PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

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COURSE AIMS

- To introduce Geometric Algebra as a new mathematical technique to add to your existing base as a theoretician or experimentalist.
- To develop applications of this new technique in the fields of classical mechanics, engineering, relativistic physics and gravitation.
- To introduce these new techniques through their applications, rather than as purely formal mathematics.
- To emphasise the generality and portability of geometric algebra through the diversity of applications.
- To promote a multi-disciplinary view of science.

All material related to this course is available from

http://www.mrao.cam.ac.uk/∼clifford/ptIIIcourse

or follow the link Cavendish → Research → Geometric Algebra → Physical Applications of Geometric Algebra 2001.

A Quick Tour

In the following weeks we will

- Discover a new, powerful technique for handling rotations in arbitrary dimensions, and analyse the insights this brings to the mathematics of Lorentz transformations.
- Uncover the links between rotations, bivectors and the structure of the Lie groups which underpin much of modern physics.
- Learn how to extend the concept of a complex analytic function in 2-d (*i.e.* a function satisfying the Cauchy-Riemann equations) to arbitrary dimensions, and how this is applied in quantum theory and electromagnetism.
- Unite all four Maxwell equations into a single equation ($\nabla F = J$), and develop new techniques for solving it.
- Combine many of the preceding ideas to construct a gauge theory of gravitation in (flat) Minkowski spacetime, which is still consistent with General Relativity.
- Use our new understanding of gravitation to quickly reach advanced applications such as black holes and cosmology.

Some History

A central problem being tackled in the first part of the 19th Century was how to represent 3-d rotations.

1844

Hamilton introduces his quaternions, which generalize complex numbers. But confusion persists over the status of vectors in his algebra — do $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ constitute the components of a vector?

1844

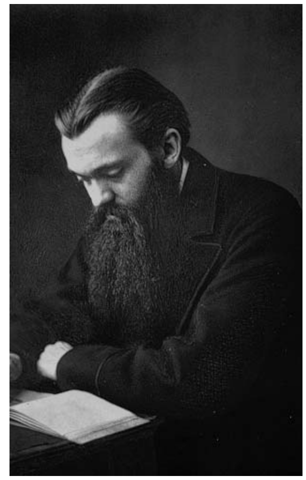
In a separate development, Grassmann. introduces the exterior product. (See later this lecture.) Largely ignored in his lifetime, his work later gave rise to differential forms and Grassmann (anticommuting) variables (used in supersymmetry and superstring theory)

1878

Clifford invents Geometric Algebra by uniting the scalar and exterior products into a single geometric product. This is invertible, so an equation such as $ab = C$ has the solution $b = a^{-1}C$. This is not possible with the separate scalar or exterior products.

Clifford could relate his product to the quaternions, and his system should have gone on to dominate mathematical physics. But . . .

- Clifford died young, at the age of just 33
- Vector calculus was heavily promoted by Gibbs and rapidly became popular, eclipsing Clifford and Grassmann's work.

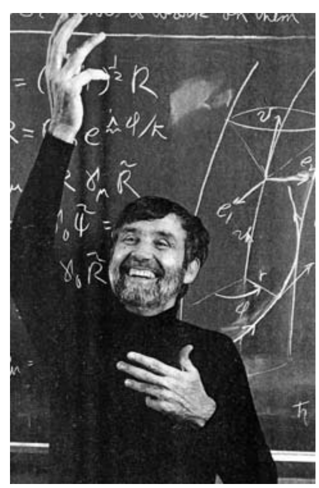


1920's

Clifford algebra resurfaces in the theory of quantum spin. In particular the algebra of the Pauli and Dirac matrices became indispensable in quantum theory. But these were treated just as algebras — the geometrical meaning was lost.

1966

David Hestenes recovers the geometrical meaning (in 3-d and 4-d respectively) underlying the Pauli and Dirac algebras. Publishes his results in the book Spacetime Algebra. Hestenes goes on to produce a fully developed geometric calculus.



In 1984, Hestenes and Sobczyk publish

Clifford Algebra to Geometric Calculus

This book describes a unified language for much for mathematics, physics and engineering. This was followed in 1986 by the (much easier!)

New Foundations for Classical Mechanics

1990's

Hestenes' ideas have been slow to catch on, but in Cambridge we now routinely apply geometric algebra to topics as diverse as

- black holes and cosmology (Astrophysics, Cavendish)
- quantum tunnelling and quantum field theory (Astrophysics, Cavendish)
- beam dynamics and buckling (Structures Group, CUED)
- computer vision (Signal Processing Group, CUED)

Exactly the same algebraic system is used throughout.

PART 1

GEOMETRIC ALGEBRA IN TWO AND THREE DIMENSIONS

LECTURE 1

In this lecture we will introduce the basic ideas behind the mathematics of geometric algebra (abbreviated to GA). The geometric product is motivated by a direct analogy with complex arithmetic, and we will understand the imaginary unit as a geometric entity.

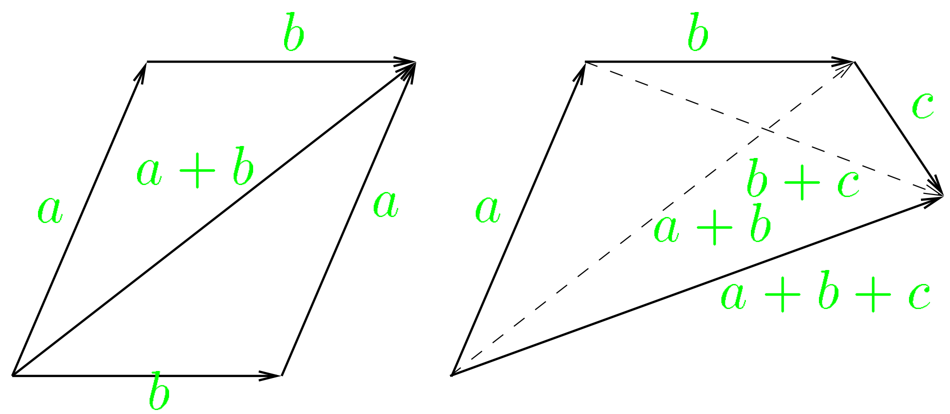
- Multiplying Vectors - The scalar, complex and quaternion products.
- The Exterior Product - Encoding the geometry of planes and higher dimensional objects.
- The Geometric Product - Axioms and basic properties
- The Geometric Algebra of 2-dimensional space.
- Complex numbers rediscovered. The algebra of rotations has a particularly simple expression in 2-d, and leads to the identification of complex numbers with GA.

VECTOR SPACES

Consist of vectors a, b, with an addition law which is

commutative: $a + b = b + a$

associative: $a + (b + c) = (a + b) + c.$



For real scalars λ, μ and vectors a and b:

1. $\lambda(a + b) = \lambda a + \lambda b$;
2. $(\lambda + \mu)a = \lambda a + \mu a$;
3. $(\lambda\mu)a = \lambda(\mu a)$;
4. If $1\lambda = \lambda$ for all scalars λ then $1a = a$ for all vectors a.

NB Two different addition operations.

Get familiar concepts of dimension, linearly independent vectors, and basis. Have no rule for multiplying vectors.

MULTIPLYING VECTORS

In your mathematical training so far, you will have various products for vectors:

The Scalar Product

The scalar, (or inner or dot) product, $a \cdot b$ returns a scalar from two vectors. In Euclidean space the inner product is positive definite,

$$|a|^2 = a \cdot a > 0 \qquad \forall a \neq 0$$

From this we recover Schwarz inequality

$$\begin{aligned} |a+\lambda b|^2 &\geq 0 \qquad \forall \lambda \\ \Longrightarrow |a|^2 + 2\lambda a \cdot b + \lambda^2 |b|^2 &\geq 0 \qquad \forall \lambda \\ \Longrightarrow (a \cdot b)^2 &\leq |a|^2 |b|^2 \end{aligned}$$

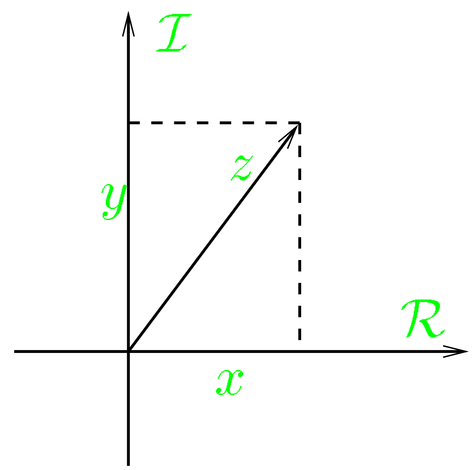
We use this to define the cosine of the angle between a and b via

$$a \cdot b = |a||b| \cos(\theta)$$

Can now do Euclidean geometry. In non-Euclidean spaces, such as Minkowski spacetime, Schwarz inequality does not hold. Can still introduce an orthonormal frame. Some vectors have squavre $+1$ and some -1.

Complex Numbers

A complex number defines a point on an Argand diagram. Complex arithmetic is a way of multiplying together vectors in 2-d.



If $z = x + iy$ then get length from

$$zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Include a second $w = u + vi$, and form

$$zw^* = (x+iy)(u-iv) = xu + vy + i(uy - vx).$$

The real part is the scalar product. For imaginary term use polar representation

$$z = |z|\,\mathrm{e}^{i\theta}, \qquad w = |w|\,\mathrm{e}^{i\phi}$$
$$zw^* = |z||w|\,\mathrm{e}^{i(\theta - \phi)}.$$

Imaginary part is $i|z||w|\sin(\theta - \phi)$. The area of the parallelogram with sides z and w. Sign is related to handedness. Second interpretation for complex addition: a sum between scalars and plane segments.

Quaternions

Quaternion algebra contains 4 objects, $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, (instead of 3). Algebra defined by

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

Define a closed algebra. (Also a division algebra — not so important). Revolutionary idea: elements anticommute

$$\boldsymbol{ij} = -\boldsymbol{ijkk} = \boldsymbol{k}$$
$$\boldsymbol{ji} = -\boldsymbol{jiijk} = -\boldsymbol{k} = -\boldsymbol{ij}$$

Problem: Where are the vectors? Hamilton used 'pure' quaternions — no real part. Gives us a new product:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \quad \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

Result of product is

$$\boldsymbol{ab} = c_0 + \boldsymbol{c}$$

c_0 is (minus) the scalar product. Vector term is

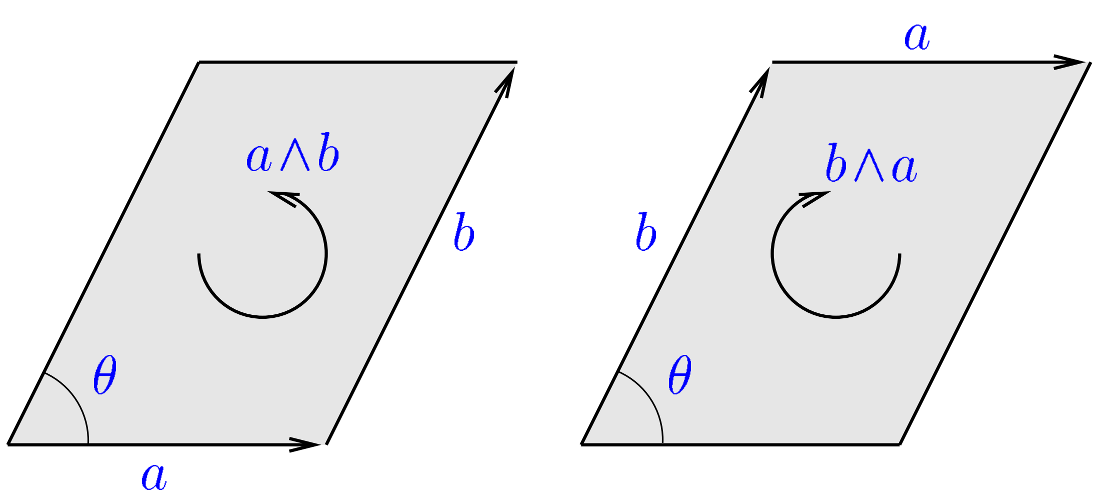
$$\boldsymbol{c} = (a_2b_3 - b_2a_3)\boldsymbol{i} + (a_3b_1 - b_3a_1)\boldsymbol{j} + (a_1b_2 - b_1a_2)\boldsymbol{k}$$

Defines the cross product $\boldsymbol{a} \times \boldsymbol{b}$. Perpendicular to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$, magnitude $ab\sin(\theta)$, and $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ form a right-handed set. The cross product was widely adopted.

THE OUTER PRODUCT

The cross product only exists in 3 dimensions. In 2-d there is nowhere else to go, in 4-d the definition is not unique. In the set $e_1 \ldots e_4$ any combination of e_3 and e_4 is perpendicular to e_1 and e_2.

Need a means of encoding a plane directly. This is what Grassmann provided. Define the outer or wedge product $a \wedge b$ as directed area swept out by a and b. Plane has area $|a||b|\sin(\theta)$, defined to be the magnitude of $a \wedge b$.



Defines an oriented plane.

Think of $a \wedge b$ as the parallelogram formed by sweeping one vector along the other. Changing the order reverses the orientation. Result is neither a scalar nor a vector. It is a bivector — a new mathematical entity encoding the notion of a plane.

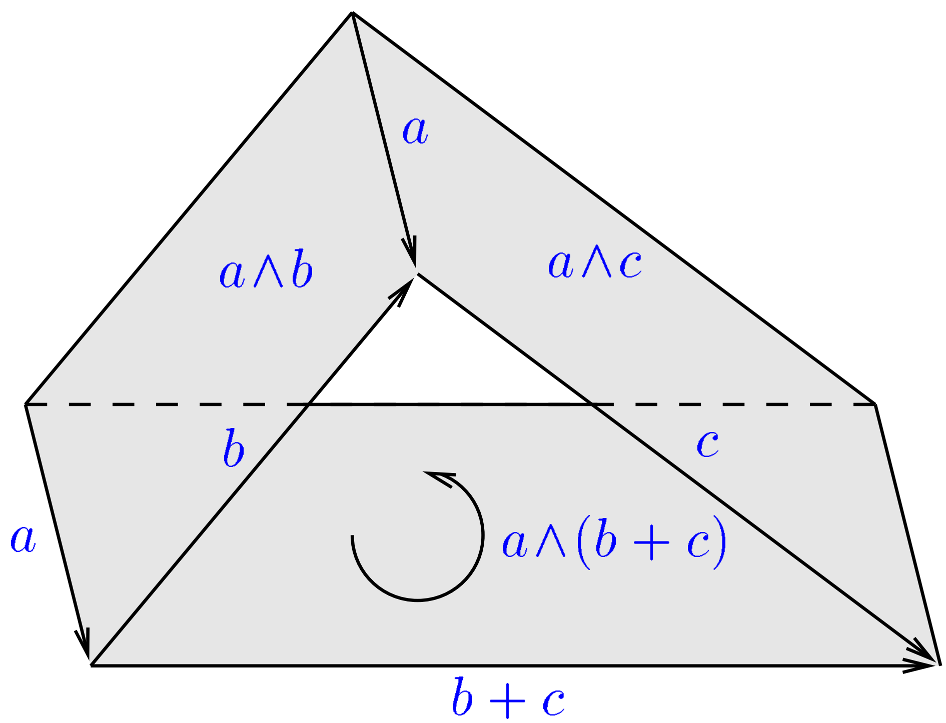
Properties

1. The outer product of two vectors is antisymmetric,

$$a \wedge b = -b \wedge a$$

This follows from the geometric definition. NB. $a \wedge a = 0$

2. Bivectors form a linear space, the same way that vectors do. In 3-d the addition of bivectors is easy to visualise. Not always so obvious in higher dimensions.



3. The outer product is distributive

$$a \wedge (b + c) = a \wedge b + a \wedge c$$

This helps to visualise the addition of bivectors.

4. The outer product does not retain information about shape.
If $a' = a + \lambda b$, have

$$a' \wedge b = (a + \lambda b) \wedge b = a \wedge b + \lambda b \wedge b = a \wedge b$$

Get same result, so cannot recover a and b from $a \wedge b$.
Sometimes better to replace the directed parallelogram with a directed circle.

Example — 2 Dimensions

Suppose e_1, e_2 are basis vectors and have

$$a = a_1 e_1 + a_2 e_2, \quad b = b_1 e_1 + b_2 e_2.$$

The outer product of these is

$$\begin{aligned}
(a_1 e_1 + a_2 e_2) &\wedge (b_1 e_1 + b_2 e_2) \\
&= a_1 b_2\, e_1 \wedge e_2 + a_2 b_1\, e_2 \wedge e_1 \\
&= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2.
\end{aligned}$$

Same as imaginary term in the complex product zw^*. In general, components are $a_{[i} b_{j]}$.

THE GEOMETRIC PRODUCT

Complex arithmetic suggests that we should combine the scalar and outer products into a single product. This is what Clifford did. He introduced the geometric product, written simply as ab, and satisfying

$$ab = a \cdot b + a \wedge b$$

Think of the right-hand side as like a complex number, with real and imaginary parts, carried round in a single entity.

From the symmetry/antisymmetry of the terms on the right-hand side, we see that

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b$$

It follows that

$$a \cdot b = \tfrac{1}{2}(ab + ba) \qquad a \wedge b = \tfrac{1}{2}(ab - ba)$$

Can define the other products in terms of the geometric product. So treat the geometric product as the primitive one and should define axioms for it. Properties of the other products then follow.

GEOMETRIC ALGEBRA IN 2-D

Consider a 2-d space (a plane) spanned by 2 orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2$,

$$\mathbf{e}_1{}^2 = \mathbf{e}_2{}^2 = 1, \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0.$$

NB writing vectors in a bold face now!

The final entity present in the 2-d algebra is the bivector $\mathbf{e}_1 \wedge \mathbf{e}_2$. The highest grade element in the algebra, often called the pseudoscalar (or directed volume element). Chosen to be right-handed, so that $\mathbf{e}_1$ sweeps onto $\mathbf{e}_2$ in a right-handed sense (when viewed from above). Use the symbol I for pseudoscalar

$$I = \mathbf{e}_1 \wedge \mathbf{e}_2$$

The full algebra is spanned by

$$1 \qquad \{\mathbf{e}_1, \mathbf{e}_2\} \qquad \mathbf{e}_1 \wedge \mathbf{e}_2$$

1 scalar   2 vectors   1 bivector.

