Bias of Amplitude Estimation Using Three-Parameter Sine Fitting in the Presence of Additive Noise

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***Abstract* – The estimation of the amplitude of a sinewave using traditional sine fitting algorithms which are based on square error minimization is biased in the presence of additive noise contrary to what happens generally in linear regression problems. An approximate closed form expression for the estimation error as a function of sinewave amplitude, additive noise standard deviation and number of data points is derived here. It is demonstrated that although the estimator is biased, it is asymptotically unbiased, that is, the estimation error vanishes as the number of data points increase to infinity. It is shown that in practical conditions the relative error in the amplitude estimation is very small – lower than 0.5% for a signal to noise ratio as low as 0 dB (with 100 data points). Only the three-parameter algorithm in the case of coherent sampling is studied.**

***Index Terms* – Sinewave fitting, uncertainty, ADC, noise, bias, error.**

I. INTRODUCTION

Sine fitting is an algorithm used to extract the parameters of a sinusoidal model from a set of noisy observations. Those parameters are the amplitude, offset, initial phase and frequency. In the case of amplitude estimation, which is the subject of this paper, sine fitting is used in variety of domains ranging from astrophysics [1], neurophysiology [2] and optics [3] to computer vision [4], analog to digital converter characterization [5], [6] and impedance measurement [7].

Algorithms for sine fitting have been standardized in the IEEE standard 1057 [5] and IEEE standard 1241 [6]. There has been a considerable work on improved methods [8]-[12] and convergence [13]-[15].

Unfortunately, when a sine fitting algorithm is used, there is, most of the time, no explicit reference or stated concern about the uncertainty of the estimations made. There are different sources that contribute to that uncertainty, namely the presence of additive noise, phase noise, jitter in the sampling instant, quantization error, harmonic distortion, spurious distortion and error in the signal or sampling clock frequency. There has been work published in the precision of the amplitude estimation in the presence of Gaussian additive noise, namely on the Cramér-Rao Bound on the variance of the estimator [16]-[20].There are, however, very few works that address the problem of the bias of the estimator in the presence of additive noise. In [21] there is reference to a bias arising in the presence of jitter but no expression for its computation is given. In [22] the IEEE 1057 algorithm is studied and it is shown how to compute the bias in the presence of jitter in the asymptotic case, that is, for an infinite number of data points.

In this paper the attention is focused on the bias of the sinewave amplitude estimation in the presence of additive noise. It is a novel study, as far as the author is aware, which is of interest for anyone using sine fitting algorithms, regardless of the domain of application.

When the sinusoidal model is defined using the in-phase (A_I) and in-quadrature (A_Q) amplitudes,

$$z(t) = C + A_I \cos(\omega t) + A_Q \sin(\omega t) , \qquad (1)$$

where C is the offset and ω is the angular frequency, it is linear in the parameters (A_I, A_Q and C) and consequently the estimators for those parameters, when the data is affected by additive noise, using a least squares procedure, are unbiased, as proven in [24], p. 15. This parameterization is not the most useful nor is it the most used one. The sinusoidal model of interest is the one that uses amplitude (A) and initial phase (φ), besides the offset (C),

$$z(t) = C + A\cos(\omega t + \varphi) . \qquad (2)$$

In this case the model is not linear in the parameters (A, φ and C) and thus it is not necessary unbiased. Note that the amplitude of a sinusoid can be obtained from the in-phase and in-quadrature amplitudes using

$$A = \sqrt{A_I^2 + A_Q^2} , \qquad (3)$$

which is a non-linear function and thus the unbiasedness of A_I and A_Q does not guarantee the unbiasedness of A.

In [17], for instance, the asymptotic Cramér-Rao Bound is derived for three and four unknown parameters using both in-phase/in-quadrature amplitudes (eq. 26 – eq. 33) and amplitude/initial phase (eq. 38 – eq. 41) models. This was an important achievement since the Cramér-Rao Bound is a lower bound on achievable variance of an unbiased estimator and thus allows the evaluation of the efficiency of the estimator. It is also very useful in the determination of the minimum number of samples required to achieve a given bound in the precision of the sine fitting estimates. In [17] it is stated clearly that "the analysis is based on an assumption

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of unbiased estimation" (last sentence in the 2nd paragraph). It is thus important to note that the lower bound on the amplitude estimation, presented in eq. 38 of [17], is not necessarily the lowest achievable value for the variance of the estimator [23], since, as will be shown in this paper, that estimator is biased in the presence of additive noise.

The main goal of this paper is, besides alerting to that fact that the amplitude estimator is biased, to derive an analytical expression for that bias so that is can be used to determine the minimum number of samples that should be used to limit the amount of estimation error or even to correct that estimations made if the additive noise standard deviation is precisely known.

As it will be shown in the end of this paper, the amount of bias of the sinewave amplitude is very small, even for large amounts of additive noise. For example, using 100 data points and having a noise standard deviation equal to the sinewave effective value, that is, a 0 dB signal to noise ratio (SNR), the amount of bias is only 0.5%. In usual practical conditions the amount of noise will be lower and the number of samples will be higher making the bias even smaller. This, however, does not invalidate the importance of the study carried out since very high accuracy estimation may be needed or unusual conditions may arise in practice. It is not inconceivable that in some applications the amount of samples acquired may be small (dozens, for instance) due to the time it takes to obtain one data point, or that the amount of noise present is even higher that the sinewave amplitude or even that accuracies on the order of parts per million may be required. In those cases it is important to be aware of the bias present and to be able to quantify it.

II. Sinewave Fitting

Consider M data points $z_1, z_2, \ldots, z_M$ given by

$$z_i = C + A\cos(\omega_x t_i + \varphi). \tag{4}$$

where φ is the initial phase and ω_x is the angular frequency ($2\pi f_x$). We consider the phase φ to be a random variable uniformly distributed in an interval with length 2π.

