Precision of Independently Based Gain and Offset Error of an ADC using the Histogram Method

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Abstract—**Two of the parameters that are determined when testing an analog to digital converter are the gain and offset error. One of the ways to define these two parameters is called "Independently Based". Here we derive the precision of the gain and offset error estimated with the Histogram Test Method affected by additive noise.**

Index Terms—**ADC, Gain, Offset Error, Histogram, Precision.**

I. Introduction

The transfer function of an analog to digital converter (ADC) relates the analog input voltage (or current) to the digital output code (Fig. 1). In the following we will consider that the ADC input is a voltage. The number of output codes depends on the ADC resolution (n_b) and is equal to 2^{n_b}. The transition voltage T_k is, by definition, the value of constant input voltage that leads to an equal amount of output codes lower than k and equal or higher than k. Different output codes are expected for a constant input voltage due to the inevitable presence of noise. The code bin width, W_k, is the distance between the two consecutive transition voltages T_k and T_{k+1}.
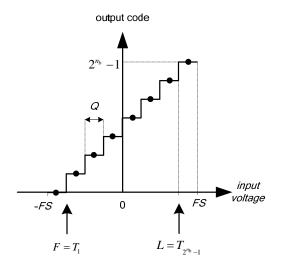

Fig. 1 – Illustration of the transfer function of a symmetrical bipolar ADC. This type of transfer function is also known as "with no true zero".

In an ideal ADC the transition voltages are equally spaced by Q, the ideal code bin width, given by

$$Q = \frac{2 \cdot FS}{2^{n_b}} \qquad (1)$$

and the ideal values of the transition voltages are

$$T_k^{ideal} = -FS + k \cdot Q \text{ with } k = 1...2^{n_b} - 1, \qquad (2)$$

where FS is the ADC Full Scale voltage.

In a real ADC, however, the transition voltages are different from the ideal ones due to manufacturing defects and environmental conditions. It is thus important, for the user of an ADC, to know how close a given ADC is behaving in relation to its ideal, and expected, behavior. To that effect several types ADC tests can be used. The most common one, for the estimation of an ADC transfer function, is the Histogram Test (also known as Code Density Test) [1-9]. This is a statistical test where a sinusoidal stimulus signal is applied and a large amount of samples are required. From the digital output codes obtained, the behavior of the ADC is inferred and its transfer function estimated, that is, the values of the transition voltages are estimated.

Fig. 1 represents the transfer function of a symmetrical bipolar ADC, also known as a bipolar with no true zero transfer function because there is no step whose middle point is zero. It is used mainly in theoretical considerations where the symmetry of transition voltages eases the derivations. There are, however other types of transfer functions like the bipolar with true zero and the unipolar, which have different values of transition voltages. The derivations carried on in this paper can easily be adapted to those cases if needed.

To better evaluate the performance of an ADC, different parameters are used to express the transfer function, namely, the gain, offset error, integral and differential non linearity. The first two, gain and offset error, express the transfer function as a straight line that ideally goes through the middle point of the quantification steps (black circles in Fig. 1). The latter two, integral and differential non linearity, pertain to the difference between the ideal and real transition voltages and code bin widths respectively.

In this paper we study the influence that the

presence of additive noise and acquisition of a finite number of samples has on the precision of the estimative of ADC gain and offset error determined with the Histogram Method. The goal is to derive expressions that let us know what is the precision of those estimates. In the case of ADC testing, as in any measurement system, the result of the measurement should be accompanied by confidence intervals that give an idea about the uncertainty of the measurement result. In this paper we will concentrate our efforts in determining the standard deviation of the estimated gain and offset error which will allow us to specify the correspondent confidence intervals.

In section II we will present the definitions of gain and offset error. In the following section (III) we address the precision of the parameters of estimated from linear regression which is used to estimate the ADC gain and offset error. In section IV we derive the covariance of the transition voltages estimated with the histogram method with the purpose of determining the standard deviation of the ADC gain and offset error which is done in section V. In section VI we present experimental results that validate the analytical derivations and approximations done. Finally, in section VIII, we present the conclusions reached with this work.

II. INDEPENDENTLY BASED GAIN AND OFFSET ERROR

There are two ways that are traditionally used to define an ADC gain and offset error. They are the *Terminal Based Definition* and the *Independently Based Definition* [10]. In this paper we will focus our attention on the second one, leaving the other one for a different publication.

In the *Independently Based Definition*, the gain and offset error are defined as the two scalars that when used to multiply the estimated transition voltages, $\hat{T}_k$, (gain, G) and add to the result of the multiplication (offset error, O) lead to corrected transition voltages

$$\hat{T}_k^{corr} = G \cdot \hat{T}_k + O \qquad (3)$$

that are as close as possible to the ideal transition voltages in a least square sense.