Denote this algebra by $\mathcal{G}_2$. To study properties of $\mathbf{e}_1 \wedge \mathbf{e}_2$ first form

$$\mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2.$$

For orthogonal vectors the geometric product is a pure

bivector. Also note that

$$e_2 e_1 = e_2 \wedge e_1 = -e_1 \wedge e_2$$

so orthogonal vectors anticommute.

Now form products involving $e_1 e_2 = I$. Multiplying vectors from the left,

$$I e_1 = (-e_2 e_1) e_1 = -e_2 e_1 e_1 = -e_2$$
$$I e_2 = (e_1 e_2) e_2 = e_1 e_2 e_2 = e_1.$$

A 90° rotation clockwise (*i.e.* in a negative sense).

From the right

$$e_1(e_1 e_2) = e_2 \qquad e_2(e_1 e_2) = -e_1.$$

a 90° rotation anticlockwise — a positive sense.

Finally form the square of I,

$$I^2 = (e_1 \wedge e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1.$$

Have discovered a geometric quantity which squares to -1! Fits with the fact that 2 successive left (or right) multiplications of a vector by I rotates the vector through 180°, equivalent to multiplying by -1.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 2

SUMMARY

In this lecture we will introduce the geometric algebra of 3-d space, and start to explore some of its features. This will enable us to build up a picture of how geometric algebra can be employed to solve interesting physical problems in geometry and mechanics.

1. Rotations in 2-d and complex numbers.
2. The geometric algebra of 3-d space.
3. Lines, planes and volumes.
4. The vector cross product rediscovered.
5. Resolving the quaternion problem.

Complex Numbers and $\mathcal{G}_2$

Clear there is a close relationship between GA in 2-d, and complex numbers. Since $I^2 = -1$, combination of a scalar and a bivector, formed via the geometric product, can be viewed as a complex number. Write

$$z = x + y\boldsymbol{e}_1\boldsymbol{e}_2 = x + Iy.$$

Complex numbers serve a dual purpose:

1. They generate rotations and dilations through their polar decomposition $r\exp(i\theta)$

2. Represent vectors as points in an Argand diagram

But in $\mathcal{G}_2$ our vectors are grade-1 objects (where grade counts number of independent vectors),

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2.$$

what is the map between this and the complex number z? Answer — pre-multiply by $\boldsymbol{e}_1$,

$$\boldsymbol{e}_1\boldsymbol{x} = x + y\boldsymbol{e}_1\boldsymbol{e}_2 = x + Iy = z.$$

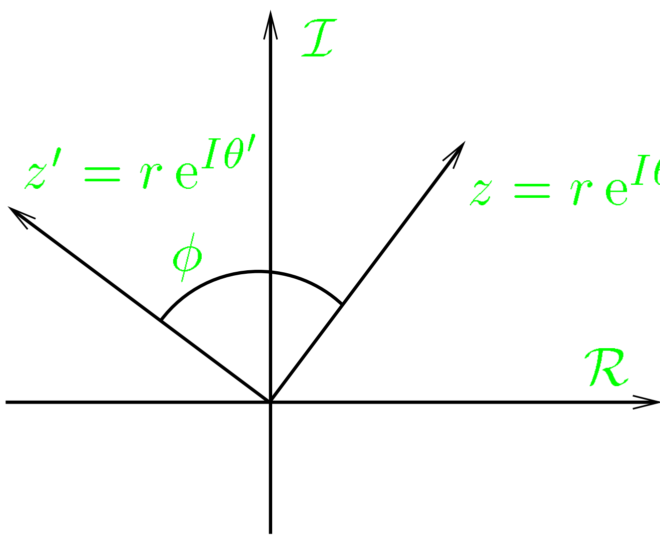
That is all there is to it! The 'preferred' vector $\boldsymbol{e}_1$ is the real axis. Vectors in $\mathcal{G}_2$ can be interchanged with complex numbers in 2-d in a natural manner.

Rotations

Know how to rotate complex numbers, so use this to find a formula for rotating vectors in 2-d. For a complex number z, a positive rotation through ϕ is achieved by

$$z \mapsto e^{I\phi} z,$$

(The exponential of a multivector is defined by power series in the normal way.)



Apply this to the vector transformation $\boldsymbol{x} \mapsto \boldsymbol{x}'$:

$$\boldsymbol{x} = \boldsymbol{e}_1 z \mapsto \boldsymbol{x}' = \boldsymbol{e}_1 z'$$

$$\boldsymbol{x}' = \boldsymbol{e}_1 e^{I\phi} z = e^{-I\phi} \boldsymbol{e}_1 z = e^{-I\phi} \boldsymbol{x}.$$

Therefore arrive at the formulae

$$\boldsymbol{x}' = e^{-I\phi} \boldsymbol{x} = \boldsymbol{x}\, e^{I\phi}.$$

Geometric Algebra in 3-d

Add a third vector $\boldsymbol{e}_3$, orthogonal to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Generate 3 independent bivectors

$$\boldsymbol{e}_1\boldsymbol{e}_2, \quad \boldsymbol{e}_2\boldsymbol{e}_3, \quad \boldsymbol{e}_3\boldsymbol{e}_1$$

The expected number of independent planes in 3-d space.

A new product to consider, all 3 orthogonal vectors

$$(\boldsymbol{e}_1\boldsymbol{e}_2)\boldsymbol{e}_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$$

Result is a trivector, the volume formed by sweeping $\boldsymbol{e}_1\boldsymbol{e}_2$ along $\boldsymbol{e}_3$. Has grade-3, as it is constructed from 3 independent vectors. (Dimension retained for size of a vector space.) Unique up to scale (*i.e.* volume) and handedness. Again, called the pseudoscalar, I,

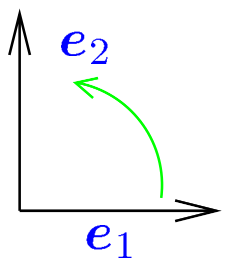
$$I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3.$$

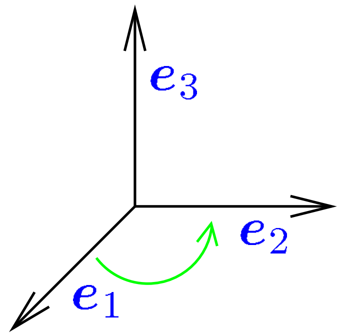
3-d algebra is spanned by

1	$\{\boldsymbol{e}_i\}$	$\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j\}$	$I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$
1 scalar	3 vectors	3 bivectors	1 trivector

A linear space of dimension $8 = 2^3$, call this $\mathcal{G}_3$. (Note the binomial coefficients.) Any 2 elements of this algebra can be added. General element is a multivector.

Pseudoscalar always chosen to be right-handed, *i.e.* $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a right-handed frame. This is a convention. In 2-d no intrinsic definition. In 3-d use right-hand rule.





PRODUCTS IN $\mathcal{G}_3$

Multiply vectors with geometric product

$$\boldsymbol{a}\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}.$$

Bivector $\boldsymbol{a}\wedge\boldsymbol{b}$ belongs to a 3-d space. In the $\{\boldsymbol{e}_i\wedge\boldsymbol{e}_j\}$ basis

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i, \quad \boldsymbol{b} = \sum_{i=1}^{3} b_i \boldsymbol{e}_i,$$

find bivector components

$$\begin{aligned}\boldsymbol{a}\wedge\boldsymbol{b} =& (a_2 b_3 - b_3 a_2)\boldsymbol{e}_2\wedge\boldsymbol{e}_3 + (a_3 b_1 - a_1 b_3)\boldsymbol{e}_3\wedge\boldsymbol{e}_1 \\ &+ (a_1 b_2 - a_2 b_1)\boldsymbol{e}_1\wedge\boldsymbol{e}_2\end{aligned}$$

Same as the cross product, but a bivector, not a vector.

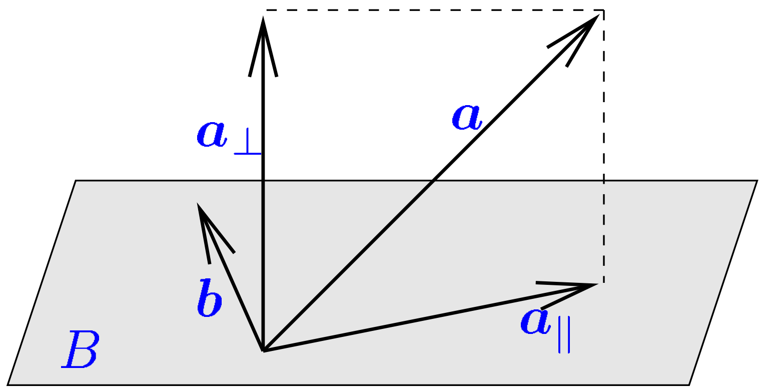
Vectors and Bivectors

3 bivectors all have negative square

$$(\boldsymbol{e}_1\boldsymbol{e}_2)^2 = (\boldsymbol{e}_2\boldsymbol{e}_3)^2 = (\boldsymbol{e}_3\boldsymbol{e}_1)^2 = -1$$

Each generates 90^{o} rotations in its own plane. But can form expressions like $\boldsymbol{a}B$. Can contain both vector and trivector terms. Decompose $\boldsymbol{a}$ into terms in and out of the plane,

$$\boldsymbol{a} = \boldsymbol{a}_{\|} + \boldsymbol{a}_{\perp}$$



now write $\boldsymbol{a}B = (\boldsymbol{a}_{\|} + \boldsymbol{a}_{\perp})B$. Also write

$$B = \boldsymbol{a}_{\|} \wedge \boldsymbol{b}$$

where $\boldsymbol{b}$ is orthogonal to $\boldsymbol{a}_{\|}$ in the B plane. See that

$$\boldsymbol{a}_{\|}B = \boldsymbol{a}_{\|}(\boldsymbol{a}_{\|} \wedge \boldsymbol{b}) = \boldsymbol{a}_{\|}(\boldsymbol{a}_{\|}\boldsymbol{b}) = (\boldsymbol{a}_{\|})^2\boldsymbol{b}$$

A vector in the $\boldsymbol{b}$ direction. But

$$\boldsymbol{a}_{\perp}B = \boldsymbol{a}_{\perp}(\boldsymbol{a}_{\|} \wedge \boldsymbol{b}) = \boldsymbol{a}_{\perp}\boldsymbol{a}_{\|}\boldsymbol{b}$$

Product of 3 orthogonal vectors, so a trivector.

Write the product

$$\boldsymbol{a}B = \boldsymbol{a}\cdot B + \boldsymbol{a}\wedge B$$

Dot and wedge generalised to mean the lowest and highest grade part of the result, respectively.

For the inner product $\boldsymbol{a}\cdot B$ see that

$$\boldsymbol{a}\cdot B = \boldsymbol{a}_{||}{}^2 \boldsymbol{b} = -(\boldsymbol{a}_{||}\boldsymbol{b})\boldsymbol{a}_{||} = -B\cdot\boldsymbol{a}$$

so dot product between a vector and a bivector is antisymmetric. This defines the inner product in this case

$$\boldsymbol{a}\cdot B = \tfrac{1}{2}(\boldsymbol{a}B - B\boldsymbol{a})$$

Result is always a vector. See this by forming

$$\begin{aligned}
\boldsymbol{a}(\boldsymbol{b}\wedge\boldsymbol{c}) &= \tfrac{1}{2}\boldsymbol{a}(\boldsymbol{b}\boldsymbol{c} - \boldsymbol{c}\boldsymbol{b}) \\
&= (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - \tfrac{1}{2}(\boldsymbol{b}\boldsymbol{a}\boldsymbol{c} - \boldsymbol{c}\boldsymbol{a}\boldsymbol{b}) \\
&= 2(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} - 2(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} + \tfrac{1}{2}(\boldsymbol{b}\boldsymbol{c} - \boldsymbol{c}\boldsymbol{b})\boldsymbol{a} \\
&= 2(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} - 2(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} + (\boldsymbol{b}\wedge\boldsymbol{c})\boldsymbol{a}
\end{aligned}$$

Made repeated use of the rearrangement

$$\boldsymbol{b}\boldsymbol{a} = 2\boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\boldsymbol{b}$$

Follows that

$$\boldsymbol{a}\cdot(\boldsymbol{b}\wedge\boldsymbol{c}) = \tfrac{1}{2}\big(\boldsymbol{a}(\boldsymbol{b}\wedge\boldsymbol{c}) - (\boldsymbol{b}\wedge\boldsymbol{c})\boldsymbol{a}\big) = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}$$

A pure vector. This is a very useful result.

Outer Product $\boldsymbol{a}\wedge B$

$\boldsymbol{a}\wedge B$ projects onto the component perpendicular to the plane, and returns a trivector. This is symmetric

$$\boldsymbol{a}\wedge B = \boldsymbol{a}_\perp \boldsymbol{a}_\| \boldsymbol{b} = \boldsymbol{a}_\| \boldsymbol{b} \boldsymbol{a}_\perp = B\wedge \boldsymbol{a}$$

Define the outer product of a vector and a bivector as

$$\boldsymbol{a}\wedge B = \tfrac{1}{2}(\boldsymbol{a}B + B\boldsymbol{a})$$

This is a pure trivector (see later). Can now define outer product of three vectors, $\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c})$ as grade-3 part of the geometric product. Denote this

$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}) = \langle \boldsymbol{a}(\boldsymbol{b}\wedge\boldsymbol{c})\rangle_3 = \langle \boldsymbol{a}(\boldsymbol{b}\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{c})\rangle_3$$

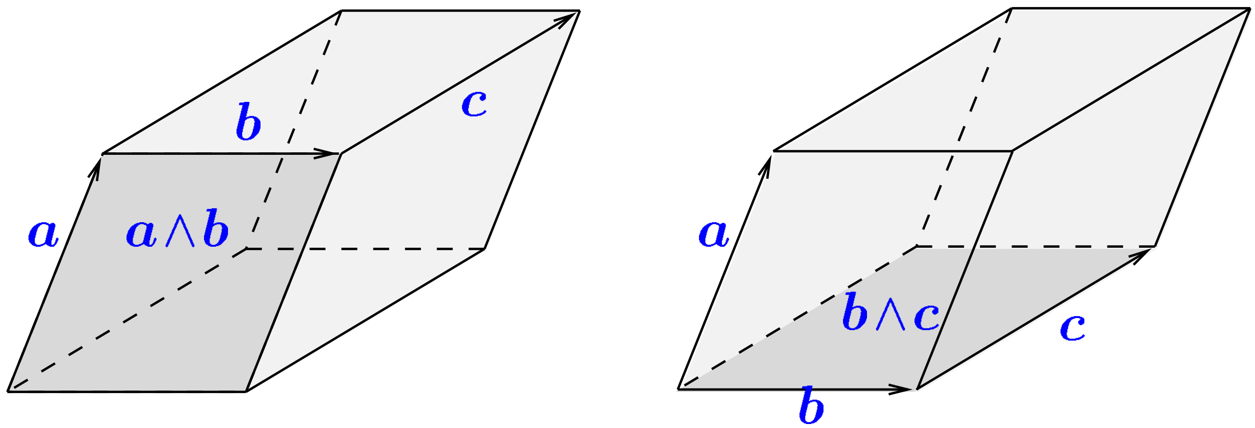
But $\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$ is a vector (grade-1) so does not contribute. Get

$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}) = \langle \boldsymbol{a}(\boldsymbol{b}\boldsymbol{c})\rangle_3 = \langle \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\rangle_3$$

Associativity of geometric product implies same for outer product

$$(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c} = \boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}) = \boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$$

True in general. The trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$ can be pictured as a parallelepiped formed by sweeping $\boldsymbol{a}\wedge\boldsymbol{b}$ along $\boldsymbol{c}$. Same obtained by sweeping $\boldsymbol{b}\wedge\boldsymbol{c}$ along $\boldsymbol{a}$.



Outer product antisymmetric on every pair of vectors,

$$\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c} = -\boldsymbol{b}\wedge\boldsymbol{a}\wedge\boldsymbol{c} = \boldsymbol{c}\wedge\boldsymbol{a}\wedge\boldsymbol{b}, \quad \textit{etc.}$$

Swapping any two vectors reverses the orientation (handedness).

The Bivector Algebra

Multiplying two different bivectors gives

$$(\boldsymbol{e}_1\wedge\boldsymbol{e}_2)(\boldsymbol{e}_2\wedge\boldsymbol{e}_3) = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_3 = \boldsymbol{e}_1\boldsymbol{e}_3$$

A third bivector. Also find that

$$(\boldsymbol{e}_2\wedge\boldsymbol{e}_3)(\boldsymbol{e}_1\wedge\boldsymbol{e}_2) = \boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{e}_3\boldsymbol{e}_1 = -\boldsymbol{e}_1\boldsymbol{e}_3$$

so antisymmetric. The symmetric contribution vanishes because the two planes are perpendicular. Define

$$B_1 = \boldsymbol{e}_2\boldsymbol{e}_3, \quad B_2 = \boldsymbol{e}_3\boldsymbol{e}_1, \quad B_3 = \boldsymbol{e}_1\boldsymbol{e}_2$$

Find that the commutator satisfies

$$B_i B_j - B_j B_i = -2\epsilon_{ijk} B_k$$

Closely linked to 3-d rotations, should be familiar from quantum theory of angular momentum.

Useful to define the commutator product

$$A \times B = \frac{1}{2}(AB - BA)$$

(NB do not confuse with cross product). Commutator product of two bivectors always results in a third bivector (or zero). (A Lie Algebra).

Basis bivectors square to -1, and anticommute — same as the quaternions. See that quaternions are actually bivectors, not vectors. Has important consequence for behaviour under reflections. Hamilton imposed condition $\boldsymbol{ijk} = -1$, but have

$$B_1 B_2 B_3 = \boldsymbol{e}_2 \boldsymbol{e}_3 \boldsymbol{e}_3 \boldsymbol{e}_1 \boldsymbol{e}_1 \boldsymbol{e}_2 = +1$$

Direct map needs a sign flip somewhere, *e.g.*

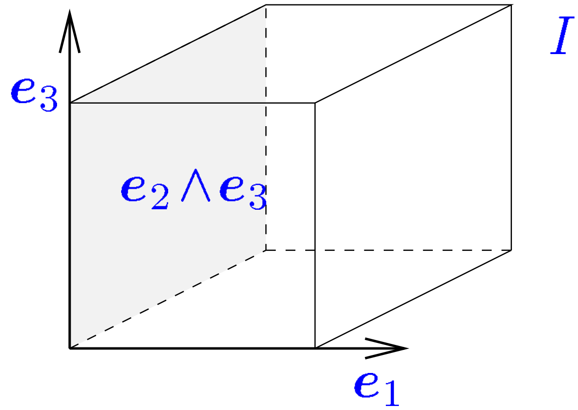
$$\boldsymbol{i} \leftrightarrow B_1, \quad \boldsymbol{j} \leftrightarrow -B_2, \quad \boldsymbol{k} \leftrightarrow B_3$$

Quaternions are left-handed bivectors, though originally interpreted as a right-handed set of vectors!