This data is affected by additive voltage white Gaussian noise, d_i, with null mean and standard deviation σ_v:

$$y_i = z_i + d_i = C + A\cos(\omega_x t_i + \varphi) + d_i\ , \tag{5}$$

We wish to estimate the sine wave that best fits, in a least square error sense, to these M points. The estimates of the sine wave are obtained, in a matrix form, with [5]

$$\begin{bmatrix} \widehat{A_\mathrm{I}} \\ \widehat{A_\mathrm{Q}} \\ \widehat{C} \end{bmatrix} = \left(D^T D\right)^{-1} D^T \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_M \end{bmatrix} \tag{6}$$

with

$$D = \begin{bmatrix} \cos(\omega_a t_1) & \sin(\omega_a t_1) & 1 \\ \cos(\omega_a t_2) & \sin(\omega_a t_2) & 1 \\ \ldots & \ldots & \ldots \\ \cos(\omega_a t_M) & \sin(\omega_a t_M) & 1 \end{bmatrix} \tag{7}$$

and

$$\hat{A} = \sqrt{\widehat{A_\mathrm{I}}^2 + \widehat{A_\mathrm{Q}}^2} \tag{8}$$

where ω_a is the angular frequency of the sinusoid we are trying to adjust to the data.

Here we will assume that the number of samples acquired (M) covers exactly an integer number of periods (J) of the sine wave we are trying to fit to the data. This means that the sine wave frequency (f_a), sampling frequency (f_s) and number of samples satisfy

$$\frac{f_a}{f_s} = \frac{J}{M}\ ,\ J \in \mathbb{N}\ . \tag{9}$$

Note that J and M should be mutually prime so that the M different samples acquired at M different time instants, correspond to M different sine wave phases. If not, one will have less than M different phases which will increase the uncertainty in the estimation of the sine wave parameters. In the case that J is a multiple of $M/2$, the sampling instants will correspond to only 2 different sine wave phases and matrix $D^T D$ will be singular and hence not invertible (you can not estimate the 3 sine wave parameters with only two data points).

Note that the sampling instants are given by $t_i = i/f_s$. The assumption is reasonable because we can choose whatever values we want for those frequencies and the number of samples. In practice, however, due to instrument inaccuracies, the actual value of those frequencies may not be exactly the values chosen and which satisfy (9) but are close enough considering typical frequency errors smaller than 100 ppm. If a non integer number of periods is acquired a bias will affect the estimator. In this work, however, we will not consider this scenario.

If the samples cover an integer number of sine wave periods, we have

$$\begin{gathered} \sum_{i=1}^{M} \cos(\omega_a t_i) = 0\ ,\ \sum_{i=1}^{M} \sin(\omega_a t_i) = 0, \\ \sum_{i=1}^{M} \cos(\omega_a t_i)\sin(\omega_a t_i) = 0 \end{gathered} \tag{10}$$

and

$$\sum_{i=1}^{M} \cos^2(\omega_a t_i) = \frac{M}{2} \qquad \sum_{i=1}^{M} \sin^2(\omega_a t_i) = \frac{M}{2}\ . \tag{11}$$

Consequently the inverse of matrix $D^T D$ becomes

$$\left(D^T D\right)^{-1} = \frac{1}{M}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (12)$$

The sine wave parameters can thus be estimated with

$$\begin{bmatrix} \widehat{A}_{\mathrm{I}} \\ \widehat{A}_{\mathrm{Q}} \\ \widehat{C} \end{bmatrix} = \begin{bmatrix} \frac{2}{M}\sum_{i=1}^{M} y_i \cos(\omega_a t_i) \\ \frac{2}{M}\sum_{i=1}^{M} y_i \sin(\omega_a t_i) \\ \frac{1}{M}\sum_{i=1}^{M} y_i \end{bmatrix}, \qquad (13)$$

and the square of the sine wave amplitude is given by

$$\widehat{A}^2 = \widehat{A}_{\mathrm{I}}^2 + \widehat{A}_{\mathrm{Q}}^2 = \frac{4}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M} y_i y_j \cos\left[\omega_a\left(t_i - t_j\right)\right]. \qquad (14)$$

The sinewave amplitude estimator is thus the square root of the value given by (14):

$$\widehat{A} = \sqrt{\frac{4}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M} y_i y_j \cos\left[\omega_a\left(t_i - t_j\right)\right]}. \qquad (15)$$

The approach taken here is to determine the mean (section III) and the variance (section IV) of the square of the estimated amplitude and then use the results obtained to write an approximate expression for the expected value of the estimated amplitude (section V).

III. Mean of Square Estimated Amplitude

The expected value of the square of the estimated sine wave amplitude is, from (14),

$$\mathrm{E}\left\{\widehat{A}^2\right\} = \frac{4}{M^2}\sum_{i,j}\mathrm{E}\left\{y_i y_j\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]. \qquad (16)$$

To simplify the notation we will abstain, from now on, to indicate the limits of the summations and assume that they span all the integers from 1 to M. In (16), for instance, we have a double summation where i and j go from 1 to M like the summations in (14).

The expected value of the product of y_i and y_j is, using (5),

$$\mathrm{E}\left\{y_i y_j\right\} = \mathrm{E}\left\{z_i z_j\right\} + \mathrm{E}\left\{d_i d_j\right\} + 2\mathrm{E}\left\{z_i d_j\right\}. \qquad (17)$$

Since the null mean additive noise d is considered here independent of the stimulus signal, $\mathrm{E}\left\{z_i d_j\right\} = \mathrm{E}\left\{z_i\right\}\mathrm{E}\left\{d_j\right\} = 0$. Equation (17) thus becomes

$$\mathrm{E}\left\{y_i y_j\right\} = \mathrm{E}\left\{z_i z_j\right\} + \mathrm{E}\left\{d_i d_j\right\}. \qquad (18)$$

Considering that the additive noise of two different samples is uncorrelated, we have

$$\mathrm{E}\left\{y_i y_j\right\} = \begin{cases} \mathrm{E}\left\{z_i^2\right\} + \sigma_v^2 \ , \ i = j \\ \mathrm{E}\left\{z_i z_j\right\} \ , \ i \neq j \end{cases}. \qquad (19)$$

It is possible now to use (19) to compute (16). Notice however that the expression to use for the argument of the double summation is different whether indices i and j are equal or not. In order to proceed with the derivation we need to have complete summations, that is, summations whose indices span all possible values, and that have in its argument a single expression for all cases of the indices. This can be achieved by splitting the summation in (16) into three summations as illustrated in Fig. 1.