The way traditionally used to determine the independently based ADC gain and offset error is to use a linear regression procedure to fit the estimated transition voltages to the ideal ones:

$$T_k^{ideal} = \hat{G} \cdot \hat{T}_k + \hat{O} . \qquad (4)$$

The estimated gain is the slope of the fitted straight line and the estimated offset error is the point of intersection of that straight line with the vertical axis (axis of the ideal transition voltages).

In the following section we will present the expressions for the standard deviation of the estimated gain and offset error obtained from the linear regression.

III. PRECISION OF THE PARAMETERS OF LINEAR REGRESSION

Consider an independent variable x (not random) and a dependent variable y that are related in the following way:

$$y_i = \beta_0 + \beta_1 \cdot x_i + u_i \quad , \quad i = 1, 2, ..., I , \qquad (5)$$

where β_1 is the slope parameter and β_0 is the intercept parameter. The term u_i represents an error, that is, it represents other factors, other than x, that influence y.

From a set of values of x and y it is possible to use the following least squares estimators for the two parameters [11]:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{I}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right)}{\sum_{i=1}^{I}\left(x_i - \overline{x}\right)^2} = \frac{s_{xy}}{s_{xx}} . \qquad (6)$$

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$$

The distance between the points y_i and the fitted straight line is called residual ($\hat{u}_i$):

$$\hat{u}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot x_i . \qquad (7)$$

The least square estimators (6) are the best linear unbiased estimators, according to the Gauss-Markov Theorem [12] if the following assumptions are satisfied:

$$E\left[u_i\right] = 0$$
$$\mathrm{var}\left[u_i\right] = \sigma^2 < \infty . \qquad (8)$$
$$\mathrm{cov}\left[u_i, u_j\right] = 0$$

These assumptions mean that the errors must have zero mean, be uncorrelated and must all have the same variance (homoskedasticity).

Unfortunately, in the case of the determination of the independently based ADC gain and offset error with the Histogram Test, in the presence of additive noise, none of these assumptions is valid. The estimated transition voltages do not have zero mean [13], do not all have the same variance [8] and are correlated as will be shown in the following section.

As a consequence, the least squares estimators (6) are not the best ones. Since the variances of the estimated voltages can be computed, it is possible to define other estimators that are better than these by using weighted least square estimation. However, since traditionally the normal least square estimators are used, we consider it important to study them even though they are not the best estimators possible. We plan, in the future, to extend the study presented here to other estimators based in weighted least square procedure.

So, given the estimators (6), we want to determine the standard deviation of the parameters β. The parameter β_1 in (6) can be written as

$$\hat{\beta}_1 = \frac{1}{s_{xx}} \sum_{i=1}^{I}\left(x_i - \overline{x}\right)\left(y_i - \overline{y}\right) = \frac{1}{s_{xx}} \sum_{i=1}^{I}\left(x_i - \overline{x}\right) y_i . \quad (9)$$

The variance of the estimator of β_1 is, using the properties of variance [14],

$$\text{var}\left[\widehat{\beta}_1\right] = \frac{1}{s_{xx}^2}\text{var}\left[\sum_{i=1}^{I}\left(x_i - \overline{x}\right)y_i\right] =$$

$$= \frac{1}{s_{xx}^2}\sum_{k=1}^{I}\sum_{l=1}^{I}\text{cov}\left[\left(x_k - \overline{x}\right)y_k, \left(x_l - \overline{x}\right)y_l\right] = . \quad (10)$$

$$= \frac{1}{s_{xx}^2}\sum_{k=1}^{N}\sum_{l=1}^{N}\left(x_k - \overline{x}\right)\left(x_l - \overline{x}\right)\text{cov}\left[y_k, y_l\right]$$

For the estimated offset error, using (6) and considering $\overline{x} = 0$, leads to

$$\text{var}\left[\widehat{\beta}_0\right] = \frac{1}{I^2}\text{var}\left[\sum_{i=1}^{I}y_i\right] = \frac{1}{I^2}\sum_{k=1}^{I}\sum_{l=1}^{I}\text{cov}\left[y_k, y_l\right]. \quad (11)$$

Since the independent variable, x, is considered as non random and given what was said in the previous section about the definition of independently based gain and offset error we will change equation (4) to

$$\widehat{T}_k = \widehat{G}' \cdot T_k^{ideal} + \widehat{O}' \quad (12)$$

where the parameters G' and O' are related to the ADC gain and offset error by

$$\widehat{G} = \frac{1}{\widehat{G}'} \quad \text{and} \quad \widehat{O} = -\frac{\widehat{O}'}{\widehat{G}'}. \quad (13)$$

From [15] we can write

$$\text{var}\left[\widehat{G}\right] \approx \text{var}\left[\widehat{G}'\right] \quad \text{and} \quad \text{var}\left[\widehat{O}\right] \approx \text{var}\left[\widehat{O}'\right] \quad (14)$$

having approximated the mean of $\widehat{G}'$ by its ideal value of 1.