Products Involving I

Form the product of I with the vector e_1,

$$
\begin{aligned}
e_1 I &= e_1(e_1 e_2 e_3) \\
&= e_2 e_3
\end{aligned}
$$



Returns a bivector — the plane perpendicular to the original vector. Product of a vector (grade-1) with the pseudoscalar (grade-3) is a bivector (grade-2). Call this a duality transformation. Can express the basis bivectors as the product of I and a dual vector,

$$e_1 e_2 = I e_3, \qquad e_2 e_3 = I e_1, \qquad e_3 e_1 = I e_2$$

Multiplying from the left, find that

$$I e_1 = e_1 e_2 e_3 e_1 = -e_1 e_2 e_1 e_3 = e_2 e_3$$

Result is independent of order — the pseudoscalar commutes with all vectors in 3-d, $Ia = aI$, so I commutes with all elements in the algebra. Always true in odd dimensions. In even dimensions I anti-commutes with vectors.

Next form the square of the pseudoscalar

$$I^2 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2 = -1$$

I is another candidate for a unit imaginary. Sometimes the correct one. Other times a bivector.

Finally, form product of a bivector and the pseudoscalar:

$$I(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = I\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_3 = II\boldsymbol{e}_3 = -\boldsymbol{e}_3$$

returns minus the vector perpendicular to the plane. Recover the cross product in 3-d

$$\boldsymbol{a} \boldsymbol{\times} \boldsymbol{b} = -I(\boldsymbol{a} \wedge \boldsymbol{b})$$

(Bold $\boldsymbol{\times}$ symbol distinct from $\times$.) Cross product requires a duality operation. Only valid in 3-d. Little use for the cross product now that we have the outer product.

Can use duality to understand geometric product $\boldsymbol{a}B$. Write $B = I\boldsymbol{b}$ so that

$$\boldsymbol{a}B = I\boldsymbol{a}\boldsymbol{b} = I(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b})$$

Symmetric part is a trivector

$$\boldsymbol{a}\wedge B = I(\boldsymbol{a}\cdot\boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{a}B + B\boldsymbol{a})$$

Antisymmetric part is a vector

$$\boldsymbol{a}\cdot B = I(\boldsymbol{a}\wedge\boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{a}B - B\boldsymbol{a})$$

FURTHER DEFINITIONS

The *revere* of a multivector A, $\tilde{A}$, formed by reversing order of all products. Scalars and vectors are invariant, but bivectors and trivectors change sign,

$$(\boldsymbol{e}_1\boldsymbol{e}_2)^{\sim} = \boldsymbol{e}_2\boldsymbol{e}_1 = -\boldsymbol{e}_1\boldsymbol{e}_2$$
$$\tilde{I} = \boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_2 = -\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 = -I$$

For a general multivector in 3-d have

$$M = \alpha + \boldsymbol{a} + B + \beta I$$
$$\tilde{M} = \alpha + \boldsymbol{a} - B - \beta I$$

Operator ordering convention:

Inner and outer products are performed before geometric products.

Helps to reduce numbers of brackets, $\boldsymbol{a}\cdot\boldsymbol{b}\,\boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$

$\langle\,\rangle_r$ denotes projection onto grade-r part. For scalar component drop the subscript 0

$$\langle AB\rangle = \langle AB\rangle_0$$

This satisfies

$$\langle AB\rangle = \langle BA\rangle, \quad \langle A\cdots BC\rangle = \langle CA\cdots B\rangle.$$

Cyclic reordering property very useful.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 3

SUMMARY

In this lecture we will reinforce our understanding of geometric algebra in 3 dimensions with some basic applications in mechanics. We will introduce the important concept of a rotor as a means for describing rotations, see how these are related to bivectors, and the types of equations they satisfy.

1. Angular momentum as a bivector.
2. Force and the angular momentum bivector.
3. Central force interactions.
4. Gravitational perturbations.
5. Reflections and Rotations in 3-d.
6. Rotors and bivectors.

CLASSICAL MECHANICS

Trajectory $\boldsymbol{x}(t)$, velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$ and momentum $\boldsymbol{p} = m\boldsymbol{v}$.

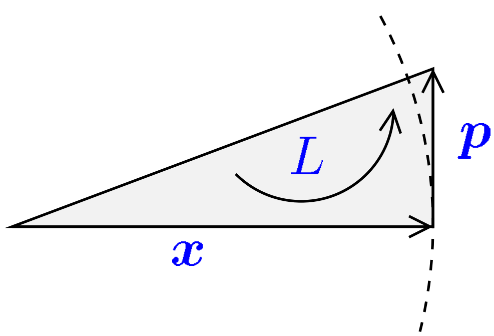
Force $\boldsymbol{f} = m\boldsymbol{a}$, (assuming m is constant).

Angular Momentum

Usually defined in terms of the cross product,

$$\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{p}$$

But concept of angular momentum concerns rate at which area is swept out by a particle moving relative to an origin (*c.f.* Kepler's second law)



Areas are encoded by bivectors, and cross product is a bivector in disguise. Replace traditional definition with the angular momentum bivector L,

$$L = \boldsymbol{x} \wedge \boldsymbol{p}.$$

Computing L is essentially same as $\boldsymbol{l}$. But can now use the geometric product to speed up derivations.

If we differentiate L we obtain

$$\dot{L} = \boldsymbol{v}\wedge(m\boldsymbol{v}) + \boldsymbol{x}\wedge(\dot{\boldsymbol{p}}) = \boldsymbol{x}\wedge\boldsymbol{f}$$

Define the torque N about the origin as the bivector

$$N = \boldsymbol{x}\wedge\boldsymbol{f}$$

Torque and angular momentum are related by $\dot{L} = N$. Torque acts over the plane defined by $\boldsymbol{f}$ and the chosen origin. Bivectors are additive, (a vector space), so add bivectors to get resultant torque.

Now define $r = |\boldsymbol{x}|$ and write

$$\boldsymbol{x} = r\hat{\boldsymbol{x}}$$

where $\hat{\boldsymbol{x}}^2 = 1$. We therefore have

$$\dot{\boldsymbol{x}} = \frac{d}{dt}(r\hat{\boldsymbol{x}}) = \dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}$$

so that

$$L = m\boldsymbol{x}\wedge(\dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}) = mr\hat{\boldsymbol{x}}\wedge(\dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}) = mr^2\,\hat{\boldsymbol{x}}\wedge\dot{\hat{\boldsymbol{x}}}$$

But since $\hat{\boldsymbol{x}}^2 = 1$ must have

$$0 = \frac{d}{dt}(\hat{\boldsymbol{x}}^2) = 2\hat{\boldsymbol{x}}\cdot\dot{\hat{\boldsymbol{x}}}$$

Can replace outer product with a geometric product:

$$L = mr^2\hat{\boldsymbol{x}}\dot{\hat{\boldsymbol{x}}} = -mr^2\dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}}$$

TWO-BODY CENTRAL FORCE INTERACTIONS

Two particles with positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, masses m_1 and m_2, subject to a central force $\boldsymbol{f}$

$$m_1\ddot{\boldsymbol{x}}_1 = \boldsymbol{f}, \qquad m_2\ddot{\boldsymbol{x}}_2 = -\boldsymbol{f}$$

Define centre of mass vector $\boldsymbol{X}$ and separation vector $\boldsymbol{x}$,

$$\boldsymbol{X} = \frac{1}{M}(m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2), \qquad \boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$$

M is the total mass $m_1 + m_2$. The force equations become

$$\mu\ddot{\boldsymbol{x}} = \boldsymbol{f}, \quad \text{and} \quad \ddot{\boldsymbol{X}} = 0$$

where μ is the reduced mass m_1m_2/M.

The angular momentum about the centre of mass $\boldsymbol{X}$ is given by

$$\begin{aligned} & m_1(\boldsymbol{X} - \boldsymbol{x}_1)\wedge(\dot{\boldsymbol{X}} - \dot{\boldsymbol{x}}_1) + m_2(\boldsymbol{X} - \boldsymbol{x}_2)\wedge(\dot{\boldsymbol{X}} - \dot{\boldsymbol{x}}_2) \\ & \quad = -\mu\boldsymbol{x}\wedge(\dot{\boldsymbol{X}} - \dot{\boldsymbol{x}}_1) + \mu\boldsymbol{x}\wedge(\dot{\boldsymbol{X}} - \dot{\boldsymbol{x}}_2) \\ & \quad = \mu\boldsymbol{x}\wedge\dot{\boldsymbol{x}} = L \end{aligned}$$

$\boldsymbol{f}$ directed along $\boldsymbol{x}$, $\boldsymbol{f} = f\hat{\boldsymbol{x}}$, so L conserved. Motion confined to the L plane. Trajectory of $\boldsymbol{x}$ sweeps out area at a constant rate (Kepler again).

The magnitude of L is

$$l^2 = |L|^2 = \langle L\tilde{L}\rangle = \mu^2 r^4 \langle \hat{\boldsymbol{x}}\dot{\hat{\boldsymbol{x}}}\dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}}\rangle = \mu^2 r^4 \dot{\hat{\boldsymbol{x}}}^2$$

so $l = \mu r^2 |\dot{\hat{\boldsymbol{x}}}|$. (Reverse one factor of L to get a positive result.) Force is conservative, write as (minus) the gradient of a potential $V(r)$:

$$\boldsymbol{f} = f\hat{\boldsymbol{x}}, \quad f = -\frac{dV}{dr}.$$

The total conserved energy is given by

$$E = \frac{\mu \dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + V(r)$$

Inverse Square Forces

Set $f = -k/r^2$ (k positive for attractive force). Basic equation is

$$\mu \ddot{\boldsymbol{x}} = -\frac{k}{r^2}\hat{\boldsymbol{x}} = -\frac{k}{r^3}\boldsymbol{x}.$$

A 2nd order vector differential equation — two constant vectors in the solution. One given by L. Find other from

$$L\dot{\boldsymbol{v}} = -\frac{k}{\mu r^2} L\hat{\boldsymbol{x}} = -k\hat{\boldsymbol{x}}\dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}} = k\dot{\hat{\boldsymbol{x}}}$$

Follows that

$$\frac{d}{dt}(L\boldsymbol{v} - k\hat{\boldsymbol{x}}) = 0$$

Motion therefore described by the simple equation

$$L\boldsymbol{v} = k(\hat{\boldsymbol{x}} + \boldsymbol{e})$$

where $\boldsymbol{e}$ is the eccentricity vector. A second constant of motion. Lies in the L plane — only 2 new constants.

To find a direct equation for the trajectory write

$$L\boldsymbol{v}\boldsymbol{x} = L(\boldsymbol{v}\cdot\boldsymbol{x} + \boldsymbol{v}\wedge\boldsymbol{x}) = \frac{1}{\mu}L\tilde{L} + \boldsymbol{v}\cdot\boldsymbol{x}\, L = k(r + \boldsymbol{e}\boldsymbol{x})$$

Scalar part gives

$$r = \frac{l^2}{k\mu(1 + \boldsymbol{e}\cdot\hat{\boldsymbol{x}})}$$

Specifies a conic surface in 3-d, with symmetry axis $\boldsymbol{e}$. Formed by rotating a 2-d conic about $\boldsymbol{e}$. Motion takes place in the L plane, so described by a conic.

Gravitational Perturbations

The eccentricity vector $\boldsymbol{e}$ is very important for celestial perturbation theory. As an example, consider general relativistic correction to the inverse square law:

$$\ddot{\boldsymbol{x}} = -\frac{GM}{r^2}\left(1 + \frac{3l^2}{\mu^2 c^2 r^2}\right)\hat{\boldsymbol{x}}$$

(*N.B* derivatives with respect to proper time, not relevant here.) Force is still central, so L is conserved.

Eccentricity vector now satisfies (exercise)

$$\dot{\boldsymbol{e}} = \frac{3l^2}{\mu^2 c^2 r^2} \dot{\hat{\boldsymbol{x}}}.$$

For bound orbits gives rise to a precession of the major axis.

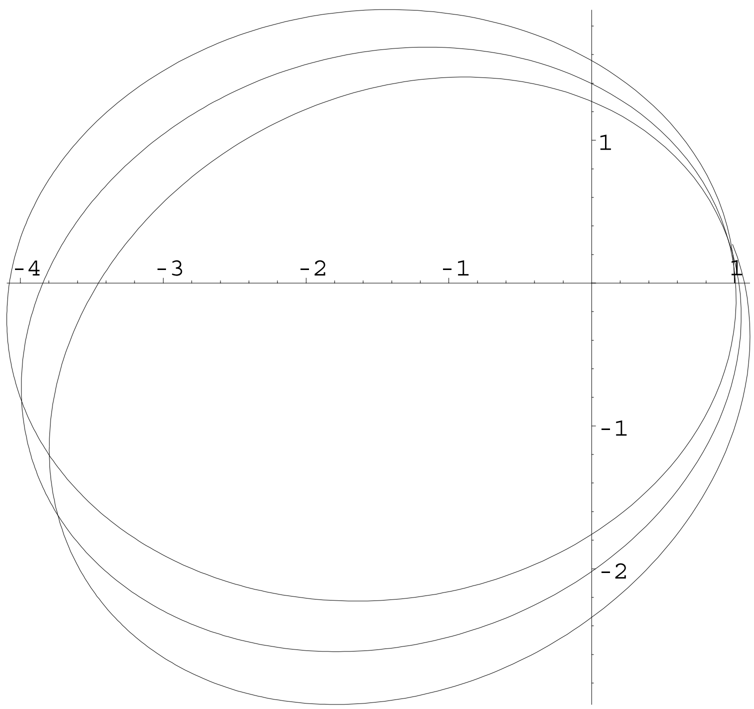
Quantity of most interest is the change in $\boldsymbol{e}$ over orbit. Get an approximate result by assuming orbit is elliptical and forming

$$\Delta \boldsymbol{e} = \int_0^T dt \frac{3l^2}{\mu^2 c^2 r^2} \dot{\hat{\boldsymbol{x}}} = -\frac{3l^2}{\mu^3 c^2} L \int_0^T dt \frac{\hat{\boldsymbol{x}}}{r^4}$$

where T is the orbital period. Orbital averages are tabulated in various books. Find that

$$\Delta \boldsymbol{e} = \frac{6\pi GM}{a(1-e^2)c^2} \boldsymbol{e} \cdot \hat{L},$$

where $\hat{L} = L/l$, a is the semi-major axis and $e = |\boldsymbol{e}|$ is the eccentricity. Get precession of $\boldsymbol{e}$ with same orientation as L — an advance. For Mercury, get perihelion advance of 43 arc seconds per century.



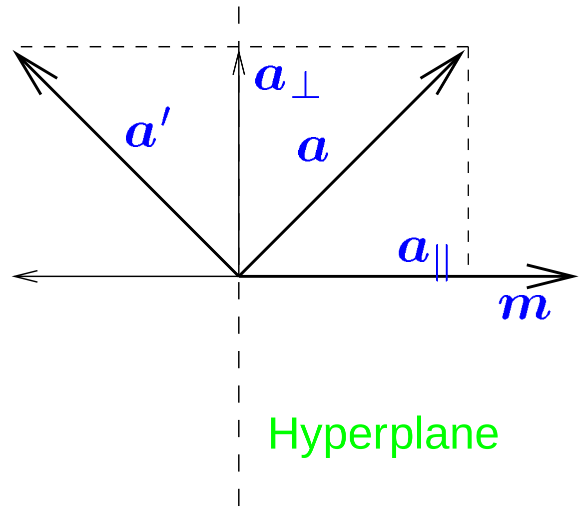
-4
-3
-2
-1
1
1
-1
-2

REFLECTIONS

Reflect the vector $\boldsymbol{a}$ in the (hyper)plane orthogonal to unit vector $\boldsymbol{m}$ ($\boldsymbol{m}^2 = 1$).

$$\boldsymbol{a} = \boldsymbol{a}_\perp + \boldsymbol{a}_{||}$$

$$\boldsymbol{a}' = \boldsymbol{a}_\perp - \boldsymbol{a}_{||}$$



Parallel component changes sign, perpendicular component unchanged. Parallel component is projection onto $\boldsymbol{m}$:

$$\boldsymbol{a}_{||} = \boldsymbol{a}\cdot\boldsymbol{m}\,\boldsymbol{m}$$

(NB operator ordering convention). Perpendicular component is remainder

$$\boldsymbol{a}_\perp = \boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{m}\,\boldsymbol{m} = (\boldsymbol{a}\boldsymbol{m} - \boldsymbol{a}\cdot\boldsymbol{m})\boldsymbol{m} = \boldsymbol{a}\wedge\boldsymbol{m}\,\boldsymbol{m}.$$

Outer product projects onto component perpendicular to a vector.

Result of the reflection is

$$\begin{aligned}
\boldsymbol{a}' = \boldsymbol{a}_\perp - \boldsymbol{a}_{||} &= -\boldsymbol{a}\cdot\boldsymbol{m}\,\boldsymbol{m} + \boldsymbol{a}\wedge\boldsymbol{m}\,\boldsymbol{m} \\
&= -(\boldsymbol{m}\cdot\boldsymbol{a} + \boldsymbol{m}\wedge\boldsymbol{a})\boldsymbol{m} = -\boldsymbol{m}\boldsymbol{a}\boldsymbol{m}
\end{aligned}$$

Unique to geometric algebra. See that geometric products arise naturally when operating on vectors.