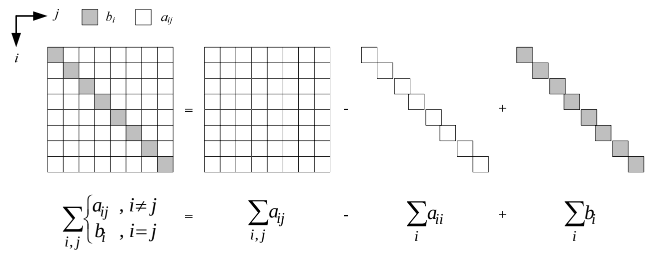

Fig. 1 – Illustration of a double summation split into three other summations (one double and two single).

Inserting (19) into (16) leads to

$$\mathrm{E}\left\{\widehat{A}^2\right\} = \frac{4}{M^2}\sum_{i,j}\mathrm{E}\left\{z_i z_j\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right] - \\ - \frac{4}{M^2}\sum_{i}\mathrm{E}\left\{z_i z_i\right\} + \frac{4}{M^2}\sum_{i}\left[\mathrm{E}\left\{z_i^2\right\} + \sigma_v^2\right]. \qquad (20)$$

This can be simplified to

$$\mathrm{E}\left\{\widehat{A}^2\right\} = \frac{4}{M^2}\sum_{i,j}\mathrm{E}\left\{z_i z_j\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right] + \frac{4}{M}\sigma_v^2. \qquad (21)$$

Using (4) we can write

$$\begin{aligned} \mathrm{E}\left\{z_i z_j\right\} &= \mathrm{E}\left\{\begin{matrix}\left[C + A\cos\left(\omega_x t_i + \varphi\right)\right]\times \\ \times\left[C + A\cos\left(\omega_x t_j + \varphi\right)\right]\end{matrix}\right\} = \\ &= C^2 + 2CA\times\mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\right\} + \\ &+ A^2\times\mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\cos\left(\omega_x t_j + \varphi\right)\right\} = \\ &= C^2 + A^2\times\mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\cos\left(\omega_x t_j + \varphi\right)\right\} \end{aligned}, \qquad (22)$$

making use of the fact that, since φ is a uniformly distributed random variable between 0 and 2π, $\mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\right\}$ is null. Transforming the product of two cosine functions into the sum of two cosine functions,

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b), \qquad (23)$$

(22) can be further simplified to

$$\mathrm{E}\left\{z_i z_j\right\} = C^2 + A^2 \frac{1}{2}\mathrm{E}\left\{\cos\left(\omega_x t_i + \omega_x t_j + 2\varphi\right)\right\} + + A^2 \frac{1}{2}\mathrm{E}\left\{\cos\left(\omega_x t_i - \omega_x t_j\right)\right\} . \quad (24)$$

Again, because φ is a uniformly distributed random variable between 0 and 2π, the second term in the second member of (24) is null, leading to

$$\mathrm{E}\left\{z_i z_j\right\} = C^2 + \frac{1}{2}A^2 \cos\left[\omega_x\left(t_i - t_j\right)\right] . \quad (25)$$

Inserting (25) into (20) leads to

$$\mathrm{E}\left\{\hat{A}^2\right\} = \frac{4}{M}\sigma_v^2 + \frac{4}{M^2}C^2 \sum_{i,j}\cos\left[\omega_a\left(t_i - t_j\right)\right] + + \frac{2}{M^2}A^2 \sum_{i,j}\cos\left[\omega_x\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_i - t_j\right)\right] . \quad (26)$$

Considering that the frequency of the sinewave model (ω_x) is known and used as the frequency of the sinewave that is being fitted to the data (ω_a), that is $\omega_x = \omega_a$, and that the data spans an integer number of periods, (26) becomes

$$\mathrm{E}\left\{\widehat{A^2}\right\} = A^2 + \frac{4}{M}\sigma_v^2 . \quad (27)$$

IV. Variance of the Estimated Square Amplitude

The variance of the square of the estimated amplitude can be expressed as the difference between two expected values:

$$\sigma^2_{\hat{A}^2} = \mathrm{E}\left\{\hat{A}^4\right\} - \mathrm{E}^2\left\{\hat{A}^2\right\} . \quad (28)$$

Using (14) we can write

$$\hat{A}^4 = \frac{16}{M^4}\sum_{i,j,k,l} y_i y_j y_k y_l \cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] . \quad (29)$$

The expected value is thus

$$\mathrm{E}\left\{\hat{A}^4\right\} = \frac{16}{M^4}\sum_{i,j,k,l}\left\{\begin{array}{l}\mathrm{E}\left\{y_i y_j y_k y_l\right\}\times \\ \times\cos\left[\omega_a\left(t_i - t_j\right)\right]\times \\ \times\cos\left[\omega_a\left(t_k - t_l\right)\right]\end{array}\right\} . \quad (30)$$

Inserting (16) and (30) into (28) leads to

$$\sigma^2_{\hat{A}^2} = \frac{16}{M^4}\sum_{i,j,k,l}\left\{\begin{array}{l}\mathrm{cov}\left\{y_i y_j , y_k y_l\right\}\times \\ \times\cos\left[\omega_a\left(t_i - t_j\right)\right]\times \\ \times\cos\left[\omega_a\left(t_k - t_l\right)\right]\end{array}\right\} , \quad (31)$$

where the definition of covariance,

$$\mathrm{cov}\{a,b\} = \mathrm{E}\{ab\} - \mathrm{E}\{a\}\mathrm{E}\{b\} , \quad (32)$$

was used.

