Fourier’s Law and the Heat Equation
Fourier’s Law

• A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium.

• Its most general (vector) form for multidimensional conduction is:

\[ \vec{q} = -k \nabla T \]

Implications:

– Heat transfer is in the direction of decreasing temperature (basis for minus sign).

– Fourier’s Law serves to define the thermal conductivity of the medium

\[ k = -\frac{\vec{q}}{\nabla T} \]

– Direction of heat transfer is perpendicular to lines of constant temperature (isotherms).

– Heat flux vector may be resolved into orthogonal components.
• Cartesian Coordinates: \[ T(x, y, z) \]
\[
\vec{q}'' = -k \frac{\partial T}{\partial x} \hat{i} - k \frac{\partial T}{\partial y} \hat{j} - k \frac{\partial T}{\partial z} \hat{k}
\]
\[
q''_x \quad q''_y \quad q''_z
\]

(2.3)

• Cylindrical Coordinates: \[ T(r, \phi, z) \]
\[
\vec{q}'' = -k \frac{\partial T}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial T}{\partial \phi} \hat{\phi} - k \frac{\partial T}{\partial z} \hat{k}
\]
\[
q''_r \quad q''_\phi \quad q''_z
\]

(2.18)

• Spherical Coordinates: \[ T(r, \phi, \theta) \]
\[
\vec{q}'' = -k \frac{\partial T}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial T}{\partial \phi} \hat{\phi} - \frac{1}{r \sin \theta} \frac{\partial T}{\partial \theta} \hat{\theta}
\]
\[
q''_r \quad q''_\theta \quad q''_\phi
\]

(2.21)
• In angular coordinates (\(\phi \) or \(\phi, \theta\)), the temperature gradient is still based on temperature change over a length scale and hence has units of \(^\circ\)C/m and not \(^\circ\)C/deg.

Heat rate for one-dimensional, radial conduction in a cylinder or sphere:

- Cylinder
  \[ q_r = A_r q_r'' = 2\pi r L q_r'' \]
  or,
  \[ q_r' = A_r q_r'' = 2\pi r q_r'' \]

- Sphere
  \[ q_r = A_r q_r'' = 4\pi r^2 q_r'' \]
The Heat Equation

- A differential equation whose solution provides the temperature distribution in a stationary medium.

- Based on applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.

- Cartesian Coordinates:

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) - q = \rho c_v \frac{\partial T}{\partial t}
\]

- Net transfer of thermal energy into the control volume (inflow-outflow)
- Thermal energy generation
- Change in thermal energy storage
• Cylindrical Coordinates:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left( k \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + q = \rho c_p \frac{\partial T}{\partial t} \]

(2.20)

• Spherical Coordinates:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( kr^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( k \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( k \sin \theta \frac{\partial T}{\partial \theta} \right) + q = \rho c_p \frac{\partial T}{\partial t} \]

(2.33)
• One-Dimensional Conduction in a Planar Medium with Constant Properties and No Generation

\[ \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

\[ \alpha \equiv \frac{k}{\rho c_p} \rightarrow \text{thermal diffusivity of the medium} \]
Boundary and Initial Conditions

- For **transient conduction**, heat equation is first order in time, requiring specification of an initial temperature distribution: \( T(x,t)_{t=0} = T(x,0) \)

- Since heat equation is second order in space, two boundary conditions must be specified. Some common cases:

  **Constant Surface Temperature:**
  \[
  T(0, t) = T_s
  \]

  **Constant Heat Flux:**
  \[
  -k \frac{\partial T}{\partial x} \bigg|_{x=0} = q_s''
  \]

  **Convection**
  \[
  -k \frac{\partial T}{\partial x} \bigg|_{x=0} = h \left[ T_\infty - T(0, t) \right]
  \]
Thermophysical Properties

Thermal Conductivity: A measure of a material’s ability to transfer thermal energy by conduction.

Thermal Diffusivity: A measure of a material’s ability to respond to changes in its thermal environment.

Property Tables:
- Solids: Tables A.1 – A.3
- Gases: Table A.4
- Liquids: Tables A.5 – A.7
Methodology of a Conduction Analysis

• Solve appropriate form of heat equation to obtain the temperature distribution.

• Knowing the temperature distribution, apply Fourier’s Law to obtain the heat flux at any time, location and direction of interest.

• Applications:
  
  Chapter 3: One-Dimensional, Steady-State Conduction
  Chapter 4: Two-Dimensional, Steady-State Conduction
  Chapter 5: Transient Conduction
Problem 2.46 Thermal response of a plane wall to convection heat transfer.

KNOWN: Plane wall, initially at a uniform temperature, is suddenly exposed to convective heating.

