sum_{i,j,k,l}Cov\{w_i w_j,w_k w_l\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]+\\&+\frac{64}{M^4}C^2\sum_{i,j,k,l}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]\end{aligned}. \qquad (65)$$

Looking at the second summation in (65) we see that the summation in l can be separated from the other summations:

$$\begin{aligned}&\sum_{i,j,k,l}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]=\\&=\sum_{i,j,k,l}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\cos(\omega_a t_k)\cos(\omega_a t_l)+\\&+\sum_{i,j,k,l}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\sin(\omega_a t_k)\sin(\omega_a t_l)=\\&=\sum_l\cos(\omega_a t_l)\sum_{i,j,k}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\cos(\omega_a t_k)+\\&+\sum_l\sin(\omega_a t_l)\sum_{i,j,k}Cov\{w_i,w_k\}\cos\left[\omega_a(t_i-t_j)\right]\sin(\omega_a t_k)\end{aligned}. \qquad (66)$$

The same could be done for the summation in j. Considering that we are fitting an integer number of periods of a sine wave to the data, the summations in l of the sine and the cosine are null which makes (66) null. Equation (65) then becomes

$$\sigma^2_{\hat{A}^2}=\frac{16}{M^4}\sum_{i,j,k,l}Cov\{w_i w_j,w_k w_l\}\cos\left[\omega_a(t_i-t_j)\right]\cos\left[\omega_a(t_k-t_l)\right]. \qquad (67)$$

APPENDIX B

In this appendix we compute the first term in second member of (42). To determine an expression for the expected value we have to consider whether some of the indices are equal because is such cases the random variables θ will cancel each other out. There are 14 different cases where one or more of the 4 indices i, j, k and l are equal. Those cases are illustrated in (68). To make it easier to read the expressions that follow, we have chosen to attribute different symbols ($\bullet\times\circ *$) to the indices. For example, the first case in (68) is identified by the symbols ($\bullet\bullet\bullet\bullet$). This means that all the 4 indices are the same. Note that this case encompasses many different possible value of the indices (they can be all equal to 1 or 2, or 3, etc…). In the second case in (68), for example, we indexes i, j and k are the same and index l is different ($\bullet\bullet\bullet\times$). In the last case in (68) all the 4 indices are different.

$$\begin{aligned}&\sum_{i,j,k,l}E\left\{\cos\left[\omega_x(t_i-t_j+t_k-t_l)+(\theta_i-\theta_j+\theta_k-\theta_l)\right]\right\}=\\&=\sum_{i,j,k,l}\begin{cases}1 & \bullet\bullet\bullet\bullet & M\\ e^{-\sigma_\theta^2}\cos(\omega_x t_k-\omega_x t_l) & \bullet\bullet\bullet\times & -M\\ e^{-\sigma_\theta^2}\cos(\omega_x t_k-\omega_x t_l) & \bullet\bullet\times\bullet & -M\\ e^{-\sigma_\theta^2}\cos(\omega_x t_i-\omega_x t_j) & \bullet\times\bullet\bullet & -M\\ e^{-\sigma_\theta^2}\cos(\omega_x t_i-\omega_x t_j) & \times\bullet\bullet\bullet & -M\\ 1 & \bullet\bullet\times\times & M(M-1)\\ e^{-4\sigma_\theta^2}\cos(2\omega_x t_i-2\omega_x t_j) & \bullet\times\bullet\times & -M\\ 1 & \bullet\times\times\bullet & M(M-1)\\ e^{-\sigma_\theta^2}\cos(\omega_x t_k-\omega_x t_l) & \bullet\bullet\times\circ & -M(M-2)\\ e^{-3\sigma_\theta^2}\cos(2\omega_x t_i-\omega_x t_j-\omega_x t_l) & \bullet\times\bullet\circ & 2M\\ e^{-\sigma_\theta^2}\cos(\omega_x t_j-\omega_x t_k) & \bullet\times\circ\bullet & -M(M-2)\\ e^{-\sigma_\theta^2}\cos(\omega_x t_i-\omega_x t_l) & \times\bullet\bullet\circ & -M(M-2)\\ e^{-3\sigma_\theta^2}\cos(\omega_x t_i-2\omega_x t_j+\omega_x t_k) & \times\bullet\circ\bullet & 2M\\ e^{-\sigma_\theta^2}\cos(\omega_x t_i-\omega_x t_j) & \times\circ\bullet\bullet & -M(M-2)\\ e^{-2\sigma_\theta^2}\cos(\omega_x t_i-\omega_x t_j+\omega_x t_k-\omega_x t_l) & \bullet\times\circ * & 2M^2-6M\end{cases}\end{aligned}. \qquad (68)$$

Each of the 15 cases in the curly bracket corresponds to a different set of values of i, j, k and l. Since all those cases are mutually exclusive the quadruple summation of the bracket in (68) can be expressed as the sum of 15 summations with different arguments (the ones in the curly bracket). The value of those summations, not considering the exponential term, is indicated in the right most column of (68).

We will look now at how the value of some of those summations was obtained.

The first summation in the curly brackets is the number of elements which in this case is M.