Comparing (5) with (12) we have $y = \widehat{T}$ and $x = T^{ideal}$. To compute (10) and (11) it is necessary to derive the covariance of the estimated transition voltages which is done in the following section.

IV. COVARIANCE OF THE ESTIMATED TRANSITION VOLTAGES

A. Histogram Test

In the Histogram Test usually a sinusoidal stimulus signal is applied to the ADC input:

$$s(t) = C - A \cdot \cos\left(2\pi \cdot f \cdot t + \varphi\right). \quad (15)$$

Due to the presence of additive noise in the stimulus signal and generated in the ADC itself we can consider the sampled voltage to be

$$v_j = s\left(\frac{j}{f_s}\right) + n_V \quad , \quad j = 0, 1, ..., M-1. \quad (16)$$

where n_V is a random variable that represents additive white Gaussian noise (AWGN) and M is the number of samples acquired. Inserting (15) into (16) leads to

$$v_j = C - A \cdot \cos\left(2\pi \cdot f \cdot \frac{j}{f_s} + \varphi\right) + n_V. \quad (17)$$

In order to simplify the derivations we will introduce the following normalized variables:

$$u_j = \frac{v_j - C}{A} \quad , \quad n = \frac{n_V}{A} \quad , \quad \gamma_j = 2\pi\frac{f}{f_s}t_j + \varphi. \quad (18)$$

Using these variables we can write the normalized sampled voltage, from (17), as

$$u_j = -\cos\left(\gamma_j\right) + n. \quad (19)$$

The M samples acquired are used to construct the cumulative histogram, c_k, which is the number of samples with output code equal to or lower than k. The transition voltages are then estimated with [1]

$$\widehat{T}_k = C - A \cdot \cos\left(\frac{\pi}{M}c_k\right) \quad (20)$$

or, using a normalized transition voltage,

$$\widehat{U}_k = \frac{\widehat{T}_k - C}{A} = -\cos\left(\frac{\pi}{M}c_k\right). \quad (21)$$

B. Probability of a sample having a given code

To compute the covariance of the transition voltages we will need first to compute the covariance of the number of counts of the cumulative histogram, c_k. To that effect we will introduce a binomial variable, w_k, which assumes the value 1 if a sample belongs to class k of the cumulative histogram and 0 otherwise. The probability that a given sample, with phase γ_j, belongs to c_k is the probability that the sampled voltage (with phase γ_j) is lower than the transition voltage U_k [8]

$$p_k\left(\gamma_j\right) = \frac{1}{2} + \frac{1}{2}erf\left(\frac{U_k + \cos\left(\gamma_j\right)}{\sqrt{2}\cdot\sigma_n}\right). \quad (22)$$

The definition of covariance is [15]

$$\text{cov}\left[w_k, w_l\right] = E\left[w_k w_l\right] - E\left[w_k\right]E\left[w_l\right]. \quad (23)$$

The expected value, $E\left[w_k w_l\right]$, of the product of w_k and w_l is the probability that a sample belongs simultaneously to c_k and c_l. which is to say that it is the probability that the sampled voltage is lower than both U_k and U_l. This probability is just the probability that the sampled voltage is lower than the minimum of U_k and U_l:

$$E\left[w_k w_l \mid \gamma_j\right] = \begin{cases} p_k\left(\gamma_j\right) & , \quad U_k < U_l \\ p_l\left(\gamma_j\right) & , \quad otherwise \end{cases}. \quad (24)$$

Since w is a binomial variable its expected value is equal to the probability of $w = 1$ which is

$$E\left[w_k \mid \gamma_j\right] = p_k\left(\gamma_j\right) \quad \text{and} \quad E\left[w_l \mid \gamma_j\right] = p_l\left(\gamma_j\right). \quad (25)$$

Inserting (24) and (25) into (23) leads to

$$\text{cov}\left[w_k w_l \mid \gamma_j\right] = \begin{cases} p_k\left(\gamma_j\right)\left[1 - p_l\left(\gamma_j\right)\right] & , \quad U_k < U_l \\ p_l\left(\gamma_j\right)\left[1 - p_k\left(\gamma_j\right)\right] & , \quad otherwise \end{cases}.$$

$$(26)$$

C. Conditional Covariance of the Cumulative Histogram

The number of counts of the cumulative histogram,

c_k, is just the sum of variable w_k for all samples acquired:

$$c_k = \sum_{j=0}^{M-1} w_{k,j} \ . \tag{27}$$

The covariance between the summations of two variables is the double summation of the covariance of those variables [14]:

$$\text{cov}\left[c_k, c_l \mid \varphi \right] = \text{cov}\left[\sum_{j=0}^{M-1} w_{k|\gamma_j}, \sum_{i=0}^{M-1} w_{l|\gamma_i} \right] =$$
$$= \sum_{j=0}^{M-1} \sum_{i=0}^{M-1} \text{cov}\left[w_{k|\gamma_j}, w_{l|\gamma_i} \right] \tag{28}$$