FIND: (a) Differential equation and initial and boundary conditions which may be used to find the temperature distribution, \( T(x,t) \); (b) Sketch \( T(x,t) \) for the following conditions: initial (\( t \leq 0 \)), steady-state (\( t \to \infty \)), and two intermediate times; (c) Sketch heat fluxes as a function of time at the two surfaces; (d) Expression for total energy transferred to wall per unit volume (J/m\(^3\)).

SCHEMATIC:
**ASSUMPTIONS:** (1) One-dimensional conduction, (2) Constant properties, (3) No internal heat generation.

**ANALYSIS: (a)** For one-dimensional conduction with constant properties, the heat equation has the form,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

and the conditions are:

$$\begin{align*}
\text{Initial, } t \leq 0 : & \quad T(x,0) = T_i \\
\text{Boundaries: } x=0 & \quad \frac{\partial T}{\partial x}(0) = 0 \\
& \quad x=L, \quad -k \frac{\partial T}{\partial x}(L) = h\left[T(L,t) - T_\infty\right] \\
\end{align*}$$

uniform temperature, adiabatic surface, surface convection

(b) The temperature distributions are shown on the sketch.

Note that the gradient at $x = 0$ is always zero, since this boundary is adiabatic. Note also that the gradient at $x = L$ decreases with time.
c) The heat flux, \( q''_x(x,t) \), as a function of time, is shown on the sketch for the surfaces \( x = 0 \) and \( x = L \).

![Graph showing heat flux](image)

\[ E_{\text{in}} = \int_0^\infty q''_{\text{conv}} A_s \, dt \]

\[ E_{\text{in}} = hA_s \int_0^\infty (T_\infty - T(L,t)) \, dt \]

d) The total energy transferred to the wall may be expressed as

\[ \frac{E_{\text{in}}}{V} = \frac{h}{L} \int_0^\infty [T_\infty - T(L,t)] \, dt \quad \text{[J/m}^2\text{]} \]
Problem 2.28  Surface heat fluxes, heat generation and total rate of radiation absorption in an irradiated semi-transparent material with a prescribed temperature distribution.

**KNOWN:** Temperature distribution in a semi-transparent medium subjected to radiative flux

**FIND:** (a) Expressions for the heat flux at the front and rear surfaces, (b) The heat generation rate $\dot{q}(x)$, and (c) Expression for absorbed radiation per unit surface area.

**SCHEMATIC:**

Laser irradiation

\[ T(x) = \frac{A}{k_0^2} e^{-ax} + Bx + C \]
ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in medium, (3) Constant properties, (4) All laser irradiation is absorbed and can be characterized by an internal volumetric heat generation term $\dot{q}(x)$.

ANALYSIS: (a) Knowing the temperature distribution, the surface heat fluxes are found using Fourier’s law,

$$ q''_x = -k \left[ \frac{dT}{dx} \right] = -k \left[ -\frac{A}{ka^2} (a^{-ax} \cdot e^{Bx}) + B \right] $$

**Front Surface, x=0:**

$$ q''_x (0) = -k \left[ \frac{A}{ka} + 1 + B \right] = -\left[ \frac{A}{a} + kB \right] < 0 $n

**Rear Surface, x=L:**

$$ q''_x (L) = -k \left[ \frac{A}{ka} e^{-al} + B \right] = -\left[ \frac{A}{a} e^{-al} + kB \right]. < 0 $$

(b) The heat diffusion equation for the medium is

$$ \frac{d}{dx} \left( \frac{dT}{dx} \right) + \frac{\dot{q}}{k} = 0 \quad \text{or} \quad \dot{q} = -k \frac{d}{dx} \left( \frac{dT}{dx} \right) $$

$$ \dot{q}(x) = -k \frac{d}{dx} \left[ \frac{A}{ka} e^{-ax} + B \right] = Ae^{-ax} $$

(c) Performing an energy balance on the medium,

$$ \dot{E}_{in} - \dot{E}_{out} + \dot{E}_{g} = 0 $$
On a unit area basis

\[ \dot{E}_{g}'' = -\dot{E}_{in}'' + \dot{E}_{out}'' = -q_{x}''(0) + q_{x}''(L) = + \frac{A}{a} \left( 1 - e^{-aL} \right). \]

Alternatively, evaluate \( \dot{E}_{g}'' \) by integration over the volume of the medium,

\[ \dot{E}_{g}'' = \int_{0}^{L} \dot{q}(x) \, dx = \int_{0}^{L} A e^{-ax} \, dx = \frac{A}{a} \left[ e^{-ax} \right]_{0}^{L} = \frac{A}{a} \left( 1 - e^{-aL} \right). \]