The second summation can be seen as a complete double summation minus the cases where $k = l$:

$$\sum_{k\neq l}\cos(\omega_x t_k-\omega_x t_l)=\sum_{k,l}\cos(\omega_x t_k-\omega_x t_l)-\sum_{k=l}\cos(\omega_x t_k-\omega_x t_l)=0-M=-M \quad (69)$$

The complete summation has a null value since we have an integer number of periods of the cosine function and there are M cases where $k = l$. The summation will thus be $–M$.

The sixth summation is

$$\sum_{i\neq j}1=\sum_{i,j}1-\sum_{i=j}1=M^2-M=M(M-1)\,. \quad (70)$$

The ninth summation is

$$\sum_{j\neq k\neq l}\cos(\omega_x t_k-\omega_x t_l)=(M-2)\sum_{k\neq l}\cos(\omega_x t_k-\omega_x t_l)=-M(M-2)\,. \quad (71)$$

The argument of this summation does not depend on j and there are $M-2$ values of j which are different from k and l. This term thus has $M-2$ times the summation on k and l which, as was seen in (69), equals $–M$.

The 13th summation is a triple summation which can be split into a complete triple summation minus the cases where two or three indices are the same. The complete summation is null so we have:

$$\begin{aligned}&\sum_{i\neq j\neq k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]=\\&=-\sum_{i=j\neq k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]-\sum_{i\neq j=k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]-\\&-\sum_{i=k\neq j}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]-\sum_{i=j=k}\cos\left[2\omega_x t_i-\omega_x t_j-\omega_x t_k\right]\end{aligned}\,. \quad (72)$$

The first 3 summations in the second member of (72) are equal to $– M$. The argument of the cosine in last summation is null since all the indices are the same. As the indices go from 1 to M, there are M terms equal to 1 (cos(0)). The last term in (72) is thus M. The 13th summation in (68) is thus equal to $2M$.

The last summation in (68) has a quadruple summation where all the indices have different values. This partial summation can be seen as a complete quadruple summation minus the cases where some or all the indices are equal. The complete summation has a null value since we have an integer number of periods of the cosine function and there are $6M-2M^2$ cases where some or all the indices are equal. This is just the sum of the cases in all the other terms.

Putting all the terms together leads to

$$\begin{aligned}&\sum_{i,j,k,l}E\left\{\cos\left[\omega_x(t_i-t_j+t_k-t_l)+(\theta_i-\theta_j+\theta_k-\theta_l)\right]\right\}=\\&=M+2M(M-1)-4Me^{-\sigma_\theta^2}-Me^{-4\sigma_\theta^2}-4M(M-2)e^{-\sigma_\theta^2}+\\&+4Me^{-3\sigma_\theta^2}+\left(2M^2-6M\right)e^{-2\sigma_\theta^2}=\\&=M\left(-1+4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+4e^{-3\sigma_\theta^2}-e^{-4\sigma_\theta^2}\right)+2M^2\left(1-2e^{-\sigma_\theta^2}+e^{-2\sigma_\theta^2}\right)\end{aligned} \quad (73)$$

APPENDIX C

In this appendix we compute the third term in second member of (42). Here we proceed as we did in Appendix B. All the 15 cases where the 4 indices can be equal to each other are enumerated and the expected value is computed individually for each of those cases.

$$\begin{aligned}&\sum_{i,j,k,l}E\left\{\cos\left[\omega_x(t_i-t_j)+(\theta_i-\theta_j+\theta_k-\theta_l)\right]\right\}=\\&=\sum_{i,j,k,l}\begin{cases}1 & \bullet\bullet\bullet\bullet & M\\ e^{-\sigma_\theta^2} & \bullet\bullet\bullet\times & M(M-1)\\ e^{-\sigma_\theta^2} & \bullet\bullet\times\bullet & M(M-1)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\bullet\bullet & -M\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\bullet\bullet\bullet & -M\\ 1 & \bullet\bullet\times\times & M(M-1)\\ e^{-4\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\bullet\times & -M\\ \cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\times\bullet & -M\\ e^{-\sigma_\theta^2} & \bullet\bullet\times\circ & M(M-1)(M-2)\\ e^{-3\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\bullet\circ & -M(M-2)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_j-\omega_x t_j\right] & \bullet\times\circ\bullet & -M(M-2)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\bullet\bullet\circ & -M(M-2)\\ e^{-3\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\bullet\circ\bullet & -M(M-2)\\ e^{-\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \times\circ\bullet\bullet & -M(M-2)\\ e^{-2\sigma_\theta^2}\cos\left[\omega_x t_i-\omega_x t_j\right] & \bullet\times\circ * & -M(M-2)(M-3)\end{cases}\end{aligned}\,. \quad (74)$$