Using (5) it is possible to write

$$\mathrm{cov}\left\{y_i y_j , y_k y_l\right\} = \mathrm{cov}\left\{\begin{array}{l} z_i z_j + d_i d_j + z_i d_j + d_i z_j , \\ z_k z_l + d_k d_l + z_k d_l + d_k z_l \end{array}\right\} , \quad (33)$$

which can also be written as

$$\begin{aligned} &\mathrm{cov}\left\{y_i y_j , y_k y_l\right\} = \\ &= \mathrm{cov}\left\{z_i z_j , z_k z_l\right\} + \mathrm{cov}\left\{z_i z_j , d_k d_l\right\} + \\ &+ \mathrm{cov}\left\{z_i z_j , z_k d_l\right\} + \mathrm{cov}\left\{z_i z_j , d_k z_l\right\} + \\ &+ \mathrm{cov}\left\{d_i d_j , z_k z_l\right\} + \mathrm{cov}\left\{d_i d_j , d_k d_l\right\} + \\ &+ \mathrm{cov}\left\{d_i d_j , z_k d_l\right\} + \mathrm{cov}\left\{d_i d_j , d_k z_l\right\} + . \\ &+ \mathrm{cov}\left\{z_i d_j , z_k z_l\right\} + \mathrm{cov}\left\{z_i d_j , d_k d_l\right\} + \\ &+ \mathrm{cov}\left\{z_i d_j , z_k d_l\right\} + \mathrm{cov}\left\{z_i d_j , d_k z_l\right\} + \\ &+ \mathrm{cov}\left\{d_i z_j , z_k z_l\right\} + \mathrm{cov}\left\{d_i z_j , d_k d_l\right\} + \\ &+ \mathrm{cov}\left\{d_i z_j , z_k d_l\right\} + \mathrm{cov}\left\{d_i z_j , d_k z_l\right\} \end{aligned} \quad (34)$$

Since d is independent on z, every term with 1 or 3 times variable d will be zero. The terms with covariance of the product of two z variables with two d variables is null because the additive noise is considered independent of the signal. The same goes for the term $\mathrm{cov}\left\{d_i d_j , z_k z_l\right\}$. Using this we have, from (34),

$$\begin{aligned} \mathrm{cov}\left\{y_i y_j , y_k y_l\right\} &= \mathrm{cov}\left\{z_i z_j , z_k z_l\right\} + \mathrm{cov}\left\{d_i d_j , d_k d_l\right\} + \\ &+ \mathrm{cov}\left\{z_i d_j , z_k d_l\right\} + \mathrm{cov}\left\{z_i d_j , d_k z_l\right\} + . \\ &+ \mathrm{cov}\left\{d_i z_j , z_k d_l\right\}\mathrm{cov}\left\{d_i z_j , d_k z_l\right\} \end{aligned} \quad (35)$$

The remaining terms with two z and two d are all equal since multiplication is commutative. We thus have

$$\begin{aligned} \mathrm{cov}\left\{y_i y_j , y_k y_l\right\} &= \mathrm{cov}\left\{z_i z_j , z_k z_l\right\} + \\ &+ \mathrm{cov}\left\{d_i d_j , d_k d_l\right\} + 4\mathrm{cov}\left\{z_i d_j , z_k d_l\right\} \end{aligned} . \quad (36)$$

Inserting into (31) leads to

$$\begin{aligned} \sigma^2_{\hat{A}^2} &= \frac{16}{M^4}\sum_{i,j,k,l}\mathrm{cov}\left\{z_i z_j , z_k z_l\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] + \\ &+ \frac{16}{M^4}\sum_{i,j,k,l}\mathrm{cov}\left\{d_i d_j , d_k d_l\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] + \\ &+ 4\frac{16}{M^4}\sum_{i,j,k,l}\mathrm{cov}\left\{z_i d_j , z_k d_l\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] \end{aligned}$$

. (37)

Each of these three terms will now be determined separately.

A. Additive Noise Term

Considering that the additive is normally distributed with standard deviation σ_v, the covariance term in the second term of the second member of (37) is

$$\operatorname{cov}\left\{d_i d_j, d_k d_l\right\} = \begin{cases} 2\sigma_v^4 & , i = j = k = l \\ \sigma_v^4 & , i = k \wedge j = l \wedge i \neq j \wedge k \neq l \\ \sigma_v^4 & , i = l \wedge j = k \wedge i \neq j \wedge k \neq l \\ 0 & , \text{otherwise} \end{cases}, \quad (38)$$

allows us to write

$$\begin{aligned} &\frac{16}{M^4}\sum_{i,j,k,l}\operatorname{cov}\left\{d_i d_j, d_k d_l\right\}\cos\left[\omega_s\left(t_i - t_j\right)\right]\cos\left[\omega_s\left(t_k - t_l\right)\right] = \\ &= \frac{16}{M^4}2\sigma_v^4 M + 2\frac{16}{M^4}\sigma_v^4\sum_{i\neq j}\cos^2\left[\omega_s\left(t_i - t_j\right)\right] = \\ &= \frac{16}{M^2}2\sigma_v^4 M + 2\frac{16}{M^4}\sigma_v^4\left(\frac{1}{2}M^2 - M\right) = \\ &= \frac{16}{M^2}\sigma_v^4 \end{aligned} . \quad (39)$$

B. Product of Signal by Additive Noise Term

Using the definition of covariance we can write, for the covariance between the products of z and d,

$$\begin{aligned} \operatorname{cov}\left\{z_i d_j, z_k d_l\right\} &= \mathrm{E}\left\{z_i d_j z_k d_l\right\} - \mathrm{E}\left\{z_i d_j\right\}\mathrm{E}\left\{z_k d_l\right\} = \\ &= \mathrm{E}\left\{z_i d_j z_k d_l\right\} \end{aligned} . \quad (40)$$

The terms $\mathrm{E}\left\{z_i d_j\right\}$ and $\mathrm{E}\left\{z_k d_l\right\}$ are null because z and d are independent and d has a null expected value. The remaining expected value in (40) is equal to

$$\mathrm{E}\left\{z_i d_j z_k d_l\right\} = \begin{cases} \mathrm{E}\left\{z_i d_j^2 z_k\right\} & , j = l \\ 0 & , j \neq l \end{cases}, \quad (41)$$

where

$$\mathrm{E}\left\{z_i d_j^2 z_k\right\} = \mathrm{E}\left\{z_i z_k\right\}\mathrm{E}\left\{d_j^2\right\} = \sigma_v^2\mathrm{E}\left\{z_i z_k\right\} . \quad (42)$$