Check that inner products are unchanged:

$$\begin{aligned}
\boldsymbol{a}'\cdot\boldsymbol{b}' = (-\boldsymbol{m}\boldsymbol{a}\boldsymbol{m})\cdot(-\boldsymbol{m}\boldsymbol{b}\boldsymbol{m}) &= \langle \boldsymbol{m}\boldsymbol{a}\boldsymbol{m}\boldsymbol{m}\boldsymbol{b}\boldsymbol{m}\rangle \\
&= \langle \boldsymbol{m}\boldsymbol{a}\boldsymbol{b}\boldsymbol{m}\rangle = \langle \boldsymbol{m}\boldsymbol{m}\boldsymbol{a}\boldsymbol{b}\rangle = \boldsymbol{a}\cdot\boldsymbol{b}.
\end{aligned}$$

Bivector Transformation Law

Reflect both vectors in the bivector $\boldsymbol{a}\wedge\boldsymbol{b}$. Obtain

$$\begin{aligned}
\boldsymbol{a}'\wedge\boldsymbol{b}' &= (-\boldsymbol{m}\boldsymbol{a}\boldsymbol{m})\wedge(-\boldsymbol{m}\boldsymbol{b}\boldsymbol{m}) \\
&= \tfrac{1}{2}(\boldsymbol{m}\boldsymbol{a}\boldsymbol{m}\boldsymbol{m}\boldsymbol{b}\boldsymbol{m} - \boldsymbol{m}\boldsymbol{b}\boldsymbol{m}\boldsymbol{m}\boldsymbol{a}\boldsymbol{m}) \\
&= \tfrac{1}{2}\boldsymbol{m}(\boldsymbol{a}\boldsymbol{b} - \boldsymbol{b}\boldsymbol{a})\boldsymbol{m} = \boldsymbol{m}\,\boldsymbol{a}\wedge\boldsymbol{b}\,\boldsymbol{m}.
\end{aligned}$$

Get a crucial sign difference. Bivectors do not transform as vectors under reflections. Reason for distinction between polar and axial vectors in 3-d. Axial vectors are really bivectors, and should be treated as such. Also explains why quaternions do not transform as vectors.

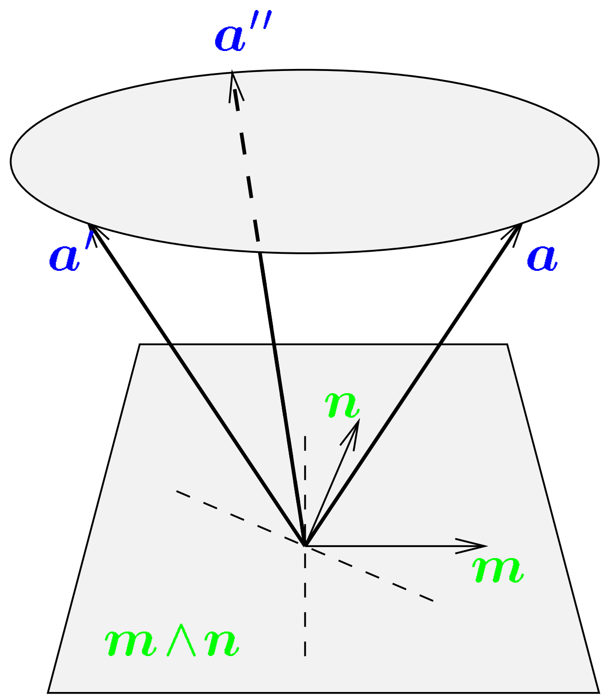
Rotations

In 2-d found rotation formula

$$\boldsymbol{a} \mapsto \boldsymbol{a}' = \exp(-\boldsymbol{e}_1\boldsymbol{e}_2\theta)\boldsymbol{a} = \boldsymbol{a}\exp(\boldsymbol{e}_1\boldsymbol{e}_2\theta)$$

Want a version for 3-d. Key result:

A rotation in the plane $\boldsymbol{m}\wedge\boldsymbol{n}$ plane is generated by successive reflections in the planes perpendicular to $\boldsymbol{m}$ and $\boldsymbol{n}$.



- $\boldsymbol{a}'$ is result of reflecting $\boldsymbol{a}$ with $\boldsymbol{m}$.
- $\boldsymbol{a}''$ is the result of reflecting $\boldsymbol{a}'$ with $\boldsymbol{n}$.
- Any component outside the $\boldsymbol{m}\wedge\boldsymbol{n}$ plane is untouched.
- Angle between $\boldsymbol{a}$ and $\boldsymbol{a}''$ is twice angle θ between $\boldsymbol{m}$ and $\boldsymbol{n}$. (Exercise.) Rotate through 2θ in $\boldsymbol{m}\wedge\boldsymbol{n}$ plane.

So how does this look in GA?

$$\boldsymbol{a}' = -\boldsymbol{mam}$$

$$\boldsymbol{a}'' = -\boldsymbol{na}'\boldsymbol{n} = -\boldsymbol{n}(-\boldsymbol{mam})\boldsymbol{n} = \boldsymbol{nmamn}$$

Very simple! Define the rotor R by

$$R = \boldsymbol{nm}$$

Note the geometric product here! Write a rotation as

$$\boxed{\boldsymbol{a} \mapsto R\boldsymbol{a}\tilde{R}}$$

Works for any grade of multivector, in any dimension, of any signature!

Now expand R as

$$R = \boldsymbol{nm} = \boldsymbol{n}\cdot\boldsymbol{m} + \boldsymbol{n}\wedge\boldsymbol{m} = \cos(\theta) + \boldsymbol{n}\wedge\boldsymbol{m}.$$

So what is the magnitude of the bivector $\boldsymbol{n}\wedge\boldsymbol{m}$?

$$(\boldsymbol{n}\wedge\boldsymbol{m})\cdot(\boldsymbol{n}\wedge\boldsymbol{m}) = \langle \boldsymbol{n}\wedge\boldsymbol{m}\,\boldsymbol{n}\wedge\boldsymbol{m}\rangle = \langle \boldsymbol{nm}\,\boldsymbol{n}\wedge\boldsymbol{m}\rangle$$
$$= \boldsymbol{n}\cdot[\boldsymbol{m}\cdot(\boldsymbol{n}\wedge\boldsymbol{m})] = \boldsymbol{n}\cdot(\boldsymbol{m}\cos(\theta) - \boldsymbol{n})$$
$$= \cos^2(\theta) - 1 = -\sin^2(\theta)$$

Define a unit bivector in the $\boldsymbol{m}\wedge\boldsymbol{n}$ plane by

$$\hat{B} = \boldsymbol{m}\wedge\boldsymbol{n}/\sin(\theta), \qquad \hat{B}^2 = -1$$

Choice of orientation ($\boldsymbol{m}\wedge\boldsymbol{n}$ rather than $\boldsymbol{n}\wedge\boldsymbol{m}$) ensures the bivector has same orientation as rotation.

Now have

$$R = \cos(\theta) - \hat{B}\sin(\theta)$$

The polar decomposition of a complex number! With unit imaginary replaced by $\hat{B}$. Write

$$R = \exp(-\hat{B}\theta)$$

(Exponential defined as a power series in normal way.) This is for a rotation through 2θ. To rotate through θ, rotor is

$$R = \exp\{-\hat{B}\theta/2\}$$

Gives us the final formula

$$\boxed{\boldsymbol{a} \mapsto \mathrm{e}^{-\hat{B}\theta/2}\,\boldsymbol{a}\,\mathrm{e}^{\hat{B}\theta/2}}$$

for a rotation through θ in the $\hat{B}$ plane, with orientation specified by $\hat{B}$.

- Think of rotations in a plane as opposed to about an axis. This is the general concept.
- Rotor works directly with the plane of interest. Hidden in matrix form.
- Computationally more efficient to use rotors.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 4

SUMMARY

In this lecture we study the geometric algebra treatment of rotations. The key objects in this are rotors and we will uncover a number of their properties.

1. Rotors and bivectors.
2. Rotating multivectors.
3. Rotating frames and the angular velocity bivector.
4. Euler angles.
5. The rotor equation.
6. Rigid body dynamics.

PROPERTIES

Lengths and Angles

Rotor R satisfies

$$R\tilde{R} = \boldsymbol{nm}(\boldsymbol{nm})^{\sim} = \boldsymbol{nmmn} = 1 = \tilde{R}R$$

Form inner product

$$\begin{aligned} \boldsymbol{a}'\cdot\boldsymbol{b}' &= (R\boldsymbol{a}\tilde{R})\cdot(R\boldsymbol{b}\tilde{R}) = \langle R\boldsymbol{a}\tilde{R}R\boldsymbol{b}\tilde{R}\rangle \\ &= \langle R\boldsymbol{a}\boldsymbol{b}\tilde{R}\rangle = \boldsymbol{a}\cdot\boldsymbol{b} \end{aligned}$$

So lengths and angles preserved — as required.

Bivectors

Rotate both vectors in bivector $B = \boldsymbol{a}\wedge\boldsymbol{b}$

$$\begin{aligned} B' = \boldsymbol{a}'\wedge\boldsymbol{b}' &= \tfrac{1}{2}(\boldsymbol{a}'\boldsymbol{b}' - \boldsymbol{b}'\boldsymbol{a}') \\ &= \tfrac{1}{2}(R\boldsymbol{a}\tilde{R}R\boldsymbol{b}\tilde{R} - R\boldsymbol{b}\tilde{R}R\boldsymbol{a}\tilde{R}) \\ &= \tfrac{1}{2}(R\boldsymbol{a}\boldsymbol{b}\tilde{R} - R\boldsymbol{b}\boldsymbol{a}\tilde{R}) = \tfrac{1}{2}R(\boldsymbol{a}\boldsymbol{b} - \boldsymbol{b}\boldsymbol{a})\tilde{R} \\ &= R\boldsymbol{a}\wedge\boldsymbol{b}\tilde{R} = RB\tilde{R}. \end{aligned}$$

The same formula as vectors! Turns out to be true for all geometric objects represented by multivectors. A very useful feature. With matrices, would need r copies for grade-r.

THE QUATERNION PROBLEM

Return to 2-d see that we can write

$$\begin{aligned}\boldsymbol{a} \mapsto \boldsymbol{a}' &= \exp(-\boldsymbol{e}_1\boldsymbol{e}_2\theta)\boldsymbol{a} = \boldsymbol{a}\exp(\boldsymbol{e}_1\boldsymbol{e}_2\theta) \\ &= \exp(-\boldsymbol{e}_1\boldsymbol{e}_2\theta/2)\boldsymbol{a}\exp(\boldsymbol{e}_1\boldsymbol{e}_2\theta/2)\end{aligned}$$

Recovers double-sided, half-angle formula. Applied on one side only, unit imaginary generates 90° rotations. Applied on both sides get 180° rotations. (Just a sign change).

In > 2 dimensions, only 2-sided formula works. Single-sided fails altogether, *e.g.*

$$\begin{aligned}\mathrm{e}^{-\boldsymbol{e}_1\boldsymbol{e}_2\theta}\boldsymbol{e}_3 &= [\cos(\theta) - \sin(\theta)\boldsymbol{e}_1\boldsymbol{e}_2]\boldsymbol{e}_3 \\ &= \cos(\theta)\boldsymbol{e}_3 - \sin(\theta)I,\end{aligned}$$

In 3-d no choice but to interpret the basis unit bivectors as generators of 180° rotations.

Hamilton and followers got this wrong. Tried to continue to work with a single-sided transformation law. Held back the subject for many years.

Euler Angles

Find the Euler angles in any textbook on mechanics. Usually encoded as a strange combination of rotations about varying axes. Want to find the GA version for this.

Take $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ as our fixed initial axes. Want final axes $\boldsymbol{e}_i' = R\boldsymbol{e}_i\tilde{R}$. Start with rotation through ϕ in $\boldsymbol{e}_1\boldsymbol{e}_2$ plane.

$$R_\phi = \mathrm{e}^{-I\boldsymbol{e}_3\phi/2} \qquad \boldsymbol{\xi} = R_\phi \boldsymbol{e}_1 \tilde{R}_\phi$$

Next we rotate through $+\theta$ about the $\boldsymbol{\xi}$ axis. The rotor for this is

$$\begin{aligned} R_\theta = \mathrm{e}^{-I\boldsymbol{\xi}\theta/2} &= \cos(\theta/2) - I\boldsymbol{\xi}\sin(\theta/2) \\ &= R_\phi\big(\cos(\theta/2) - I\boldsymbol{e}_1\sin(\theta/2)\big)\tilde{R}_\phi \\ &= R_\phi\,\mathrm{e}^{-I\boldsymbol{e}_1\theta/2}\,\tilde{R}_\phi \end{aligned}$$

Finally we rotate around the $\boldsymbol{\zeta}'$ axis, where

$$\boldsymbol{\zeta}' = R_\theta R_\phi \boldsymbol{e}_3 \tilde{R}_\phi \tilde{R}_\theta$$

This time the rotor is

$$R_\psi = \mathrm{e}^{-I\boldsymbol{\zeta}'\psi/2} = R_\theta R_\phi\,\mathrm{e}^{-I\boldsymbol{e}_3\psi/2}\,\tilde{R}_\phi\tilde{R}_\theta$$

Euler Angles II

The product of the rotors is therefore

$$\begin{aligned}
R_\psi R_\theta R_\phi &= R_\theta R_\phi \, \mathrm{e}^{-I\boldsymbol{e}_3\psi/2} \, \tilde{R}_\phi \tilde{R}_\theta R_\theta R_\phi \\
&= R_\phi \, \mathrm{e}^{-I\boldsymbol{e}_1\theta/2} \, \tilde{R}_\phi R_\phi \, \mathrm{e}^{-I\boldsymbol{e}_3\psi/2} \\
&= \mathrm{e}^{-I\boldsymbol{e}_3\phi/2} \, \mathrm{e}^{-I\boldsymbol{e}_1\theta/2} \, \mathrm{e}^{-I\boldsymbol{e}_3\psi/2}
\end{aligned}$$

The overall rotation to the new frame is therefore described by

$$\boldsymbol{e}_i' = R\boldsymbol{e}_i\tilde{R}$$

where

$$R = \mathrm{e}^{-I\boldsymbol{e}_3\phi/2} \, \mathrm{e}^{-I\boldsymbol{e}_1\theta/2} \, \mathrm{e}^{-I\boldsymbol{e}_3\psi/2}$$

- Puts the rotations in their logical order!
- Do ψ first.
- Much simpler than the matrix equivalent.
- But not the best way to handle rotations in GA.

Rotating Frames and Angular Velocity

Frame of vectors $\{\boldsymbol{f}_k\}$ rotating in space. Relate to a fixed orthonormal frame $\{\boldsymbol{e}_k\}$ by the time-dependent rotor $R(t)$:

$$\boldsymbol{f}_k(t) = R(t)\boldsymbol{e}_k\tilde{R}(t)$$

Angular velocity vector $\boldsymbol{\omega}$ is traditionally defined by

$$\dot{\boldsymbol{f}}_k = \boldsymbol{\omega} \times \boldsymbol{f}_k$$

Now know that this is

$$\boldsymbol{\omega} \times \boldsymbol{f}_k = (-I\boldsymbol{\omega}) \cdot \boldsymbol{f}_k = \boldsymbol{f}_k \cdot (I\boldsymbol{\omega})$$

Define the angular velocity bivector Ω by

$$\Omega = I\boldsymbol{\omega}$$

Choice ensures that rotation has orientation of Ω. Now differentiate to get

$$\dot{\boldsymbol{f}}_k = \dot{R}\boldsymbol{e}_k\tilde{R} + R\boldsymbol{e}_k\dot{\tilde{R}} = \dot{R}\tilde{R}\boldsymbol{f}_k + \boldsymbol{f}_k R\dot{\tilde{R}}$$

All rotors are normalised, $R\tilde{R} = 1$, so

$$0 = \frac{d}{dt}(R\tilde{R}) = \dot{R}\tilde{R} + R\dot{\tilde{R}}$$

Differentiation and reversion are interchangeable (commuting) operations.

Follows that,

$$\dot{R}\tilde{R} = -R\dot{\tilde{R}} = -(\dot{R}\tilde{R})^{\sim}.$$

$\dot{R}\tilde{R}$ is equal to minus its own reverse and has even grade, so must be a pure bivector. Equation for $\boldsymbol{f}_k$ becomes

$$\dot{\boldsymbol{f}}_k = \dot{R}\tilde{R}\boldsymbol{f}_k - \boldsymbol{f}_k\dot{R}\tilde{R} = (2\dot{R}\tilde{R})\cdot\boldsymbol{f}_k$$

See that $2\dot{R}\tilde{R}$ must equal minus the angular velocity bivector Ω, so

$$2\dot{R}\tilde{R} = -\Omega$$

The dynamics is therefore contained in the single rotor equation

$$\dot{R} = -\tfrac{1}{2}\Omega R \quad \text{or} \quad \dot{\tilde{R}} = \tfrac{1}{2}\tilde{R}\Omega$$

Surprisingly common type of equation. Rotors are elements of a Lie group, and the bivectors form their Lie algebra.

Can use either space Ω or body Ω_B angular velocities. Body angular velocity is Ω rotated back to the fixed reference frame,

$$\Omega = R\Omega_B\tilde{R}, \qquad \Omega_B = \tilde{R}\Omega R$$

In terms of these we have

$$\dot{R} = -\tfrac{1}{2}\Omega R = -\tfrac{1}{2}R\Omega_B, \qquad \dot{\tilde{R}} = \tfrac{1}{2}\Omega_B\tilde{R}$$

CONSTANT Ω

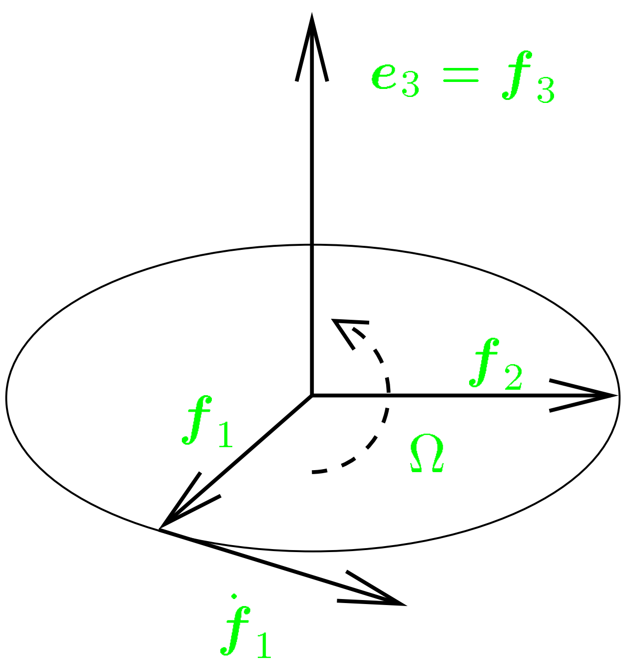
For constant Ω, rotor equation integrates immediately

$$R = e^{-\Omega t/2} R_0$$

Constant frequency rotation in the Ω plane, with handedness defined by Ω. The frame rotates according to

$$\boldsymbol{f}_k(t) = e^{-\Omega t/2} R_0 \boldsymbol{e}_k \tilde{R}_0 e^{\Omega t/2}$$

R_0 describes the orientation of the frame at $t = 0$, relative to the $\{\boldsymbol{e}_k\}$ frame.