A sample having or not a certain output code is independent of another sample, with a different phase, of having the same or another output code. From this it follows that the covariance in (28) is only different from 0 when $i = j$. Eq. (28) can thus be written as

$$\text{cov}\left[c_k, c_l \mid \varphi \right] = \sum_{j=0}^{M-1} \text{cov}\left[w_{k|\gamma_j}, w_{l|\gamma_j} \right]. \tag{29}$$

D. Total Covariance of the Cumulative Histogram

Usually the stimulus signal and the sampling signal are not synchronized which means that the phase at the origin of the stimulus signal, φ, can assume any value. Considering that it has a uniform distribution in the interval $[-\pi, \pi]$ the total covariance of the number of counts of the cumulative histogram can be computed from the conditional covariance given by (29) [14]:

$$\text{cov}\left[c_k, c_l \right] =$$
$$= E\left\{ \text{cov}\left[c_k, c_l \mid \varphi \right] \right\} + \text{cov}\left[E\left[c_k \mid \varphi \right], E\left[c_l \mid \varphi \right] \right] \tag{30}$$

The first term of the right side of eq. (30) is

$$E\left\{ \text{cov}\left[c_k, c_l \mid \varphi \right] \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{cov}\left[c_k, c_l \mid \varphi \right] d\varphi . \tag{31}$$

Inserting (29) leads to

$$E\left\{ \text{cov}\left[c_k, c_l \mid \varphi \right] \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{M-1} \text{cov}\left[w_{k|\gamma_j}, w_{l|\gamma_j} \right] d\varphi . \tag{32}$$

Considering the definition of γ_j, given by (18), and that (22) is periodic on γ_j, eq. (32) becomes

$$E\left\{ \text{cov}\left[c_k, c_l \mid \varphi \right] \right\} = \frac{M}{2\pi} \int_{-\pi}^{\pi} \text{cov}\left[w_{k|\gamma_j}, w_{l|\gamma_j} \right] d\varphi . \tag{33}$$

Inserting (26) leads to

$$E\left\{ \text{cov}\left[c_k, c_l \mid \varphi \right] \right\} =$$
$$= \frac{M}{2\pi} \begin{cases} \int_{-\pi}^{\pi} p_k\left(\gamma_j \right)\left[1 - p_l\left(\gamma_j \right) \right] d\varphi & , \ U_k < U_l \\ \int_{-\pi}^{\pi} p_l\left(\gamma_j \right)\left[1 - p_k\left(\gamma_j \right) \right] d\varphi & , \ otherwise \end{cases} \tag{34}$$

In relation to the second term of the right side of (30), we can use the definition of covariance to write

$$\text{cov}\left[E\left[c_k \mid \varphi \right], E\left[c_l \mid \varphi \right] \right] =$$
$$E\left\{ E\left[c_k \mid \varphi \right] \cdot E\left[c_l \mid \varphi \right] \right\} - E\left\{ E\left[c_k \mid \varphi \right] \right\} \cdot E\left\{ E\left[c_l \mid \varphi \right] \right\} \tag{35}$$

Considering the uniform distribution of φ between $-\pi$ and π leads to

$$\text{cov}\left[E\left[c_k \mid \varphi \right], E\left[c_l \mid \varphi \right] \right] =$$
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} E\left[c_k \mid \varphi \right] \cdot E\left[c_l \mid \varphi \right] d\varphi -$$
$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} E\left[c_k \mid \varphi \right] d\varphi \cdot \int_{-\pi}^{\pi} E\left[c_l \mid \varphi \right] d\varphi \tag{36}$$

The conditional expected value of c_k is the sum of the conditional expected values of all variables w_k

$$E\left[c_k \mid \varphi \right] = \sum_{j=0}^{M-1} E\left[w_{k,j} \right] = \sum_{j=0}^{M-1} p_k\left(\gamma_j \right). \tag{37}$$

Substituting in (36) leads to

$$\text{cov}\left[E\left[c_k \mid \varphi \right], E\left[c_l \mid \varphi \right] \right] =$$
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{M-1} p_k\left(\gamma_j \right) \cdot \sum_{j=0}^{M-1} p_l\left(\gamma_j \right) d\varphi -$$
$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \sum_{j=0}^{M-1} p_k\left(\gamma_j \right) d\varphi \cdot \int_{-\pi}^{\pi} \sum_{j=0}^{M-1} p_l\left(\gamma_j \right) d\varphi \tag{38}$$