Equation (40) can thus be written as

$$\operatorname{cov}\left\{z_i d_j, z_k d_l\right\} = \begin{cases} \sigma_v^2\mathrm{E}\left\{z_i z_k\right\} & , j = l \\ 0 & , j \neq l \end{cases}. \quad (43)$$

Inserting it into the quadruple summation of the last term in (37) leads to

$$\begin{aligned} &\frac{16}{M^4}\sum_{i,j,k,l}\operatorname{cov}\left\{z_i d_j, z_k d_l\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] = \\ &= \frac{16}{M^4}\sum_{i,j,k}\sigma_v^2\mathrm{E}\left\{z_i z_k\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_j\right)\right] = \\ &= \frac{8}{M^3}\sigma_v^2\sum_{i,k}\mathrm{E}\left\{z_i z_k\right\}\cos\left[\omega_a\left(t_i - t_k\right)\right] \end{aligned} . \quad (44)$$

Using the same reasoning as before leads to

$$\begin{aligned} &\frac{16}{M^4}\sum_{i,j,k,l}\operatorname{cov}\left\{z_i d_j, z_k d_l\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] = \\ &\quad = \frac{2}{M}\sigma_v^2 A^2 \end{aligned} . \quad (45)$$

C. Signal Term

The value of sample voltage without noise, given by (4), can be written as

$$z_i = C + w_i . \quad (46)$$

with

$$w_i = A\cos\left(\omega_x t_i + \varphi\right) . \quad (47)$$

The covariance in the first term of the second member of (37) can thus be expressed as

$$\begin{aligned} \operatorname{cov}\left\{z_i z_j, z_k z_l\right\} &= \operatorname{cov}\left\{\left(C + w_i\right)\left(C + w_j\right), \left(C + w_k\right)\left(C + w_l\right)\right\} = \\ &= \operatorname{cov}\left\{C^2 + C\left(w_i + w_j\right) + w_i w_j, C^2 + C\left(w_k + w_l\right) + w_k w_l\right\} \end{aligned} . \quad (48)$$

Using the properties of the covariance this can be written as

$$\begin{aligned} &\operatorname{cov}\left\{z_i z_j, z_k z_l\right\} = \\ &= 4C^2\operatorname{cov}\left\{w_i, w_k\right\} + 4C\times\operatorname{cov}\left\{w_i, w_k w_l\right\} + \operatorname{cov}\left\{w_i w_j, w_k w_l\right\} \end{aligned} . \quad (49)$$

In the Appendix each of these covariances has been calculated. Inserting (63), (68) and (73) into (49) leads to

$$\begin{aligned} \operatorname{cov}\left\{z_i z_j, z_k z_l\right\} &= 2C^2A^2\cos\left(\omega_x t_i - \omega_x t_k\right) + \\ &+ \frac{1}{8}A^4\cos\left(\omega_x t_i + \omega_x t_j - \omega_x t_k - \omega_x t_l\right) \end{aligned} . \quad (50)$$

Inserting this into the quadruple summation in the first term of the second member of (37) and considering that the data points span an integer number of periods leads to the same conclusion that this term is null:

$$\frac{16}{M^4}\sum_{i,j,k,l}\operatorname{cov}\left\{z_i z_j, z_k z_l\right\}\cos\left[\omega_x\left(t_i - t_j\right)\right]\cos\left[\omega_x\left(t_k - t_l\right)\right] = 0 . \quad (51)$$

Inserting (39), (45) and (51) into (37) leads finally to

$$\sigma^2_{\hat{A}^2} = \frac{16}{M^2}\sigma_v^4 + \frac{8}{M}\sigma_v^2 A^2 . \quad (52)$$

V. MEAN OF ESTIMATED AMPLITUDE

We now know the mean and the variance of $\hat{A}^2$ and are going to use it to compute the mean of $\hat{A}$ using the following approximation:

$$\mu_{\hat{A}} \simeq \sqrt{\mu_{\hat{A}^2}} - \frac{\sigma^2_{\hat{A}^2}}{8\sqrt{\mu^3_{\hat{A}^2}}} . \quad (53)$$

This comes from a second order Taylor series of the non-linear function that is the square root.

Inserting (27) and (52) into (53) leads to

$$\mu_{\hat{A}} \simeq \sqrt{A^2 + \frac{4}{M}\sigma_v^2} - \frac{\frac{16}{M^2}\sigma_v^4 + \frac{8}{M}\sigma_v^2 A^2}{8\left(A^2 + \frac{4}{M}\sigma_v^2\right)^{\frac{3}{2}}} . \quad (54)$$

A simpler expression may be derived if we are not interested in the extreme cases of high random noise standard deviation and low number of samples, in which case some approximations can be made:

- The A^2 term in the denominator is usually much larger than the σ_v^2 term;
- The $\sigma_v^2 A^2$ term in the numerator is usually much larger than the σ_v^4 term.
- The square root can be substituted by a Taylor series approximation $\left(\sqrt{1+x} \approx 1 + x/2\right)$.

The simplified expression than becomes

$$\mu_{\hat{A}} \approx A + \frac{1}{M}\frac{\sigma_v^2}{A} . \quad (55)$$

This shows that the amplitude estimation is biased for a finite number of samples but asymptotically unbiased, that is, unbiased in the case of an infinite number of samples. It also shows that the expected value does not depend on the signal offset (*C*).

An interesting parameter to define is the relative error of the estimated amplitude:

$$\varepsilon_A \approx \frac{\mu_{\hat{A}} - A}{A} . \quad (56)$$

Inserting (55) into (56) leads to

$$\varepsilon_A \approx \frac{1}{2M \cdot SNR^2} , \quad (57)$$

where

$$SNR = \frac{A}{\sqrt{2}\sigma_v} . \quad (58)$$

VI. MONTE CARLO ANALYSIS

In order to validate the approximations made, namely in (53) and (55), a Monte Carlo analysis was done using simulated data on a computer. The data points were corrupted with additive noise with different standard deviation values. Different number of samples were used. In all cases the relative estimation error obtained through simulation was in accordance with the theoretical value given by the approximate expression (57).