Example: motion about the $\boldsymbol{e}_3$ axis.

$$\Omega = \omega I \boldsymbol{e}_3 = \omega \boldsymbol{e}_1 \boldsymbol{e}_2$$

and set $R_0 = 1$.

Motion is described by

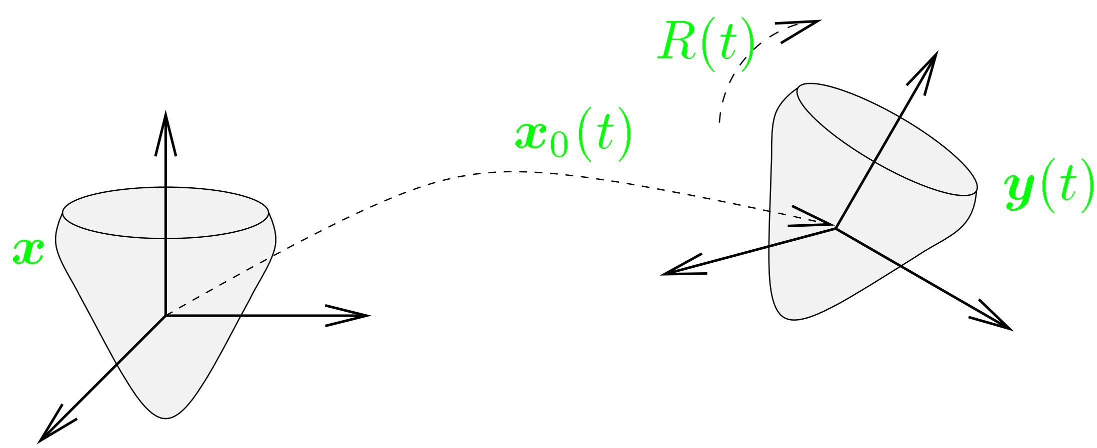
$$\boldsymbol{f}_k(t) = \exp\left(-\tfrac{1}{2}\boldsymbol{e}_1\boldsymbol{e}_2\omega t\right)\boldsymbol{e}_k \exp\left(\tfrac{1}{2}\boldsymbol{e}_1\boldsymbol{e}_2\omega t\right)$$

so that the $\boldsymbol{f}_1$ axis rotates as

$$\boldsymbol{f}_1 = \boldsymbol{e}_1 \exp(\boldsymbol{e}_1\boldsymbol{e}_2\omega t) = \cos(\omega t)\boldsymbol{e}_1 + \sin(\omega t)\boldsymbol{e}_2$$

Defines a right-handed (anticlockwise) rotation in the $\boldsymbol{e}_1\boldsymbol{e}_2$ plane, as defined by orientation of Ω.

Rigid Body Dynamics



- Vector $\boldsymbol{x}_0(t)$ is position of the centre of mass, relative to the origin
- Rotor $R(t)$ defines the orientation of the body, relative to a fixed copy imagined at the origin.
- $\boldsymbol{x}_i$ is a constant vector in the reference body.
- $\boldsymbol{y}_i$ is the vector in space of the equivalent point on the moving body.

Points related by

$$\boldsymbol{y}_i(t) = R(t)\boldsymbol{x}_i\tilde{R}(t) + \boldsymbol{x}_0(t)$$

Places all rotational dynamics in the rotor. Velocity of point $\boldsymbol{y}$ is

$$\begin{aligned}\boldsymbol{v}(t) &= \dot{R}\boldsymbol{x}\tilde{R} + R\boldsymbol{x}\dot{\tilde{R}} + \dot{\boldsymbol{x}}_0 \\ &= -\tfrac{1}{2}R\Omega_B\boldsymbol{x}\tilde{R} + \tfrac{1}{2}R\boldsymbol{x}\Omega_B\tilde{R} + \boldsymbol{v}_0 \\ &= R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0\end{aligned}$$

$\boldsymbol{v}_0$ is velocity of the centre of mass. NB ordering convention

$$R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} = R(\boldsymbol{x}\cdot\Omega_B)\tilde{R}$$

Working in terms of the 'body' angular velocity Ω_B.

Use continuum approximation with density $\rho = \rho(\boldsymbol{x})$. Since $\boldsymbol{x}$ taken relative to centre of mass have

$$\int d^3x\,\rho = M, \quad \text{and} \quad \int d^3x\,\rho\boldsymbol{x} = 0.$$

The momentum of the rigid body is

$$\int d^3x\,\rho\boldsymbol{v} = \int d^3x\,\rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0) = M\boldsymbol{v}_0,$$

Determined entirely by the motion of the centre of mass.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

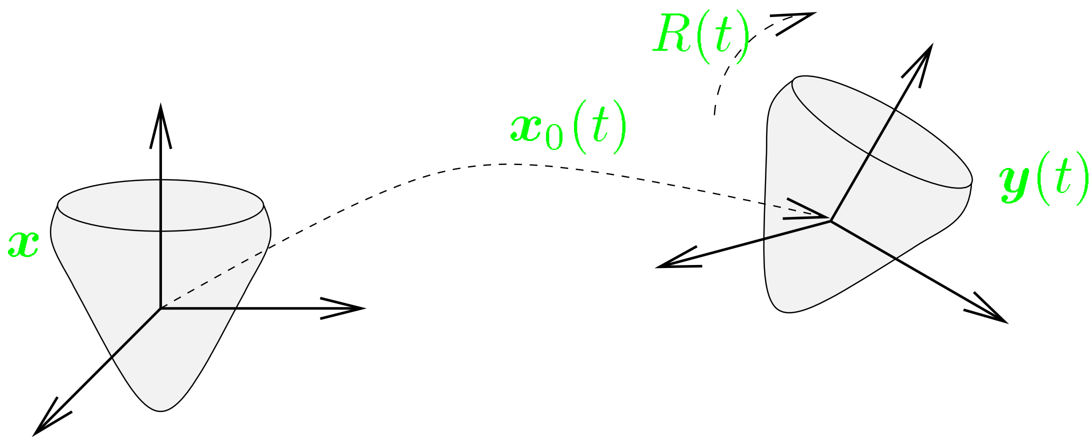
LECTURE 5

SUMMARY

In this lecture apply the rotor formalism for rotations to rigid body dynamics. This simplifies the equations and, for the case of the symmetric top, provides a new way of obtaining the solution. The physical role of the inertia tensor is also clarified.

1. Rigid body bynamics.
2. The inertia tensor
3. Torque-free motion.
4. The symmetric top.

Rigid Body Dynamics



- Vector $\boldsymbol{x}_0(t)$ is position of the centre of mass, relative to the origin
- Rotor $R(t)$ defines the orientation of the body, relative to a fixed copy imagined at the origin.
- $\boldsymbol{x}_i$ is a constant vector in the reference body.
- $\boldsymbol{y}_i$ is the vector in space of the equivalent point on the moving body.

Points related by

$$\boldsymbol{y}_i(t) = R(t)\boldsymbol{x}_i\tilde{R}(t) + \boldsymbol{x}_0(t)$$

Places all rotational dynamics in the rotor.

Velocity

The velocity of a point $\boldsymbol{y}$ is

$$\begin{aligned}\boldsymbol{v}(t) &= \dot{R}\boldsymbol{x}\tilde{R} + R\boldsymbol{x}\dot{\tilde{R}} + \dot{\boldsymbol{x}}_0 \\ &= -\tfrac{1}{2}R\Omega_B\boldsymbol{x}\tilde{R} + \tfrac{1}{2}R\boldsymbol{x}\Omega_B\tilde{R} + \boldsymbol{v}_0 \\ &= R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0\end{aligned}$$

$\boldsymbol{v}_0$ is velocity of the centre of mass. NB ordering convention

$$R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} = R(\boldsymbol{x}\cdot\Omega_B)\tilde{R}$$

Working in terms of the 'body' angular velocity Ω_B.

Momentum

Use continuum approximation with density $\rho = \rho(\boldsymbol{x})$. Since $\boldsymbol{x}$ taken relative to centre of mass have

$$\int d^3x\,\rho = M, \quad \text{and} \quad \int d^3x\,\rho\boldsymbol{x} = 0.$$

The momentum of the rigid body is

$$\int d^3x\,\rho\boldsymbol{v} = \int d^3x\,\rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0) = M\boldsymbol{v}_0,$$

Determined entirely by the motion of the centre of mass.

THE INERTIA TENSOR

Next need angular momentum bivector L for the body about its centre of mass.

$$\begin{aligned} L &= \int d^3x\, \rho(\boldsymbol{y} - \boldsymbol{x}_0) \wedge \boldsymbol{v} \\ &= \int d^3x\, \rho(R\boldsymbol{x}\tilde{R}) \wedge (R\, \boldsymbol{x} \cdot \Omega_B\, \tilde{R} + \boldsymbol{v}_0) \\ &= R\Big(\int d^3x\, \rho \boldsymbol{x} \wedge (\boldsymbol{x} \cdot \Omega_B)\Big)\tilde{R} \end{aligned}$$

Integral in brackets refers only to the fixed copy, so is a time-independent function of Ω_B. Define the inertia tensor $\mathcal{I}(B)$ by

$$\mathcal{I}(B) = \int d^3x\, \rho \boldsymbol{x} \wedge (\boldsymbol{x} \cdot B)$$

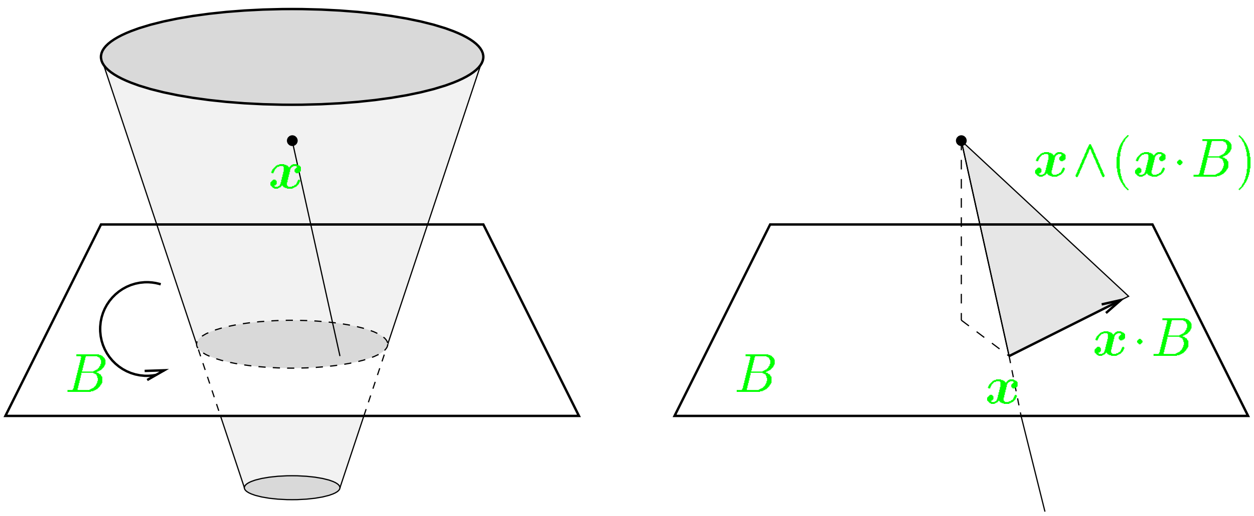
A linear function mapping bivectors to bivectors. For any bivectors A and B have properties

Linearity: $\mathcal{I}(\lambda A + \mu B) = \lambda \mathcal{I}(A) + \mu \mathcal{I}(B)$

Symmetry: $\langle A \mathcal{I}(B) \rangle = \langle \mathcal{I}(A) B \rangle.$

Bivectors belong to a 3-dimensional linear space, so no additional complexity.

Inertia tensor $\mathcal{I}(B)$ receives as input the bivector B. Imagine the body were to rotate about centre of mass in the B plane.



- $\boldsymbol{x}$ would move with velocity $\boldsymbol{x}\cdot B$, momentum density $\rho\boldsymbol{x}\cdot B$ and angular momentum density $\boldsymbol{x}\wedge(\rho\boldsymbol{x}\cdot B)$.
- Integrate over the body to get total angular momentum. Result usually will not lie in B plane.
- As motion proceeds, Ω is back-rotated to the body system to give $\Omega_B = \tilde{R}\Omega R$.
- Ω_B is fed into the fixed inertia tensor.
- Result $\mathcal{I}(\Omega_B)$, is then rotated onto the space angular momentum L.

Equations of Motion

Angular momentum bivector formed by rotating $\mathcal{I}(\Omega_B)$ onto the space configuration

$$L = R\,\mathcal{I}(\Omega_B)\,\tilde{R}$$

Equations of motion are $\dot{L} = N$, where N is the external torque. Inertia tensor is time-independent, so

$$\begin{aligned}
\dot{L} &= \dot{R}\mathcal{I}(\Omega_B)\tilde{R} + R\mathcal{I}(\Omega_B)\dot{\tilde{R}} + R\mathcal{I}(\dot{\Omega}_B)\tilde{R} \\
&= R[\mathcal{I}(\dot{\Omega}_B) - \tfrac{1}{2}\Omega_B\mathcal{I}(\Omega_B) + \tfrac{1}{2}\mathcal{I}(\Omega_B)\Omega_B]\tilde{R} \\
&= R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B)]\tilde{R}.
\end{aligned}$$

NB Commutator of the bivectors Ω_B and $\mathcal{I}(\Omega_B)$ results in a third bivector. Equations usually seen in component form. Introduce a set of principal axes $\{\boldsymbol{e}_k\}$ which satisfy

$$\mathcal{I}(I\boldsymbol{e}_k) = i_k I\boldsymbol{e}_k \quad \text{no sum}$$

The i_k are the 3 principal moments of inertia. (Symmetry of $\mathcal{I}(B)$ ensures these exist). Now write

$$\Omega = \sum_{k=1}^{3} \omega_k I\boldsymbol{f}_k, \quad \Omega_B = \sum_{k=1}^{3} \omega_k I\boldsymbol{e}_k$$

$$L = \sum_{k=1}^{3} i_k\omega_k I\boldsymbol{f}_k, \quad N = \sum_{k=1}^{3} N_k I\boldsymbol{f}_k$$

Expanding out recovers the Euler equations for a rigid body

$$
\begin{aligned}
i_1\dot{\omega}_1 - \omega_2\omega_3(i_2 - i_3) &= N_1 \\
i_2\dot{\omega}_2 - \omega_3\omega_1(i_3 - i_1) &= N_2 \\
i_3\dot{\omega}_3 - \omega_1\omega_2(i_1 - i_2) &= N_3.
\end{aligned}
$$

Torque-free motion

Torque-free equation $\dot{L} = 0$ reduces to

$$\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B) = 0$$

A first-order constant coefficient differential equation for Ω_B. Closed form solutions exist, but can be messy. Consider conserved quantities, *e.g.* L. Another is rotational kinetic energy,

$$T = \frac{1}{2}\int d^3x\,\rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R})^2 = \frac{1}{2}\int d^3x\,\rho(\boldsymbol{x}\cdot\Omega_B)^2$$

Use rearrangement

$$(\boldsymbol{x}\cdot\Omega_B)^2 = \langle \boldsymbol{x}\cdot\Omega_B\boldsymbol{x}\Omega_B\rangle = -\Omega_B\cdot\big(\boldsymbol{x}\wedge(\boldsymbol{x}\cdot\Omega_B)\big)$$

to get conserved energy

$$T = -\frac{1}{2}\Omega_B\cdot\mathcal{I}(\Omega_B) = \frac{1}{2}\tilde{\Omega}_B\cdot\mathcal{I}(\Omega_B)$$

Now introduce the components $L_k = i_k \omega_k$,

$$L = \sum_{k=1}^{3} L_k I \boldsymbol{f}_k$$

the components of L in the rotating $\boldsymbol{f}_k$ frame. In terms of these the magnitude of L is

$$L\tilde{L} = L_1^2 + L_2^2 + L_3^3$$

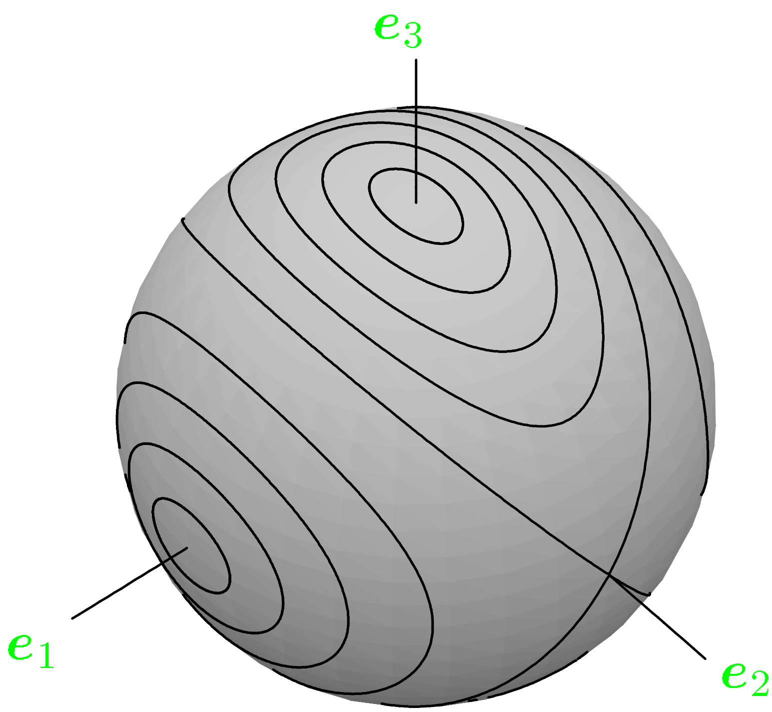
and the kinetic energy is

$$T = \frac{L_1^2}{2i_1} + \frac{L_2^2}{2i_2} + \frac{L_3^2}{2i_3}.$$

Both $|L|$ and T are constants of motion. Two constraints on the three components L_k. Visualise in terms of vector $\boldsymbol{l}$ with components L_k,

$$\boldsymbol{l} = \sum_{k=1}^{3} L_k \boldsymbol{e}_k = -I\tilde{R}LR$$

A rotating vector in the fixed reference body. Constrained to lie on the surface of a sphere and an ellipsoid. Orbits around the axes with smallest and largest principal moments are stable. Orbits around the middle axis are unstable.