Using the same arguments employed when deriving eq. (33) we can write

$$\text{cov}\left[E\left[c_k \mid \varphi \right], E\left[c_l \mid \varphi \right] \right] =$$
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{M-1} p_k\left(\gamma_j \right) \cdot \sum_{j=0}^{M-1} p_l\left(\gamma_j \right) d\varphi -$$
$$- \frac{M^2}{(2\pi)^2} \int_{-\pi}^{\pi} p_k\left(\varphi \right) \cdot \int_{-\pi}^{\pi} p_l\left(\varphi \right) d\varphi \tag{39}$$

Inserting (34) and (39) into (30) leads to the expression for the covariance of the number of counts of the cumulative histogram:

$$\text{cov}\left[c_k, c_l \right] =$$
$$\frac{M}{2\pi} \begin{cases} \int_{-\pi}^{\pi} p_k\left(\gamma_j \right)\left[1 - p_l\left(\gamma_j \right) \right] d\varphi & , \ U_k < U_l \\ \int_{-\pi}^{\pi} p_l\left(\gamma_j \right)\left[1 - p_k\left(\gamma_j \right) \right] d\varphi & , \ otherwise \end{cases} +$$
$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{M-1} p_k\left(\gamma_j \right) \cdot \sum_{j=0}^{M-1} p_l\left(\gamma_j \right) d\varphi -$$
$$- \frac{M^2}{(2\pi)^2} \int_{-\pi}^{\pi} p_k\left(\varphi \right) \cdot \int_{-\pi}^{\pi} p_l\left(\varphi \right) d\varphi \tag{40}$$

As can be seen in Fig. 2, the covariance of the expected values of the number of counts of the cumulative histogram tends to 0 as the standard deviation of noise increases. This can also be seen for the variance of the expected values of the number of counts as shown in [8]. The condition found in [8] was that this term can be considered approximately null for situations where $M \cdot \sigma_n > 1$. To give an idea of the actual values one might encounter for this product, consider a stimulus signal with an amplitude equal to the ADC Full Scale ($A = FS$) and that the additive noise standard deviation is 0.5 LSB ($\sigma_{n_V} = Q/2$). The referred condition is thus

$$M \cdot \sigma_n > 1 \Leftrightarrow M \cdot \frac{Q}{2A} > 1 \Leftrightarrow M > 2^{nb} \qquad (41)$$

which corresponds to having more than 1 sample per output code. Even if the additive noise is greater, for instance, 2 LSB, this condition corresponds to at least 4 samples per output code which is a common situation in actual tests using the Histogram Method. This approximation will be used later to derive an approximate expression for the variance of the estimated gain and offset error.
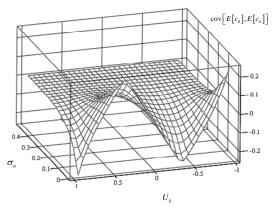

Fig. 2 – Representation of the covariance between the expected value of the number of counts of class 0 and of class k of the cumulative histogram as a function of the normalized transition voltage of class k and the standard deviation of additive noise. $M=4$.

E. Covariance of the Transition Voltages

The estimated value of the transition voltages is obtained from the number of counts of the cumulative histogram with (20). The covariance of the estimated transition voltages is given by [15]

$$\text{cov}\left[\hat{T}_k, \hat{T}_l\right] \approx \frac{d\hat{T}_k}{dc_k}\bigg|_{\hat{T}_k = \mu_{\hat{T}_k}} \frac{d\hat{T}_l}{dc_l}\bigg|_{\hat{T}_l = \mu_{\hat{T}_l}} \text{cov}\left[c_k, c_l\right]. \qquad (42)$$

Differentiating (20) and inserting into (42) leads to

$$\text{cov}\left[\hat{T}_k, \hat{T}_l\right] \approx \left(\frac{\pi}{M}\right)^2 \sqrt{A^2 - \mu_{\hat{T}_k}^2} \times \\ \times \sqrt{A^2 - \mu_{\hat{T}_l}^2} \, \text{cov}\left[c_k, c_l\right] \qquad (43)$$

We considered, to simplify the computations, that the stimulus signal has no offset ($C=0$). The same results would have been obtained if the stimulus signal had an offset.