In Fig. 2, the relative estimation error is plotted against the additive noise standard deviation. It is seen that, as expected, the error increases with the amount of noise. The vertical bars represent the confidence intervals in estimating the relative error using the Monte Carlo analysis with *J* repetitions. A 99.9 % confidence level was used to compute the intervals. It is assumed that the relative error estimation is normally distributed.

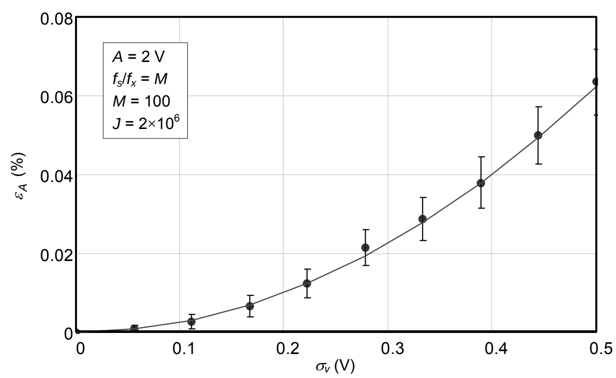

Fig. 2 – Expected value of the estimated sine wave amplitude as a function of the additive noise standard deviation. The circles represent the values obtained with the Monte Carlo analysis. The confidence intervals for a confidence level of 99.9% are represented by the vertical bars. The solid line represents the theoretical value given by (57).

In Fig. 3, the relative error is represented as a function of the number of samples. The higher the number of samples, the lower the error.

The parameters chosen for the test, and which are presented inside the boxes in the figure, were the ones that best illustrate the dependence of the relative error on the different parameters and which correspond to typical conditions encountered in practice.

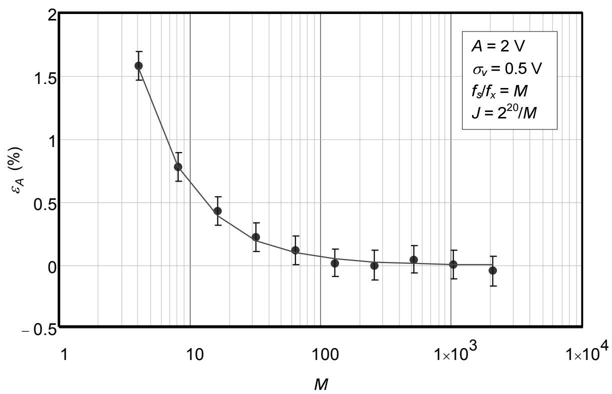

Fig. 3 – Expected value of the estimated sine wave amplitude as a function of the number of samples. The circles represent the values obtained with the Monte Carlo analysis. The confidence intervals for a confidence level of 99.9% are represented by the vertical bars. The solid line represents the theoretical value given by (57).

VII. Experimental Validation

The results of the previous section were meant to validate the mathematical approximations made in deriving (55). In this section we are going to present some experimental results meant to validate that the model used, namely (5), does in fact represent practical conditions encountered in real world experiments. The referred model takes into account a perfect sine wave with a random initial phase and corrupted by additive noise. All other non-ideal effects, like jitter, phase noise, harmonic distortion, frequency error, quantization, etc., are left out in this model. Putting it differently, the experimental data presented here, serves to show that the results obtained in this paper are in fact applicable in practice and that the assumptions made, like the uniformity of the distribution of the random sine wave initial phase or the independence of additive noise and sine wave, do hold in real conditions.

To gather the experimental data, a test setup like the one depicted in Fig. 4 was built. A data acquisition board for a personal computer was used to sample and digitize the samples of a sinusoidal voltage signal created by a function generator to which normally distributed additive noise generated by another function generator was subtracted. This subtraction was achieved through the use of a differential input to the data acquisition board.

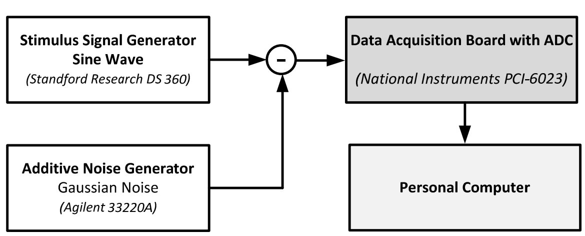

Fig. 4 – Test Bench. The personal computer controls all instruments using GPIB. The combination of the sinusoidal stimulus signal and the normally distributed noise is carried out inside the data acquisition board through the use of one of its differential inputs.

In Tab. 1 the pertinent characteristics of the instruments and the ADC used are presented.

Tab. 1 – Manufacturer specifications for the instruments and ADC used.

Specification	*Value*
Signal Generator (Stanford Research DS360) [25]	
Sine Wave Amplitude Accuracy	1 %
Sine Wave Offset Accuracy	1% + 25 mV
Sine Wave Frequency Accuracy	25 ppm+4 mHz
Total Harmonic Distortion	-93 dB
Noise Generator (Agilent AG 33220A) [26]	
Noise Bandwidth	≈ 10 MHz
ADC (National Instruments PCI-6023) [27]	
Number of Bits	12
Range	±5 V
Integral Non Linearity (INL)	1.5 LSB
Differential Non Linearity (DNL)	1 LSB
Gain Error	0.02 %
Offset Error	0.5 mV
Bandwidth (small signal)	500 kHz
Quantization Step (Q)	2.44 mV
Noise standard deviation	0.1 LSB
Jitter	Not specified
Timing Accuracy (relative to sample rate)	50 ppm

One of the goals of the test setup is to be able to add a given amount of normally distributed noise to the signal that is sampled by the analog to digital converter. The function generator that produces the noise has a noise bandwidth of approximately 10 MHz (Tab. 1) while the data acquisition board has a small signal bandwidth of 500 kHz (this bandwidth is not, however, an equivalent noise bandwidth). Due to the bandwidths mismatch, in order to have a given value of noise standard deviation one has to produce an higher value of noise effective value that the desired one. To calibrate this gain a value of 100 mVrms of noise was generated and the IEEE 1057 random noise test of waveform digitizers was performed in order to measure the actual noise that was present at the ADC input. Note that the amount of internal noise in the data acquisition board (244 μV) is negligible compared to the values of additive noise that were injected. The same can be said of the quantization error of the ADC.