THE SYMMETRIC TOP

Suppose body has a single symmetry axis $\boldsymbol{e}_3$, so two equal moments of inertia, $i_1 = i_2$, with i_3 different. Action of the inertia tensor on Ω_B is

$$\begin{aligned}\mathcal{I}(\Omega_B) &= i_1\omega_1\boldsymbol{e}_2\boldsymbol{e}_3 + i_1\omega_2\boldsymbol{e}_3\boldsymbol{e}_1 + i_3\omega_3\boldsymbol{e}_1\boldsymbol{e}_2 \\ &= i_1\Omega_B + (i_3 - i_1)\omega_3\boldsymbol{I}\boldsymbol{e}_3\end{aligned}$$

The final Euler equation tells us that ω_3 is constant. Rotating both sides of this equation with R gives

$$\Omega = R\Omega_B\tilde{R} = \frac{1}{i_1}L + \frac{i_1 - i_3}{i_1}\omega_3 R\boldsymbol{I}\boldsymbol{e}_3\tilde{R}$$

The rotor equation is now

$$\dot{R} = -\tfrac{1}{2}\Omega R = -\frac{1}{2i_1}(LR + R(i_1 - i_3)\omega_3 I\boldsymbol{e}_3)$$

Involves two constant bivectors, multiplying R from the left and right. Define

$$\Omega_l = \frac{1}{i_1}L, \qquad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I\boldsymbol{e}_3,$$

so that the rotor equation becomes

$$\dot{R} = -\tfrac{1}{2}\Omega_l R - \tfrac{1}{2}R\Omega_r$$

Integrates immediately to give

$$R(t) = \exp(-\tfrac{1}{2}\Omega_l t)R_0 \exp(-\tfrac{1}{2}\Omega_r t)$$

Fully describes the motion of a symmetric top.

- An internal rotation in the $\boldsymbol{e}_1\boldsymbol{e}_2$ plane. Responsible for precession.
- R_0 defines the attitude of the body at $t = 0$ and can be set to 1.
- The resultant body is rotated in the angular momentum plane to obtain final attitude in space.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 6

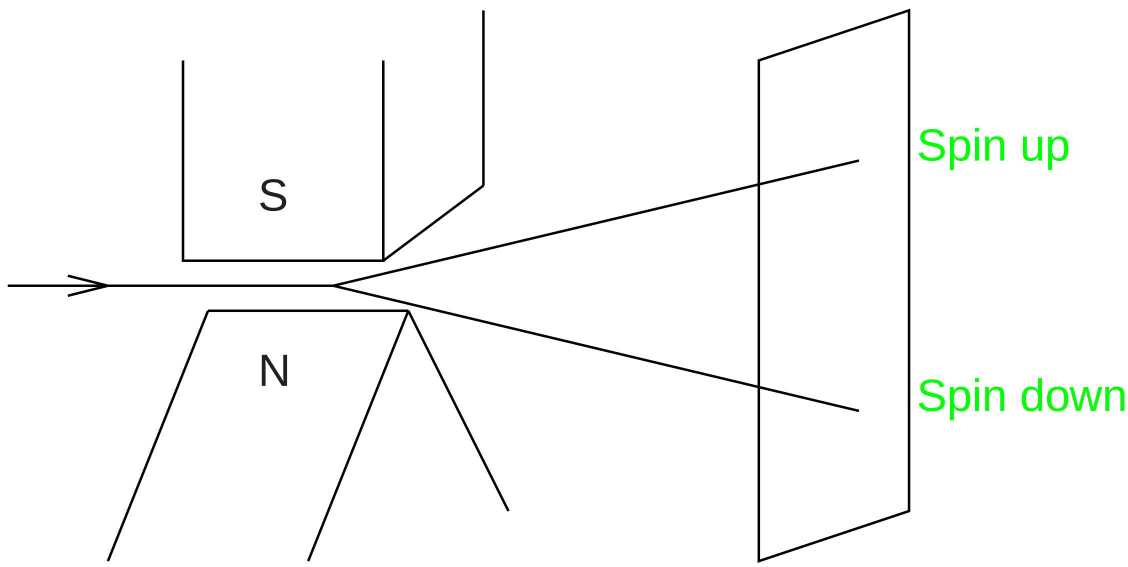
SUMMARY

In this lecture we study the application of GA to non-relativistic quantum physics, focusing attention on quantum spin. The Pauli matrix algebras is a Clifford algebra, so quantum spin has a natural expression in GA. But this has some surprising consequences, as we start to formulate both classical and quantum physics in a single framework.

- Non-relativistic quantum spin.
- Pauli matrices, 2 state systems and spinors.
- Spinors, rotors and observables.
- Particle in a magnetic field. The quantum Hamiltonian and its rotor equation.
- Magnetic Resonance Imaging.

Non-Relativistic Quantum Spin

The Stern-Gerlach experiment reveals the quantum nature of the magnetic moment.



Expect a continuum from $\boldsymbol{f} = \boldsymbol{\mu} \cdot \boldsymbol{\nabla} \boldsymbol{B}$. Instead see evenly-spaced discrete bands. The magnetic moment is quantised, like angular momentum.

With silver atoms (Z=47, single electron in outermost shell), only two beams emerge. Electrons have an intrinsic angular momentum — the spin — which can take only two values. Electron wavefunction is a superposition of two spin states

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

α and β are complex numbers.

Represent the state in matrix form as a spinor

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Align z-axis with spin-up direction. Operator returning spin along z-axis must be

$$\hat{s}_3 = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with λ to be determined. Spin and orbital contributions combine to give a conserved total angular momentum operator $\hat{\boldsymbol{j}} = \hat{\boldsymbol{l}} + \hat{\boldsymbol{s}}$. Spin operators must have same commutation relations as angular momentum $\hat{l}_i$

$$\hat{l}_i = -i\hbar\epsilon_{ijk}x_j\partial_k, \quad [\hat{l}_i, \hat{l}_j] = i\hbar\epsilon_{ijk}\hat{l}_k.$$

Find the spin operators

$$\hat{s}_k = \tfrac{1}{2}\hbar\hat{\sigma}_k,$$

where the $\hat{\sigma}_k$ are the Pauli matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These operate on spinors.

The Pauli matrices satisfy

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$$

Also different matrices anticommute,

$$\hat{\sigma}_1\hat{\sigma}_2 + \hat{\sigma}_2\hat{\sigma}_1 = 0, \quad \textit{etc.}$$

and all matrices square to the identity matrix

$$\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = \hat{\sigma}_3^2 = 1$$

The defining relations for a set of orthonormal basis vectors in $\mathcal{G}_3$! The Pauli matrices are a matrix representation of the geometric algebra of 3-d space.

- Any multivector in $\mathcal{G}_3$ can be written as a 2×2 complex matrix.
- Scalars are represented by multiples of the identity matrix.
- The pseudoscalar I is replaced by i times the identity.
- 2×2 complex matrices form a linear space of real dimension 8.
- Multivector products can be computed by finding the equivalent matrices and multiplying these.
- This is usually slower.

SPINORS AND MULTIVECTORS

The $|\psi\rangle$'s form a two-dimensional complex vector space. Would a multivector equivalent. Consider the observables

$$\begin{aligned} n_1 &= \langle\psi|\hat{\sigma}_1|\psi\rangle = \alpha\beta^* + \alpha^*\beta \\ n_2 &= \langle\psi|\hat{\sigma}_2|\psi\rangle = i(\alpha\beta^* - \alpha^*\beta) \\ n_3 &= \langle\psi|\hat{\sigma}_3|\psi\rangle = \alpha\alpha^* - \beta\beta^* \end{aligned}$$

Form the components of a vector $\boldsymbol{n}$ with length

$$|\boldsymbol{n}|^2 = (|\alpha|^2 + |\beta|^2)^2 = \langle\psi|\psi\rangle^2 = 1$$

Write this in polar coordinates as

$$\begin{aligned} n_1 &= \sin(\theta)\cos(\phi) \\ n_2 &= \sin(\theta)\sin(\phi) \\ n_3 &= \cos(\theta) \end{aligned}$$

Comparing with above, must have

$$\alpha = \cos(\theta/2)\,\mathrm{e}^{i\gamma}, \qquad \beta = \sin(\theta/2)\,\mathrm{e}^{i\delta}$$

where $\delta - \gamma = \phi$. Now write the spinor as

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-i\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{i\phi/2} \end{pmatrix} \mathrm{e}^{i(\gamma+\delta)/2}$$

Rotors and Spinors

Double-sided construction of observables is similar to rotor form of rotation. Write

$$\boldsymbol{n} = \sin(\theta)\big(\cos(\phi)\boldsymbol{\sigma}_1 + \sin(\phi)\boldsymbol{\sigma}_2\big) + \cos(\theta)\boldsymbol{\sigma}_3$$

Using $\{\boldsymbol{\sigma}_k\}$ as our orthonormal frame. Vector equivalents of the Pauli matrix operators. Can write

$$\boldsymbol{n} = R\boldsymbol{\sigma}_3\tilde{R}$$

where

$$R = \mathrm{e}^{-\phi I\boldsymbol{\sigma}_3/2}\,\mathrm{e}^{-\theta I\boldsymbol{\sigma}_2/2}$$

Expanded out

$$R = \cos(\theta/2)\,\mathrm{e}^{-\phi I\boldsymbol{\sigma}_3/2} - I\boldsymbol{\sigma}_2\sin(\theta/2)\,\mathrm{e}^{\phi I\boldsymbol{\sigma}_3/2}$$

Comparing with $|\psi\rangle$, establish to 1-to-1 map

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I\boldsymbol{\sigma}_k$$

Can now use the multivector ψ in place of the quantum spinor $|\psi\rangle$.

Pauli Operators

Action of the quantum operators $\{\hat{\sigma}_k\}$ on states $|\psi\rangle$ has an analogous operation on the multivector ψ:

$$\hat{\sigma}_k|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3 \quad (k=1,2,3)$$

$\boldsymbol{\sigma}_3$ on the right-hand side ensures result stays in the even subalgebra. Verify by computation

$$\hat{\sigma}_1|\psi\rangle = \begin{pmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{pmatrix}$$

maps to

$$-a^2 + a^1 I\boldsymbol{\sigma}_3 - a^0 I\boldsymbol{\sigma}_2 + a^3 I\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1\psi\boldsymbol{\sigma}_3$$

Also need equivalent of multiplication by the unit imaginary i. Do this using

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

so arrive at the operator equivalence

$$i|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\psi(\boldsymbol{\sigma}_3)^3 = \psi I\boldsymbol{\sigma}_3$$

Unit imaginary of quantum theory is replaced by right multiplication by the bivector $I\boldsymbol{\sigma}_3$. Very suggestive!

Pauli Observables

Hermitian adjoint has $\hat{\sigma}_k^\dagger = \hat{\sigma}_k$ and reverses the order of all products. The same as the reversion operation, so

$$M^\dagger = \tilde{M}$$

For spinor inner product have

$$\langle\psi|\psi\rangle = (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 = \langle\psi\tilde{\psi}\rangle$$

(No spatial integral.) Apply to $|\psi + \lambda\phi\rangle$ find

$$\langle\psi|\phi\rangle + \langle\phi|\psi\rangle = \langle\psi\tilde{\phi}\rangle + \langle\phi\tilde{\psi}\rangle$$

so real part is

$$\mathsf{R}\langle\psi|\phi\rangle = \langle\tilde{\psi}\phi\rangle$$

Since

$$\langle\psi|\phi\rangle = \mathsf{R}\langle\psi|\phi\rangle - i\mathsf{R}\langle\psi|i\phi\rangle$$

the full inner product can be written

$$\langle\psi|\phi\rangle \;\leftrightarrow\; \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3$$

Right hand side projects out the 1 and $I\boldsymbol{\sigma}_3$ components from $\tilde{\psi}\phi$. Write this $\langle A\rangle_q$. For even-grade multivectors have

$$\langle A\rangle_q = \tfrac{1}{2}(A + \boldsymbol{\sigma}_3 A \boldsymbol{\sigma}_3)$$

The Spin Vector

Expectation value of the spin in the k-direction requires

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \leftrightarrow \langle\tilde{\psi}\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\rangle - \langle\tilde{\psi}\boldsymbol{\sigma}_k\psi I\rangle I\boldsymbol{\sigma}_3$$

$\tilde{\psi}I\boldsymbol{\sigma}_k\psi$ reverses to give minus itself, so has zero scalar part. (The $\hat{\sigma}_k$ are Hermitian operators.) $\psi\boldsymbol{\sigma}_3\tilde{\psi}$ is odd grade and reverses to itself, so is a pure vector. Define the spin vector

$$\boldsymbol{s} = \tfrac{1}{2}\hbar\psi\boldsymbol{\sigma}_3\tilde{\psi}$$

The quantum expectation now reduces to

$$\langle\psi|\hat{s}_k|\psi\rangle = \boldsymbol{\sigma}_k\cdot\boldsymbol{s}$$

A different interpretation now! Rather than forming expectations, we are projecting out the kth component of the vector $\boldsymbol{s}$.

Gain further insight by defining the scalar

$$\rho = \psi\tilde{\psi}$$

The spinor ψ then decomposes into

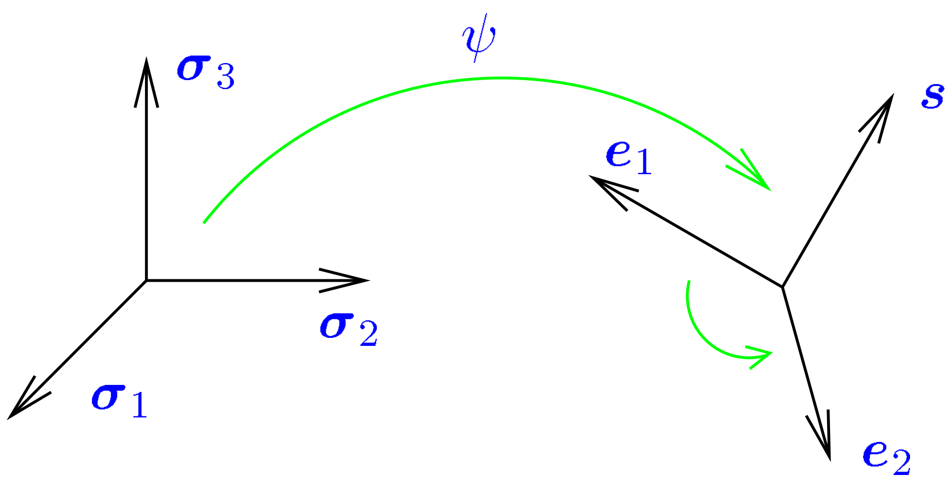
$$\psi = \rho^{1/2}R$$

where $R = \rho^{-1/2}\psi$. R satisfies $R\tilde{R} = 1$, so is a rotor. In this approach, Pauli spinors are unnormalised rotors!

The spin-vector $\boldsymbol{s}$ can now be written as

$$\boldsymbol{s} = \tfrac{1}{2}\hbar\rho R\boldsymbol{\sigma}_3\tilde{R}$$

The double-sided construction of the expectation value is an instruction to rotate the fixed $\boldsymbol{\sigma}_3$ axis into the spin direction and dilate it. Similar to rigid body dynamics.



A phase transformation of ψ is a rotation in the $\boldsymbol{e}_1\boldsymbol{e}_2$ plane. These are unobservable, so $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ also unobservable.

Now rotate $\boldsymbol{s}$ to a new vector $R_0\boldsymbol{s}\tilde{R}_0$. ψ must transform as

$$\psi \mapsto R_0\psi$$

Suppose we set $R_0 = \exp(-\hat{B}\theta/2)$ and increase θ from 0 to 2π. See that

$$\psi' = \mathrm{e}^{-\hat{B}\pi}\psi = (\cos\pi - \hat{B}\sin\pi)\psi = -\psi$$

Change sign under 360^{o} rotations. This is the 'spin-1/2' nature of spinor wave functions.

APPLICATION — MAGNETIC FIELDS

Particles with non-zero spin have a magnetic moment, expressed in terms of operators as

$$\hat{\mu}_k = \gamma \hat{s}_k$$

$\hat{\mu}_k$ is the magnetic moment operator, γ is the gyromagnetic ratio and $\hat{s}_k$ is the spin operator. Usually write

$$\gamma = g \frac{q}{2m}$$

m is particle mass, q is charge and g is reduced gyromagnetic ratio. Find these experimentally to be

electron	$g_e = 2$	(actually $2(1 + \alpha/2\pi + \cdots)$)
proton	$g_p = 5.587$	
neutron	$g_n = -3.826$	(use proton charge)

Neutron has negative g_n because its spin and magnetic moment are anti-parallel. All of the above are spin-$1/2$ particles, so $\hat{s}_k = \frac{1}{2}\hbar\hat{\sigma}_k$.

The Hamiltonian operator for a particle in a magnetic field is

$$\hat{H} = -\tfrac{1}{2}\gamma\hbar B_k \hat{\sigma}_k = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B}$$

Operates on a spin states.

Dynamics from time-dependent Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{d|\psi\rangle}{dt} = -\tfrac{1}{2}\gamma\hbar B_k\hat{\sigma}_k|\psi\rangle$$

A pair of coupled differential equations. Can be hard.

To analyse in our new setup, replace $|\psi\rangle$ by multivector ψ. Action of $i\hat{\sigma}_k$ becomes

$$i\hat{\sigma}_k|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3(I\boldsymbol{\sigma}_3) = I\boldsymbol{\sigma}_k\psi$$

Our version of the Schrödinger equation is now

$$\dot{\psi} = \tfrac{1}{2}\gamma B_k I\boldsymbol{\sigma}_k\psi = \tfrac{1}{2}\gamma I\boldsymbol{B}\psi$$

Now decompose ψ into $\rho^{1/2}R$

$$\dot{\psi}\tilde{\psi} = \tfrac{1}{2}\dot{\rho}\rho + \rho\dot{R}\tilde{R} = \tfrac{1}{2}\rho\gamma I\boldsymbol{B}$$

The right-hand side is a bivector, so ρ is constant (unitary evolution).