V. GAIN PRECISION

The variance of the estimated ADC gain is given by (10), with $y = T$, as seen previously. Inserting (43) gives rise to

$$\text{var}\left[\hat{G}\right] \approx \frac{1}{s_{xx}^2} \sum_{k=1}^{2^{nb}-1} \sum_{l=1}^{2^{nb}-1} \left\{ \begin{array}{l} \left(T_k^{ideal} - \overline{T^{ideal}}\right)\left(T_l^{ideal} - \overline{T^{ideal}}\right) \times \\ \left(\frac{\pi}{M}\right)^2 \sqrt{A^2 - \mu_{\hat{T}_k}^2} \times \\ \times \sqrt{A^2 - \mu_{\hat{T}_l}^2} \, \text{cov}\left[c_k, c_l\right] \end{array} \right\}$$

$$(44)$$

Making the variable substitution $T = UA$, leads to

$$\text{var}\left[\hat{G}\right] \approx \frac{A^4}{s_{xx}^2}\left(\frac{\pi}{M}\right)^2 \times \\ \times \sum_{k=1}^{2^{nb}-1} \sum_{l=1}^{2^{nb}-1} \left\{ \begin{array}{l} \left(U_k^{ideal} - \overline{U^{ideal}}\right)\left(U_l^{ideal} - \overline{U^{ideal}}\right) \times \\ \sqrt{1 - \mu_{\hat{U}_k}^2} \sqrt{1 - \mu_{\hat{U}_l}^2} \, \text{cov}\left[c_k, c_l\right] \end{array} \right\} \qquad (45)$$

Using (6) and considering that x represents the ideal transition voltage, we can write

$$s_{xx} = \sum_{i=1}^{2^{nb}-1} \left(T_i^{ideal} - \overline{T^{ideal}}\right)^2. \qquad (46)$$

For a sufficient number of bits (in general more than 4) we can approximate the summation in (46) by an integral:

$$s_{xx} \approx \frac{2^{nb}-1}{2(FS-Q)} \int_{-FS+Q}^{FS-Q} T^2 dT = \frac{2^{nb}-1}{3}(FS-Q)^2. \quad (47)$$

We considered that it was a bipolar with no zero ADC, which means that the transition voltages are symmetric and thus their mean is null. Inserting into (45) leads to

$$\text{var}\left[\hat{G}\right] \approx \left(\frac{\pi}{M}\right)^2 \frac{9A^4}{\left(2^{nb}-1\right)^2 (FS-Q)^4} \times \\ \times \sum_{k=1}^{2^{nb}-1} \sum_{l=1}^{2^{nb}-1} \left\{ \begin{array}{l} \left(U_k^{ideal} - \overline{U^{ideal}}\right)\left(U_l^{ideal} - \overline{U^{ideal}}\right) \times \\ \sqrt{1 - \mu_{\hat{U}_k}^2} \sqrt{1 - \mu_{\hat{U}_l}^2} \, \text{cov}\left[c_k, c_l\right] \end{array} \right\} \qquad (48)$$

The double summation in (48) can also be approximated by a double integration in U_k and U_l:

$$\text{var}\left[\hat{G}\right] \approx \frac{9}{4}\left(\frac{\pi}{M}\right)^2 \times \left(\frac{A}{FS-Q}\right)^4 \\ \times \int_{-1}^{1}\int_{-1}^{1} \left\{ \begin{array}{l} U_k U_l \sqrt{1 - \mu_{\hat{U}_k}^2} \times \\ \times \sqrt{1 - \mu_{\hat{U}_l}^2} \, \text{cov}\left[c_k, c_l\right] \end{array} \right\} dU_k dU_l \qquad (49)$$

Note that the integration limits should be $(-FS+Q)/A$ and $(FS-Q)/A$ which are the normalized values of the first and last transition voltage. In practice the value of stimulus amplitude used, A, is slightly larger than $FS-Q$ because the ADC must be overdriven to minimize the bias introduced in the transition voltage estimates due to the presence of additive noise [13]. We chose to use the limits ± 1 in the integrals in (49) in order to have its value independent of both A and FS. As a consequence the value of the integration becomes slightly smaller.

Using the approximation justified in the previous section, $M \cdot \sigma_n > 1$, we can substitute the covariance in (49) by the expected value of the covariance term given by (34)

$$\text{var}\left[\widehat{G}\right] \approx \frac{9}{4}\frac{\pi}{M}\left(\frac{A}{FS-Q}\right)^4 \times$$
$$\times \int_{-1}^{1}\int_{-1}^{U_l} \left\{ \begin{array}{l} U_k U_l \sqrt{1-U_k^2}\sqrt{1-U_l^2} \times \\ \times \int_{-\pi}^{\pi} p_k(\varphi)\left[1-p_l(\varphi)\right]d\varphi \end{array}\right\} dU_k dU_l \tag{50}$$

In the previous expression we have taken into account that for small amounts of additive noise ($\sigma_n < 0.1$) usually encountered in practical situations, and considering the use of overdrive, we can assume the mean of the estimated transition voltages to be unbiased [13].