A Monte Carlo analysis was carried on to estimate the relative error of the sine fitting amplitude estimation as a function of additive noise standard deviation. The unprocessed results are presented in Fig. 5. The vertical bars represent the confidence interval of the relative error estimation considering a normal distribution and a confidence level of 99.9%. Note that the theoretical curve, given by (57), is not in agreement with the experimental data. This discrepancy is most likely due to the error in the exact value of the amplitude of the sine wave that was generated. The function generator manufacturer specifies an accuracy for it of 1%, which is significant larger than the discrepancy that is observed in the experimental data (around 0.13 %).

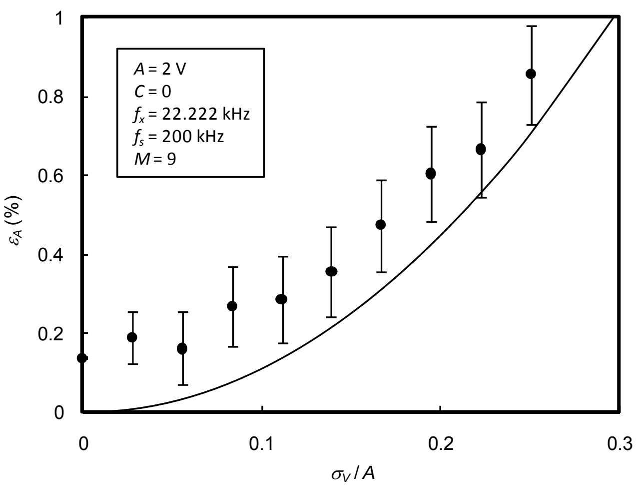

Fig. 5 – Experimentally determined relative error of the estimated amplitude as a function of the normalized additive noise standard deviation. The circles represent the values obtained with the Monte Carlo analysis done with a number of repetitions ranging from 10^3 (left most point) to 10^5 (right most point). The confidence intervals for a confidence level of 99.9% are represented by the vertical bars. The solid line represents the theoretical value given by (57).

In Fig. 6 the value of the relative error of the first data point was subtracted from all the data points in an attempt to compensate for the error allegedly introduced by the sine wave generator. The value of the relative error of the first point (left most) in Fig. 6 is thus 0 which is the value one expects when no noise is present. It can be seen that all the other data points are in agreement with the theoretical result since the confidence intervals all contain the solid line.

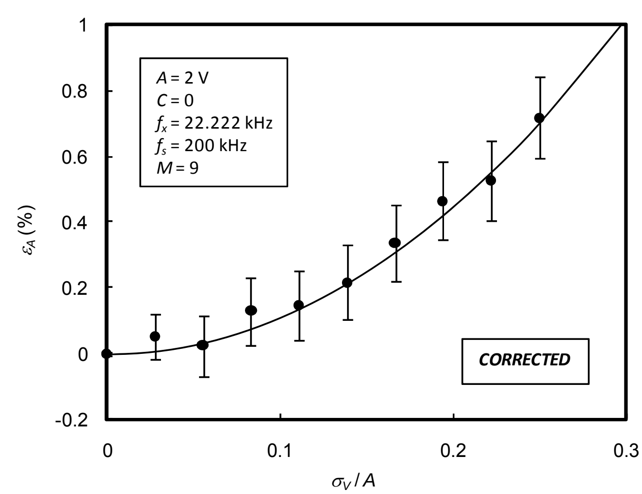

Fig. 6 – Experimentally determined relative error of the estimated amplitude as a function of the normalized additive noise standard deviation. The circles represent the values obtained with the Monte Carlo analysis done with a number of repetitions ranging from 10^3 (left most point) to 10^5 (right most point). The confidence intervals for a confidence level of 99.9% are represented by the vertical bars. The solid line represents the theoretical value given by (57). An offset was added to the experimental data so that the first data point had a value of 0 which is the theoretical value for the case of no additive noise present.

VIII. Conclusions

The experimental data presented does validate expression (57) proposed here to be used for the computation of the relative error of the estimation of the amplitude of a sine wave using the three-parameter sine fitting algorithm.

The expression derived is a simple one, due to some approximations done but which were thoroughly validated with simulated and experimental tests. It is a novel expression which takes into accounts the effects of additive noise and a limit number of samples. In the future further work should be carried out to include other important effects like jitter, phase noise, quantization error and harmonic distortion. Note that expression (57) is an approximate expression which not valid for the extreme cases of high random noise standard deviation (relative to signal amplitude) and low number of samples. In those cases, expression (54) should be used.

It should be pointed out that what the expression gives is the expected value of the error in the amplitude estimation and not the error itself which is random in nature. It thus cannot be used to correct the error but only to specify a confidence interval for the measurements made. It is also necessary, in the future, to demonstrate what the statistical distribution of the estimator is. Given the large number of samples generally used and the central limit theorem applied to (14), it is presumably approximately a normal distribution.

The work presented here is also useful for the calculation of the minimum number of samples required to achive a given bound on the error of the amplitude estimation. As was shown here, that error decreases as the number of samples increase. In fact, it was demonstrated that the estimator is asymptotically unbiased in the presence of additive noise.

Appendix

In this appendix, a simplified expression for

$$\begin{aligned} \operatorname{cov}\{z_i z_j, z_k z_l\} &= 4C^2 \operatorname{cov}\{w_i, w_k\} + \\ &+ 4C \times \operatorname{cov}\{w_i, w_k w_l\} + , \\ &+ Cov\{w_i w_j, w_k w_l\} \end{aligned} \quad (59)$$

is derived.