Dynamics reduces to

$$\dot{R} = \tfrac{1}{2}\gamma I\boldsymbol{B}R$$

The quantum theory of a spin-$1/2$ particle in a magnetic field reduces to another rotor equation! Now easy to analyse.

Magnetic Resonance Imaging

Constant $\boldsymbol{B}$-field along z-axis, $B_0\boldsymbol{\sigma}_3$, plus oscillating field $(B_1\cos(\omega t), B_1\sin(\omega t), 0)$. Induces transitions (spin-flips) between the up and down states. To study this, write

$$\begin{aligned}\boldsymbol{B} &= B_1(\cos(\omega t)\boldsymbol{\sigma}_1 + \sin(\omega t)\boldsymbol{\sigma}_2) + B_0\boldsymbol{\sigma}_3 \\ &= \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2}(B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3)\,\mathrm{e}^{\omega t I\boldsymbol{\sigma}_3/2}\end{aligned}$$

Now define

$$S = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2} \qquad \boldsymbol{B}_c = B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3$$

so that $\boldsymbol{B} = S\boldsymbol{B}_c\tilde{S}$. The rotor equation is now

$$\tilde{S}\dot{\psi} = \tfrac{1}{2}\gamma I\boldsymbol{B}_c\tilde{S}\psi$$

(Have pre-multiplied by $\tilde{S}$ and still using ψ). Next use

$$\dot{\tilde{S}} = \tfrac{1}{2}\omega I\boldsymbol{\sigma}_3\tilde{S}$$

Now have

$$\frac{d}{dt}(\tilde{S}\psi) = \tfrac{1}{2}(\gamma I\boldsymbol{B}_c + \omega I\boldsymbol{\sigma}_3)\tilde{S}\psi$$

$\tilde{S}\psi$ satisfies a rotor equation with a constant field. Solution is straightforward,

$$\tilde{S}\psi(t) = \exp\left(\frac{t}{2}(\gamma I\boldsymbol{B}_c + \omega I\boldsymbol{\sigma}_3)\right)\psi_0$$

$\psi(t)$ given by

$$\exp\left(-\frac{wt}{2}I\boldsymbol{\sigma}_3\right)\exp\left(\frac{t}{2}((\omega_0+\omega)I\boldsymbol{\sigma}_3+\omega_1 I\boldsymbol{\sigma}_1)\right)\psi_0$$

where $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$. Three separate frequencies in this solution.

EXPERIMENT

Switch on the oscillating field at $t = 0$, with particle initially spin-up $\psi_0 = 1$. Probability that at time t the particle is spin-down is

$$P_\downarrow = |\langle\downarrow|\psi(t)\rangle|^2$$

Form the inner product

$$\begin{aligned}\langle\downarrow|\psi(t)\rangle \leftrightarrow \langle I\boldsymbol{\sigma}_2\psi\rangle_q &= \langle I\boldsymbol{\sigma}_2\psi\rangle - I\boldsymbol{\sigma}_3\langle I\boldsymbol{\sigma}_2\psi I\boldsymbol{\sigma}_3\rangle \\ &= \langle I\boldsymbol{\sigma}_2\psi\rangle - I\boldsymbol{\sigma}_3\langle I\boldsymbol{\sigma}_1\psi\rangle\end{aligned}$$

To find this write

$$\psi(t) = \mathrm{e}^{-wtI\boldsymbol{\sigma}_3/2}\left[\cos(\alpha t/2) + I\hat{\boldsymbol{B}}\sin(\alpha t/2)\right]$$

where

$$\hat{\boldsymbol{B}} = \frac{(\omega_0+\omega)\boldsymbol{\sigma}_3 + \omega_1\boldsymbol{\sigma}_1}{\alpha}, \quad \alpha = \sqrt{(w+w_0)^2+\omega_1^2}$$

Only the $\omega_1 I\boldsymbol{\sigma}_1/\alpha$ term contributes, so have

$$\langle I\boldsymbol{\sigma}_2\psi\rangle_q = \frac{\omega_1 \sin(\alpha t/2)}{\alpha}\, \mathrm{e}^{-wtI\boldsymbol{\sigma}_3/2}\, I\boldsymbol{\sigma}_3$$

The probability is now

$$P_\downarrow = \left(\frac{\omega_1 \sin(\alpha t/2)}{\alpha}\right)^2$$

Maximum value at $\alpha t = \pi$, probability maximised by choosing α as small as possible. Gives spin resonance condition $\omega = -\omega_0 = -\gamma B_0$.

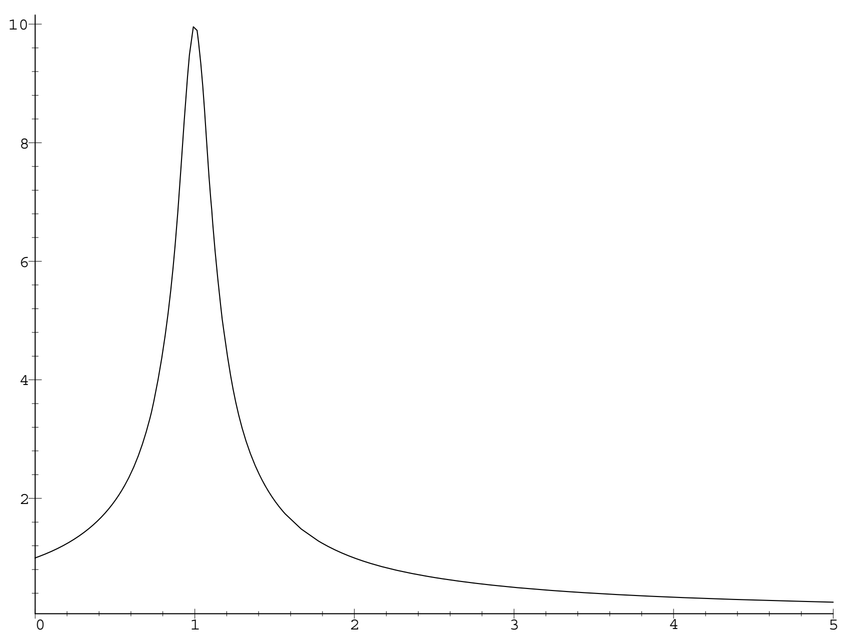


Figure 1: $1/\alpha$ vs ω with $\omega_1 = 0.1$, $\omega_0 = -1$

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 7

SUMMARY

In this lecture we introduce the spacetime algebra, the geometric algebra of spacetime. This forms the basis for most of the remaining course, which deals with applications of geometric algebra in relativistic physics.

- Magnetic Resonance Imaging.
- Adding a vector for time
- The 4-d spacetime algebra — consequences of a mixed signature metric.
- Paths, observers and frames.
- Projective splits for observers.
- Handling Lorentz transformations with rotors.

AN ALGEBRA FOR SPACETIME

Aim — to construct the geometric algebra of spacetime.

Invariant interval is

$$s^2 = c^2t^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2$$

(The 'particle physics' choice. GR often flips signs. No observable consequences). Work in natural units, $c = 1$.

Need four vectors $\{e_0, e_i\}, i = 1 \ldots 3$ with properties

$${e_0}^2 = 1, \qquad {e_i}^2 = -1$$

$$e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}$$

Summarised by

$$e_\mu \cdot e_\nu = \text{diag}(+ - - -)\,, \quad \mu, \nu = 0 \ldots 3$$

The reciprocal vectors e^μ have

$$e^\mu \cdot e_\nu = \delta^\mu_\nu$$

so $e^0 = e_0$ and $e^i = -e_i$.

There are $4 \times 3/2 = 6$ bivectors in algebra. Two types

1. Those containing e_0, *e.g.* $\{e_1 \wedge e_0, e_2 \wedge e_0, e_3 \wedge e_0\}$,
2. Those not containing e_0, *e.g.* $\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}$.

Have different properties.

Properties of Bivectors

For any pair of vectors a and b, with $a\cdot b = 0$, have

$$(a\wedge b)^2 = abab = -abba = -a^2b^2$$

The two types have different squares

$$(e_i\wedge e_j)^2 = -e_i{}^2 e_j{}^2 = -1$$

Spacelike Euclidean bivectors, generate rotations in a plane.

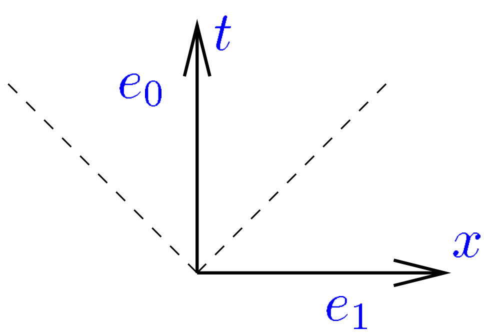
$$(e_i\wedge e_0)^2 = -e_i{}^2 e_0{}^2 = 1$$

Timelike bivectors. Generate hyperbolic geometry:

$$\begin{aligned} e^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \alpha^2/2! + \alpha^3/3!\, e_1 e_0 + \cdots \\ &= \mathrm{ch}(\alpha) + \mathrm{sh}(\alpha) e_1 e_0 \end{aligned}$$

Crucial to treatment of Lorentz transformations.

Traditionally draw spacetime diagrams as



'right-handed' bivector for this is $e_1 e_0$.

THE PSEUDOSCALAR

Define the pseudoscalar I

$$I = e_0 e_1 e_2 e_3$$

The e_1, e_2, e_3 chosen to be right-handed. I is grade 4 so $\tilde{I} = I$. Compute the square of I :

$$I^2 = I\tilde{I} = (e_0 e_1 e_2 e_3)(e_3 e_2 e_1 e_0) = -1$$

Multiply bivector by I, get grade $4 - 2 = 2$ — another bivector. Provides map between bivectors with positive and negative square:

$$I e_1 e_0 = e_1 e_0 I = e_1 e_0 e_0 e_1 e_2 e_3 = -e_2 e_3$$

Define $B_i = e_i e_0$. Bivector algebra is

$$B_i \times B_j = \epsilon_{ijk}\, I B_k, \quad (IB_i) \times (IB_j) = -\epsilon_{ijk}\, I B_k$$
$$(IB_i) \times B_j = -\epsilon_{ijk} B_k$$

Have four vectors, and four trivectors in algebra. Interchanged by duality

$$e_1 e_2 e_3 = e_0 e_0 e_1 e_2 e_3 = e_0 I = -I e_0$$

NB I anticommutes with vectors and trivectors. (In space of even dimensions). I always commutes with even-grade.

The Spacetime algebra

Putting terms together, get an algebra with 16 terms:

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	I
1	4	6	4	1
scalar	vectors	bivectors	trivectors	pseudoscalar

The spacetime algebra or STA. Use $\{\gamma_\mu\}$ for preferred orthonormal frame (analogy with QM). Also define

$$\boldsymbol{\sigma}_i = \gamma_i \gamma_0$$

Not used i for the pseudoscalar. Potentially confusing. Reciprocal frame has

$$\gamma^0 = \gamma_0, \quad \gamma^i = -\gamma_i$$

The $\{\gamma_\mu\}$ satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$$

This is the Dirac matrix algebra (with identity matrix on right). A matrix representation of the STA. Explains notation, but $\{\gamma_\mu\}$ are vectors, not a set of matrices in ‘isospace’.

Frames and Trajectories

$x(\lambda)$ a spacetime trajectory. Tangent vector is

$$x' = \frac{\partial x(\lambda)}{\partial \lambda}$$

Two cases to consider:

Timelike, $x'^2 > 0$. Introduce proper time τ:

$$v = \partial_\tau x = \dot{x}, \qquad v^2 = 1$$

Observers measure this. Unit vector v defines the instantaneous rest frame.

Null, $x'^2 = 0$. Describes a null trajectory. Taken by massless particles, (photons, *etc.*). Proper distance/time = 0. Photons do carry an intrinsic clock (their frequency), but can tick at arbitrary rate.

Consider observer with velocity $v = e_0$ at rest in some inertial frame. What do we measure? Construct a rest frame $\{e_i\}$,

$$e_i \cdot v = 0, \qquad i = 1 \ldots 3$$

Then general event x can be written as

$$x = te_0 + x^i e_i$$

Event x has time coordinate $t = x \cdot v$ and space coordinates $x^i = x \cdot e^i$ ($\{e^i\}$ is reciprocal frame).

Spatial part of x in rest frame of v is just

$$x^i e_i = x\cdot e^\mu\, e_\mu - x\cdot e^0\, e_0 = x - x\cdot v\, v = x\wedge v\, v$$

Wedge product with v projects onto components of x in rest frame of v. Define relative vector by spacetime bivector $x\wedge v$:

$$\boldsymbol{x} = x\wedge v$$

With these definitions have

$$xv = x\cdot v + x\wedge v = t + \boldsymbol{x}$$

Invariant distance decomposes as

$$\begin{aligned} x^2 = xvvx &= (x\cdot v + x\wedge v)(x\cdot v + v\wedge x) \\ &= (t+\boldsymbol{x})(t-\boldsymbol{x}) = t^2 - \boldsymbol{x}^2 \end{aligned}$$

Recovers usual result. Built into definition of STA.

THE EVEN SUBALGEBRA

Each inertial frame defines a set of relative vectors. Model these as spacetime bivectors. Take timelike vector γ_0, relative vectors $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$. Satisfy

$$\begin{aligned} \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j &= \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij} \end{aligned}$$

Generators for a 3-d algebra!

This is GA of the 3-d relative space in rest frame of γ_0.

Volume element

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I$$

so 3-d subalgebra shares same pseudoscalar as spacetime.

Still have

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \epsilon_{ijk} I \boldsymbol{\sigma}_k$$

relative vectors and relative bivectors are spacetime bivectors.

Projected onto the even subalgebra of the STA.

$$\begin{array}{ccccccccc}
1 & \cdots & \{\gamma_\mu\} & \cdots & \{\boldsymbol{\sigma}_i, I\boldsymbol{\sigma}_i\} & \cdots & \{I\gamma_\mu\} & \cdots & I & \quad 4-d \\
1 & & \{\boldsymbol{\sigma}_i\} & & & & \{I\boldsymbol{\sigma}_i\} & & I & \quad 3-d
\end{array}$$

The 6 spacetime bivectors split into relative vectors and relative bivectors. This split is observer dependent. A very useful technique.

Conventions

Expression like $\boldsymbol{a}\wedge\boldsymbol{b}$ potentially confusing.

- Spacetime bivectors used as relative vectors are written in bold. Includes the $\{\boldsymbol{\sigma}_i\}$.
- If both arguments bold, dot and wedge symbols drop down to their 3-d meaning.
- Otherwise, keep spacetime definition.

Important 4-Vectors

i. Velocity

Observer, with constant velocity v. Measures relative velocity of a particle with proper velocity $u(\tau)$, $u^2 = 1$. Form

$$uv = \partial_\tau [x(\tau)v] = \partial_\tau (t + \boldsymbol{x})$$

So that

$$\partial_\tau t = u \cdot v, \quad \partial_\tau \boldsymbol{x} = u \wedge v$$

The relative velocity is

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{u \wedge v}{u \cdot v}$$

Lorentz factor $\gamma = (1 - \boldsymbol{u}^2)^{-1/2}$ is just $u \cdot v$:

$$\gamma^{-2} = 1 + (u \cdot v)^{-2} [(uv - u \cdot v)(vu - v \cdot u)] = (u \cdot v)^{-2}$$

Can then write

$$u = uvv = (u \cdot v + u \wedge v)v = \gamma(1 + \boldsymbol{u})v$$

ii. Momentum and Wave Vectors

For particle of mass m and velocity u, momentum 4-vector is $p = mu$. Observer with velocity v measures energy E and relative momentum $\boldsymbol{p}$:

$$E = \gamma m = p \cdot v, \qquad \boldsymbol{p} = \gamma m \boldsymbol{u} = p \wedge v$$

Follows that

$$pv = p\cdot v + p\wedge v = E + \boldsymbol{p}$$

Recover the invariant

$$m^2 = p^2 = pvvp = (E+\boldsymbol{p})(E-\boldsymbol{p}) = E^2 - \boldsymbol{p}^2$$

Similarly, for a photon wave-vector k,

$$kv = k\cdot v + k\wedge v = \omega + \boldsymbol{k}$$

with frequency ω and relative wave-vector $\boldsymbol{k}$. For photons in empty space $k^2 = 0$ so

$$0 = kvvk = (\omega+\boldsymbol{k})(\omega-\boldsymbol{k}) = \omega^2 - \boldsymbol{k}^2$$

Recovers $|\boldsymbol{k}| = \omega$. Holds in all frames.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 8

SUMMARY

In this lecture we will concentrate on the rotor representation of Lorentz transformations. An analogy with rigid-body mechanics leads to a new rotor-based technique for analysing the relativistic equations of motion of a point particle.

- Handling Lorentz transformations with rotors.
- Doppler shifts and aberration.
- Fixed points and the celestial sphere.
- Pure boosts and acceleration as a bivector.

Lorentz Transformations

Usually expressed as a coordinate transformation, *e.g.*

$$\mathrm{x}' = \gamma(\mathrm{x} - \beta t) \qquad t' = \gamma(t - \beta \mathrm{x})$$
$$\mathrm{x} = \gamma(\mathrm{x}' + \beta t') \qquad t = \gamma(t' + \beta \mathrm{x}')$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and β is scalar velocity.

Vector x decomposed in two frames, $\{e_\mu\}$ and $\{e'_\mu\}$,

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu$$

with

$$t = e^0 \cdot x, \qquad t' = e^{0\prime} \cdot x$$

Concentrating on the 0, 1 components:

$$t e_0 + \mathrm{x} e_1 = t' e'_0 + \mathrm{x}' e'_1$$

Derive vector relations

$$e'_0 = \gamma(e_0 + \beta e_1), \quad e'_1 = \gamma(e_1 + \beta e_0)$$

Gives new frame in terms of the old.