Inserting (22) into (50) leads to

$$\text{var}\left[\widehat{G}\right] \approx \frac{9}{4M} \times \left(\frac{A}{FS-Q}\right)^4$$
$$\times \int_{-1}^{1}\int_{-1}^{U_l} \left\{ \begin{array}{l} \pi U_k U_l \sqrt{1-U_k^2}\sqrt{1-U_l^2} \times \\ \times \int_{-\pi}^{\pi} \left[\frac{1}{2}+\frac{1}{2}erf\left(\frac{U_k+\cos(\varphi)}{\sqrt{2}\cdot\sigma_n}\right)\right] \times \\ \times \left[\frac{1}{2}-\frac{1}{2}erf\left(\frac{U_l+\cos(\varphi)}{\sqrt{2}\cdot\sigma_n}\right)\right]d\varphi \end{array}\right\} dU_k dU_l \tag{51}$$

These integrals were solved numerically and an approximated value of $1.21\sigma_n^2$ was obtained.

$$\text{var}\left[\widehat{G}\right] \approx \frac{9}{4}\frac{1.21\sigma_n^2}{M}\left(\frac{A}{FS-Q}\right)^4. \tag{52}$$

Using (18) we can write, for the estimated gain standard deviation,

$$\boxed{\sigma_{\widehat{G}} \approx 1.1\frac{3}{2}\frac{A}{(FS-Q)^2}\frac{\sigma_{n_v}}{\sqrt{M}}.} \tag{53}$$

VI. Offset Error Precision

The variance of the estimated ADC offset is given by (11) with $y = T$, as seen previously. Inserting (43) gives rise to

$$\text{var}\left[\widehat{O}\right] \approx \frac{\sum_{k=1}^{2^{n_b}-1}\sum_{l=1}^{2^{n_b}-1}\left\{\begin{array}{l}\left(\frac{\pi}{M}\right)^2\sqrt{A^2-\mu_{\widehat{T}_k}^2} \times \\ \sqrt{A^2-\mu_{\widehat{T}_l}^2}\,\text{cov}\left[c_k,c_l\right]\end{array}\right\}}{\left(2^{n_b}-1\right)^2}. \tag{54}$$

Making the variable substitution $T = UA$, leads to

$$\text{var}\left[\widehat{O}\right] \approx \frac{A^2}{\left(2^{n_b}-1\right)^2}\left(\frac{\pi}{M}\right)^2 \times$$
$$\times \sum_{k=1}^{2^{n_b}-1}\sum_{l=1}^{2^{n_b}-1}\sqrt{1-\mu_{\widehat{U}_k}^2}\sqrt{1-\mu_{\widehat{U}_l}^2}\,\text{cov}\left[c_k,c_l\right] \tag{55}$$

The double summation in (55) can be approximated, for an ADC with a sufficiently high number of bits, by a double integration between the first and last normalized transition voltages:

$$\text{var}\left[\widehat{O}\right] \approx \frac{A^2}{4(FS-Q)^2}\left(\frac{\pi}{M}\right)^2 \times A^2$$
$$\times \int_{-\frac{FS-Q}{A}}^{\frac{FS-Q}{A}}\int_{-\frac{FS-Q}{A}}^{\frac{FS-Q}{A}}\left\{\sqrt{1-\mu_{\widehat{U}_k}^2}\sqrt{1-\mu_{\widehat{U}_l}^2}\,\text{cov}\left[c_k,c_l\right]\right\}dU_k dU_l \tag{56}$$

As was done for the estimated gain variance, we will replace the limits of integration with ± 1:

$$\text{var}\left[\widehat{O}\right] \approx \frac{A^2}{4(FS-Q)^2}\left(\frac{\pi}{M}\right)^2 \times A^2$$
$$\times \int_{-1}^{1}\int_{-1}^{1}\left\{\sqrt{1-\mu_{\widehat{U}_k}^2}\sqrt{1-\mu_{\widehat{U}_l}^2}\,\text{cov}\left[c_k,c_l\right]\right\}dU_k dU_l \tag{57}$$

Using the approximation justified previously, $M\cdot\sigma_n > 1$, we can substitute the covariance in (56) by the expected value of the covariance term given by (34),

$$\text{var}\left[\widehat{O}\right] \approx \frac{A^4}{4(FS-Q)^2}\frac{\pi}{M} \times$$
$$\times \int_{-1}^{1}\int_{-1}^{U_l}\left\{\begin{array}{l}\sqrt{1-U_k^2}\sqrt{1-U_l^2} \times \\ \times \int_{-\pi}^{\pi} p_k(\varphi)\left[1-p_l(\varphi)\right]d\varphi\end{array}\right\}dU_k dU_l \tag{58}$$

Note that the limits of integration on U_k have changed due to the piecewise definition used in (34).

In the previous expression we have taken into account that for small amounts of additive noise ($\sigma_n < 0.1$) usually encountered in practical situations, and considering the use of overdrive, we can assume the mean of the estimated transition voltages to be unbiased [13].