Again, in the right most column of (74) we indicate the value of the summations without considering the exponential terms. Equation (74) thus becomes

$$\begin{aligned}&\sum_{i,j,k,l}E\left\{\cos\left[\omega_x(t_i-t_j)+(\theta_i-\theta_j+\theta_k-\theta_l)\right]\right\}=\\&\left[M+M(M-1)-M\right]+\left[2M(M-1)-2M+M(M-1)(M-2)-3M(M-2)\right]e^{-\sigma_\theta^2}+\\&+\left[-M(M-2)(M-3)\right]e^{-2\sigma_\theta^2}+\left[-2M(M-2)\right]e^{-3\sigma_\theta^2}-Me^{-4\sigma_\theta^2}=\\&=\left[M^2-M\right]+\left[4M-4M^2+M^3\right]e^{-\sigma_\theta^2}+\\&+\left[-6M+5M^2-M^3\right]e^{-2\sigma_\theta^2}+\left[4M-2M^2\right]e^{-3\sigma_\theta^2}-Me^{-4\sigma_\theta^2}\end{aligned} \quad (75)$$

This can be further simplified to

$$\begin{aligned}&\sum_{i,j,k,l}E\left\{\cos\left[\omega_x(t_i-t_j)+(\theta_i-\theta_j+\theta_k-\theta_l)\right]\right\}=\\&=M\left(-1+4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+4e^{-3\sigma_\theta^2}-e^{-4\sigma_\theta^2}\right)+\\&+M^2\left(1-4e^{-\sigma_\theta^2}+5e^{-2\sigma_\theta^2}-2e^{-3\sigma_\theta^2}\right)+\\&+M^3\left(e^{-\sigma_\theta^2}-e^{-2\sigma_\theta^2}\right)\end{aligned}\,. \quad (76)$$

APPENDIX D

In this appendix we compute the fifth term in second member of (42) as was done in Appendix B and C. The 15 different cases are:

$$\sum_{i,j,k,l} E\left\{\cos\left[\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}=$$

$$=\sum_{i,j,k,l}\begin{cases} 1 & \bullet\bullet\bullet\bullet & M \\ e^{-\sigma_\theta^2} & \bullet\bullet\bullet\times & M(M-1) \\ e^{-\sigma_\theta^2} & \bullet\bullet\times\bullet & M(M-1) \\ e^{-\sigma_\theta^2} & \bullet\times\bullet\bullet & M(M-1) \\ e^{-\sigma_\theta^2} & \times\bullet\bullet\bullet & M(M-1) \\ 1 & \bullet\bullet\times\times & M(M-1) \\ e^{-4\sigma_\theta^2} & \bullet\times\bullet\times & M(M-1) \\ 1 & \bullet\times\times\bullet & M(M-1) \\ e^{-\sigma_\theta^2} & \bullet\bullet\times\circ & M(M-1)(M-2) \\ e^{-3\sigma_\theta^2} & \bullet\times\bullet\circ & M(M-1)(M-2) \\ e^{-\sigma_\theta^2} & \bullet\times\circ\bullet & M(M-1)(M-2) \\ e^{-\sigma_\theta^2} & \times\bullet\bullet\circ & M(M-1)(M-2) \\ e^{-3\sigma_\theta^2} & \times\bullet\circ\bullet & M(M-1)(M-2) \\ e^{-\sigma_\theta^2} & \times\circ\bullet\bullet & M(M-1)(M-2) \\ e^{-2\sigma_\theta^2} & \bullet\times\circ\ast & M(M-1)(M-2)(M-3) \end{cases} \quad . \qquad (77)$$

This summation thus becomes

$$\begin{aligned} &\sum_{i,j,k,l} E\left\{\cos\left[\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}= \\ &=\left[M+2M(M-1)\right]+ \\ &+\left[4M(M-1)+4M(M-1)(M-2)\right]e^{-\sigma_\theta^2}+ \\ &+\left[M(M-1)(M-2)(M-3)\right]e^{-2\sigma_\theta^2}+ \\ &+\left[2M(M-1)(M-2)\right]e^{-3\sigma_\theta^2}+\left[M(M-1)\right]e^{-4\sigma_\theta^2}= \\ &=\left[-M+2M^2\right]+\left[4M(M-1)(M-1)\right]e^{-\sigma_\theta^2}+ \\ &+\left[M(M-1)(M-2)(M-3)\right]e^{-2\sigma_\theta^2}+ \\ &+\left[2M(M-1)(M-2)\right]e^{-3\sigma_\theta^2}+\left[M(M-1)\right]e^{-4\sigma_\theta^2} \end{aligned} \quad . \qquad (78)$$

Simplifying leads to

$$\begin{aligned} &\sum_{i,j,k,l} E\left\{\cos\left[\left(\theta_i-\theta_j+\theta_k-\theta_l\right)\right]\right\}= \\ &=M\left(-1+4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+4e^{-3\sigma_\theta^2}-e^{-4\sigma_\theta^2}\right)+ \\ &+M^2\left(2-8e^{-\sigma_\theta^2}+11e^{-2\sigma_\theta^2}-6e^{-3\sigma_\theta^2}+e^{-4\sigma_\theta^2}\right)+ \\ &+M^3\left(4e^{-\sigma_\theta^2}-6e^{-2\sigma_\theta^2}+2e^{-3\sigma_\theta^2}\right)+ \\ &+M^4\left(e^{-2\sigma_\theta^2}\right) \end{aligned} \quad . \qquad (79)$$

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