A. First term

The first covariance term in the second member can be written, using (47), as

$$\operatorname{cov}\{w_i, w_k\} = \operatorname{cov}\{A\cos(\omega_x t_i + \varphi), A\cos(\omega_x t_k + \varphi)\} . \quad (60)$$

Using (32) and (23) it is possible to arrive at

$$\begin{aligned} \operatorname{cov}\{w_i, w_k\} &= A^2 \mathrm{E}\{\cos(\omega_x t_i + \varphi)\cos(\omega_x t_k + \varphi)\} - \\ &- A^2 \mathrm{E}\{\cos(\omega_x t_i + \varphi)\}\mathrm{E}\{\cos(\omega_x t_k + \varphi)\} \end{aligned} . \quad (61)$$

Transforming the product of cosines into a sum of cosines leads to

$$\text{cov}\{w_i, w_k\} = \frac{1}{2}A^2 \text{E}\{\cos(\omega_x t_i - \omega_x t_k)\} + \\ + \frac{1}{2}A^2 \text{E}\{\cos(\omega_x t_i + \omega_x t_k + 2\varphi)\} - \\ - A^2 \text{E}\{\cos(\omega_x t_i + \varphi)\}\text{E}\{\cos(\omega_x t_k + \varphi)\} \quad . \quad (62)$$

Since φ is a uniformly distributed random variable between 0 and 2π, $\text{E}\{\cos(\omega_x t_i + \varphi)\}$ is null, which leads to

$$\text{cov}\{w_i, w_k\} = \frac{1}{2}A^2 \cos(\omega_x t_i - \omega_x t_k) \; . \quad (63)$$

B. Second term

The second term in (59) can be written, with the help of (47), as

$$\text{cov}\{w_i, w_k w_l\} = \text{cov}\left\{\begin{matrix} A\cos(\omega_x t_i + \varphi), \\ A^2 \cos(\omega_x t_k + \varphi)\cos(\omega_x t_l + \varphi) \end{matrix}\right\} . \quad (64)$$

Transforming the product of cosines into a sum of cosines leads to

$$\text{cov}\{w_i, w_k w_l\} = \frac{A^3}{2}\text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) + \\ + \cos(\omega_x t_k - \omega_x t_l) \end{matrix}\right\} . \quad (65)$$

Splitting the covariance of the sum into the sum of covariances leads to

$$\text{cov}\{w_i, w_k w_l\} = \frac{1}{2}A^3 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) \end{matrix}\right\} + \\ + \frac{1}{2}A^3 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \varphi), \\ \cos(\omega_x t_k - \omega_x t_l) \end{matrix}\right\} \quad . \quad (66)$$

Substituting the covariances by expected values, as in (32), leads to

$$\text{cov}\{w_i, w_k w_l\} = \frac{1}{2}A^3 \text{E}\{\cos(\omega_x t_i + \varphi)\cos(\omega_x t_k + \omega_x t_l + 2\varphi)\} \\ - \frac{1}{2}A^3 \text{E}\{\cos(\omega_x t_i + \varphi)\}\text{E}\{\cos(\omega_x t_k + \omega_x t_l + 2\varphi)\} + \\ + \frac{1}{2}A^3 \text{E}\{\cos(\omega_x t_i + \varphi)\cos(\omega_x t_k - \omega_x t_l)\} - \\ - \frac{1}{2}A^3 \text{E}\{\cos(\omega_x t_i + \varphi)\}\text{E}\{\cos(\omega_x t_k - \omega_x t_l)\} \quad (67)$$

Since φ is a uniformly distributed random variable between 0 and 2π, $\text{E}\{\cos(\omega_x t_i + \varphi)\}$ is null, which leads to

$$\text{cov}\{w_i, w_k w_l\} = 0 \; . \quad (68)$$

C. Third term

Again inserting (47) into the third term of the second member of (59) leads to

$$\text{cov}\{w_i w_j, w_k w_l\} = \text{cov}\left\{\begin{matrix} A^2 \cos(\omega_x t_i + \varphi)\cos(\omega_x t_j + \varphi), \\ A^2 \cos(\omega_x t_k + \varphi)\cos(\omega_x t_l + \varphi) \end{matrix}\right\} . \quad (69)$$

Transforming the product of cosines into a sum of cosines leads to

$$\text{cov}\{w_i w_j, w_k w_l\} = \frac{1}{4}A^4 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \omega_x t_j + 2\varphi) + \\ + \cos(\omega_x t_i - \omega_x t_j), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) + \\ + \cos(\omega_x t_k - \omega_x t_l) \end{matrix}\right\} . \quad (70)$$

Substituting the covariance of the sum into the sum of covariances leads to

$$\text{cov}\{w_i w_j, w_k w_l\} = \frac{1}{4}A^4 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \omega_x t_j + 2\varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) \end{matrix}\right\} + \\ + \frac{1}{4}A^4 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \omega_x t_j + 2\varphi), \\ \cos(\omega_x t_k - \omega_x t_l) \end{matrix}\right\} + \\ + \frac{1}{4}A^4 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i - \omega_x t_j), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) \end{matrix}\right\} + \\ + \frac{1}{4}A^4 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i - \omega_x t_j), \\ \cos(\omega_x t_k - \omega_x t_l) \end{matrix}\right\} \; . \quad (71)$$

Since φ is a uniformly distributed random variable between 0 and 2π, the 2nd and 3rd terms are null. Furthermore, since the argument of the covariance in the last term is not a random variable, this term is also null. What remains is thus

$$\text{cov}\{w_i w_j, w_k w_l\} = \frac{1}{4}A^4 \text{cov}\left\{\begin{matrix} \cos(\omega_x t_i + \omega_x t_j + 2\varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) \end{matrix}\right\} . \quad (72)$$

This covariance finally leads to

$$\text{cov}\{w_i w_j, w_k w_l\} = \frac{1}{8}A^4 \cos(\omega_x t_i + \omega_x t_j - \omega_x t_k - \omega_x t_l) \; . \quad (73)$$

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