Inserting (22) leads to

$$\text{var}\left[\widehat{O}\right] \approx \frac{A^4}{4(FS-Q)^2}\frac{1}{M} \times$$
$$\times \int_{-1}^{1}\int_{-1}^{U_l}\left\{\begin{array}{l}\pi\sqrt{1-U_k^2}\sqrt{1-U_l^2} \times \\ \times \int_{-\pi}^{\pi}\left[\frac{1}{2}+\frac{1}{2}erf\left(\frac{U_k+\cos(\varphi)}{\sqrt{2}\cdot\sigma_n}\right)\right] \times \\ \times \left[\frac{1}{2}-\frac{1}{2}erf\left(\frac{U_l+\cos(\varphi)}{\sqrt{2}\cdot\sigma_n}\right)\right]d\varphi\end{array}\right\}dU_k dU_l \tag{59}$$

These integrals were solved numerically and a value of approximately $5\sigma_n^2$ was obtained.

$$\text{var}\left[\widehat{O}\right] \approx \frac{A^4}{4(FS-Q)^2}\frac{5\sigma_n^2}{M}. \tag{60}$$

Using (47) and (18) we can write, for the estimated gain standard deviation,

$$\boxed{\sigma_{\widehat{O}} \approx 1.1\frac{A}{FS-Q}\frac{\sigma_{n_v}}{\sqrt{M}}.} \tag{61}$$

VII. EXPERIMENTAL RESULTS

To demonstrate the validity of the expressions presented here we tested a 12-bit ADC using the Standard Histogram Test. Only the 8 most significant bits were used to so that the ADC could be considered ideal. Using a Monte Carlo procedure with 1000 repetitions (N) we computed the standard deviation of the estimated independently based gain and offset error. We varied the additive noise standard deviation, number of samples, stimulus signal amplitude and ADC Full Scale range.
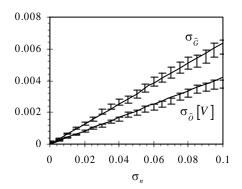

Fig. 3 – Representation of the estimated gain and offset error standard deviations as a function of the normalized additive noise standard deviation. We considered n_b=8, A=1.2V, FS=1V, M=1000, f=200Hz, f_s=200kHz and N=1000. The solid lines represent the approximation given by (53) and (61), and the vertical bars represent the experimental results.
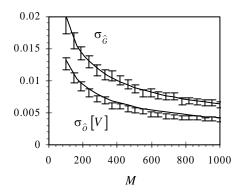

Fig. 4 – Representation of the estimated gain and offset error standard deviations as a function of the number of samples. We considered n_b=8, A=1.2V, FS=1V, σ_n=0.1, f=f_s/M, f_s=200kHz and N=1000. The solid lines represent the approximation given by (53) and (61), and the vertical bars represent the experimental results.

The results obtained for the standard deviation of the estimated ADC gain and offset error are presented in Fig. 3 through Fig. 6.

The vertical bars in the figures translate the 99.9% confidence interval to account for the Monte Carlo uncertainty due to the 1000 test repetitions (N) carried out.
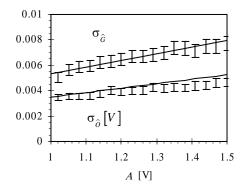

Fig. 5 – Representation of the estimated gain and offset error standard deviations as a function of the stimulus signal amplitude. We considered n_b=8, M=1000, FS=1V, σ_n=0.1, f=200Hz, f_s=200kHz and N=1000. The solid lines represent the approximation given by (53) and (61), and the vertical bars represent the experimental results.
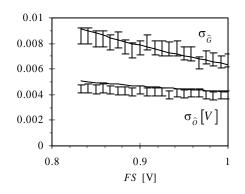

Fig. 6 – Representation of the estimated gain and offset error standard deviations as a function of the ADC Full Scale Range. We considered n_b=8, M=1000, A=1.2V, σ_n=0.1, f=200Hz, f_s=200kHz and N=1000. The solid lines represent the approximation given by (53) and (61), and the vertical bars represent the experimental results.

It can be seen, by observation of the previous figures, that the experimental results, represented by vertical bars, are below the value given by the approximate expressions (53) and (61). This validates the derivations presented here and justifies the use of those expressions to determine an upper bound for estimated ADC gain and offset error.

VIII. CONCLUSIONS

In this paper we analyzed the precision of the estimates of the independently based ADC gain and offset error obtained with the Standard Histogram Method. The main results were expressions (53) and (61) that can be used to determine the expanded uncertainty, and corresponding uncertainty interval, for the estimated independently based gain and offset error. This has the same importance as in any measurement system where the quality of the measurements should be expressed in terms of confidence intervals for the results.

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