Hyperbolic Geometry

Introduce 'hyperbolic angle' α,

$$\tanh\alpha = \beta, \qquad (\beta < 1)$$

Gives

$$\gamma = (1 - \tanh^2\alpha)^{-1/2} = \cosh\alpha$$

Vector e_0' is now

$$\begin{aligned} e_0' &= \mathrm{ch}(\alpha)e_0 + \mathrm{sh}(\alpha)e_1 \\ &= [\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)e_1e_0]e_0 = \mathrm{e}^{\alpha\, e_1 e_0}\, e_0 \end{aligned}$$

Similarly, we have

$$e_1' = \mathrm{ch}(\alpha)e_1 + \mathrm{sh}(\alpha)e_0 = \mathrm{e}^{\alpha\, e_1 e_0}\, e_1$$

Two other frame vectors unchanged. Since e_2 and e_3 commute with e_1e_0, relationship between the frames is

$$e_\mu' = Re_\mu\tilde{R}, \quad e^{\mu\prime} = Re^\mu\tilde{R}, \quad R = \mathrm{e}^{\alpha\, e_1 e_0/2}$$

Same rotor prescription works for boosts as well as rotations! Spacetime is a unified entity now.

The Restricted Lorentz Group

Transformation $a \mapsto Ra\tilde{R}$ preserves a^2. Also preserves causal ordering, so that time order of causally connected events is Lorentz invariant. If we take

$$\Delta x = \Delta t \gamma_0 \mapsto \Delta x' = \Delta t R \gamma_0 \tilde{R}$$

need the γ_0 component of $\Delta x'$ to have same sign as Δt, i.e. for $\Delta t > 0$,

$$\gamma_0 \cdot \Delta x' = \Delta t \langle \gamma_0 R \gamma_0 \tilde{R} \rangle > 0$$

Decomposing in γ_0 frame

$$R = \alpha + \boldsymbol{a} + I\boldsymbol{b} + I\beta$$

find that

$$\langle \gamma_0 R \gamma_0 \tilde{R} \rangle = \alpha^2 + \boldsymbol{a}^2 + \boldsymbol{b}^2 + \beta^2 > 0$$

as required.

- Rotor transformation law $a \mapsto Ra\tilde{R}$ defines the restricted Lorentz group.
- Physically most relevant operations.
- Full Lorentz group includes reflections and inversions.

EXAMPLES

i. Addition of Velocities

Two objects separating, velocities

$$v_1 = \mathrm{e}^{\alpha_1 e_1 e_0}\, e_0, \qquad v_2 = \mathrm{e}^{-\alpha_2 e_1 e_0}\, e_0$$

What is the relative velocity they see for each other? Form

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{\langle \mathrm{e}^{(\alpha_1 + \alpha_2) e_1 e_0} \rangle_2}{\langle \mathrm{e}^{(\alpha_1 + \alpha_2) e_1 e_0} \rangle_0} = \frac{\sinh(\alpha_1 + \alpha_2) e_1 e_0}{\cosh(\alpha_1 + \alpha_2)}$$

Both observers measure relative velocity

$$\tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 - \tanh\alpha_1 \tanh\alpha_2}$$

Addition of velocities is achieved by adding hyperbolic angles. Recovers familiar formula.

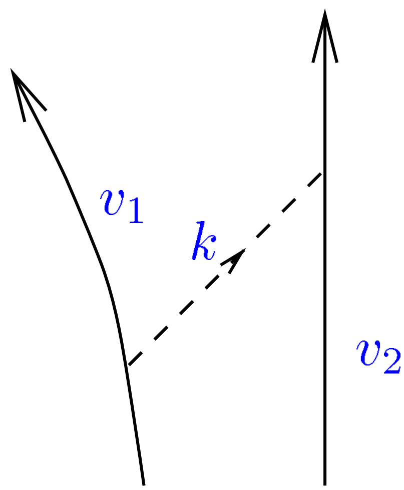
ii. Photons and Redshifts

Two particles on different worldlines. Particle 1 emits a photon, received by particle 2

Frequency for particle 1 is $\omega_1 = v_1 \cdot k$, for particle 2 is $\omega_2 = v_2 \cdot k$.

Ratio describes the Doppler effect, often expressed as a redshift:

$$1 + z = \omega_1 / \omega_2$$



Can be applied in many ways. If emitter receding in e_1 direction, and $v_2 = e_0$, have

$$k = \omega_2(e_0 + e_1), \qquad v_1 = \cosh\alpha\, e_0 - \sinh\alpha\, e_1$$

so that

$$1 + z = \frac{\omega_2(\cosh\alpha + \sinh\alpha)}{\omega_2} = \mathrm{e}^{\alpha}$$

Velocity of emitter in e_0 frame is $\tanh\alpha$, and

$$\mathrm{e}^{\alpha} = \left(\frac{1 + \tanh\alpha}{1 - \tanh\alpha}\right)^{1/2}$$

Aberration formulae obtained same way.

Invariant Decomposition

Restricted Lorentz transformation $a \mapsto Ra\tilde{R}$. Every spacetime rotor can be written as

$$R = \pm\, \mathrm{e}^{B/2}$$

The minus sign rarely needed. Can find Lorentz invariant decomposition. Write

$$B^2 = \langle B^2 \rangle_0 + \langle B^2 \rangle_4 = \rho\, \mathrm{e}^{I\phi}$$

(assume $\rho \neq 0$). Define

$$\hat{B} = \rho^{-1/2}\, \mathrm{e}^{-I\phi/2}\, B$$

So that

$$\hat{B}^2 = \rho^{-1}\, \mathrm{e}^{-I\phi}\, B^2 = 1$$

Now have

$$B = \rho^{1/2}\, \mathrm{e}^{I\phi/2}\, \hat{B} = \alpha\hat{B} + \beta I\hat{B}$$

Since

$$\hat{B}\, I\hat{B} = I\hat{B}\, \hat{B} = I$$

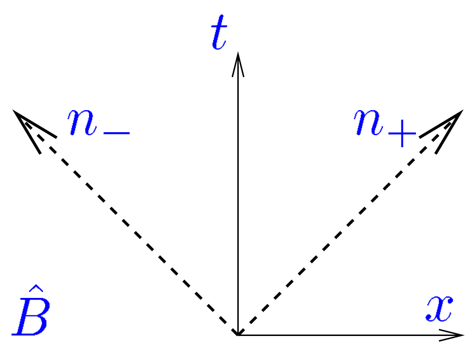
Have commuting blades $\alpha\hat{B}$, $\beta I\hat{B}$. Write

$$R = \, \mathrm{e}^{\alpha\hat{B}/2}\, \mathrm{e}^{\beta I\hat{B}/2} \, = \, \mathrm{e}^{\beta I\hat{B}/2}\, \mathrm{e}^{\alpha\hat{B}/2}$$

Invariant split into a boost and a rotation.

Fixed Points

Timelike bivector $\hat{B}$, $\hat{B}^2 = 1$, has two null vectors $n_\pm$.



Satisfy (exercise)

$$\hat{B}\cdot n_\pm = \pm n_\pm$$

Necessarily null, since

$$(\hat{B}\cdot n_\pm)\cdot n_\pm = 0 = \pm n_\pm^2$$

$n_\pm$ chosen so that

$$n_+ \wedge n_- = 2\hat{B}$$

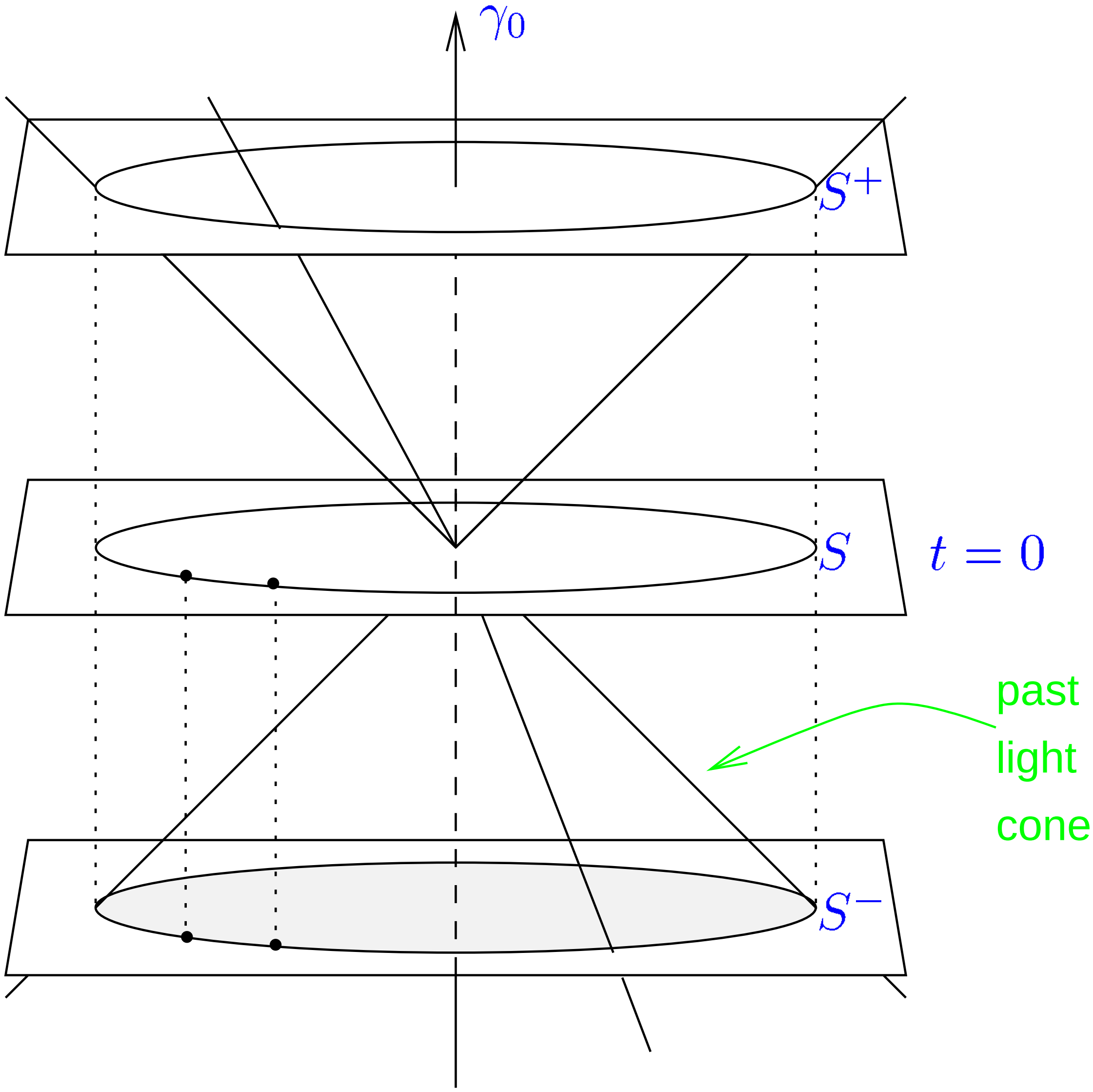
Form a null basis for $\hat{B}$ plane. $n_\pm$ anticommute with $\hat{B}$ since

$$\hat{B}\wedge(\hat{B}\cdot n_\pm) = 0 = \pm\hat{B}\wedge n_\pm$$

So $n_\pm$ commute with $I\hat{B}$. Simply scale under Lorentz transformation:

$$Rn_\pm\tilde{R} = \mathrm{e}^{\alpha\hat{B}/2}\, n_\pm\, \mathrm{e}^{-\alpha\hat{B}/2} = \mathrm{e}^{\alpha\hat{B}}\, n_\pm$$
$$= \mathrm{ch}(\alpha) n_\pm + \mathrm{sh}(\alpha)\hat{B}\cdot n_\pm = \mathrm{e}^{\pm\alpha}\, n_\pm$$

THE CELESTIAL SPHERE



Visualise Lorentz transformations through effect on the past light sphere — the celestial sphere S^-.

Observer γ_0 receives light along null vector n. Form relative vector $n\wedge\gamma_0$, with

$$(n\wedge\gamma_0)^2 = (n\cdot\gamma_0)^2 - n^2\gamma_0^2 = (n\cdot\gamma_0)^2$$

Form projective unit relative vector $\boldsymbol{n}$

$$\boldsymbol{n} = n\wedge\gamma_0/n\cdot\gamma_0$$

Maps all past events onto a sphere. Second observer $v = R\gamma_0\tilde{R}$ forms vectors $n\wedge v/n\cdot v$. Transform back to γ_0 frame for comparison:

$$\boldsymbol{n}' = \tilde{R}\frac{n\wedge v}{n\cdot v}R = \frac{n'\wedge\gamma_0}{n'\cdot\gamma_0}$$

$n' = \tilde{R}nR$. See effect by moving points on sphere with $n \mapsto \tilde{R}nR$. Two fixed points.

Pure Boosts and Observer Splits

Velocity u boosted v. Need rotor with no additional rotation component:

$$v = Lu\tilde{L}$$

$La_\perp\tilde{L} = a_\perp$ for $a_\perp$ outside $u\wedge v$. Bivector generator is multiple of $u\wedge v$. Anticommutes with u and v, so

$$v = Lu\tilde{L} = L^2u \qquad \Rightarrow L^2 = vu$$

Solution is (check!)

$$L = \frac{1+vu}{[2(1+u\cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge u}{|v\wedge u|}\right)$$

where $\mathrm{ch}(\alpha) = u\cdot v$.

Now take arbitrary rotor R. Decompose in γ_0 frame, so $R\gamma_0\tilde{R} = v$. Pure boost is

$$L = \frac{1 + v\gamma_0}{[2(1 + v\cdot\gamma_0)]^{1/2}} = \exp\left(\frac{\alpha}{2} \frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right)$$

$v\cdot\gamma_0 = \operatorname{ch}(\alpha)$. Define rotor U,

$$U = \tilde{L}R, \qquad U\tilde{U} = 1$$

Satisfies

$$U\gamma_0\tilde{U} = \tilde{L}vL = \tilde{L}L\gamma_0\tilde{L}L = \gamma_0$$

so $U\gamma_0 = \gamma_0 U$, and $U = e^{I\boldsymbol{b}/2}$ — a pure rotation in γ_0 frame.

$$R = LU$$

Frame dependent decomposition. Do not commute.

Spacetime Rotor Equations

Trajectory $x(\tau)$, future-pointing velocity $v = \partial_\tau x, v^2 = 1$. Cf rigid-body dynamics. Write

$$v = R\gamma_0\tilde{R}$$

Put dynamics in rotor R! Compute acceleration

$$\dot{v} = \partial_\tau(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}$$

$\dot{R}\tilde{R}$ is a bivector. Have

$$\dot{v} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R})\cdot v$$

Consistent with $v^2 = 1 \Rightarrow v\cdot\dot{v} = 0$. Now have

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot vv$$

$\dot{v}v$ is acceleration bivector — $\dot{v}$ projected into instantaneous rest frame. Determines $\dot{R}\tilde{R}$ up to a bivector orthogonal to v. Remaining freedom describes pure rotation in instantaneous rest frame.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 9

SUMMARY

In this lecture we use the rotor representation of Lorentz transformations to simplify relativistic dynamics. We borrow an idea from rigid-body mechanics to produce a new, rotor-based technique for analysing the relativistic equations of motion of a point particle.

- Pure boosts and acceleration as a bivector.
- Relativistic equations of motion for a point particle described by rotors.
- The Lorentz force law and the Faraday bivector.
- Point particle in a constant field.
- The gyromagnetic ratio.
- Thomas Precession.

Spacetime Rotor Equations

Trajectory $x(\tau)$, future-pointing velocity $v = \partial_\tau x, v^2 = 1$. Cf rigid-body dynamics. Write

$$v = R\gamma_0\tilde{R}$$

Put dynamics in rotor R! Compute acceleration

$$\dot{v} = \partial_\tau(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}$$

$\dot{R}\tilde{R}$ is a bivector. Have

$$\dot{v} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R})\cdot v$$

Consistent with $v^2 = 1 \Rightarrow v\cdot\dot{v} = 0$. Now have

$$\dot{v}\wedge v = \dot{v}v = 2(\dot{R}\tilde{R})\cdot vv$$

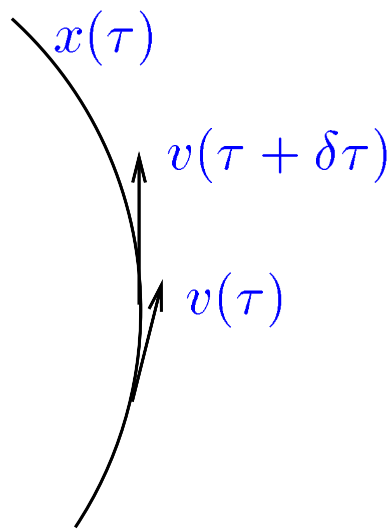
$\dot{v}v$ is acceleration bivector — $\dot{v}$ projected into instantaneous rest frame. Determines $\dot{R}\tilde{R}$ up to a bivector orthogonal to v. Remaining freedom describes pure rotation in instantaneous rest frame.

Fermi Transport

Often useful to fix R so describes pure boost from one instant to next. Rotated vectors

$$e_i = R\gamma_i\tilde{R}$$

are orthogonal to v so span instantaneous rest space. Then $\{e_i\}$ are Fermi transported — “constant as possible” with constraint $e_i \cdot v = 0$. Physically e_i can represent direction of inertial guidance gyroscopes (supported at c.o.m.).



To first order

$$v(\tau+\delta\tau) = v(\tau) + \delta\tau\,\dot{v}$$

Proper boost between $v(\tau)$ and $v(\tau+\delta\tau)$ is

$$L = \frac{1+v(\tau+\delta\tau)v(\tau)}{[2(1+v(\tau+\delta\tau)\cdot v(\tau))]^{1/2}} = 1 + \tfrac{1}{2}\delta\tau\,\dot{v}v$$

But since

$$R(\tau + \delta\tau) = R(\tau) + \delta\tau \dot{R}(\tau) = (1 + \delta\tau \dot{R}\tilde{R})R(\tau)$$

rotor taking v and $\{e_i\}$ from τ to $\tau + \delta\tau$ is $1 + \delta\tau \dot{R}R$.
Fermi transport requires

$$\dot{R}\tilde{R} = \frac{1}{2}\dot{v}v$$

Acceleration bivector is generator for R.

e_i transported as

$$\dot{e}_i = 2(\dot{R}\tilde{R})\cdot e_i = -e_i\cdot(\dot{v}v)$$

For general vector $a(\tau)$, Fermi derivative is

$$\dot{a} + a\cdot(\dot{v}v)$$

Zero for Fermi transport — preserves a^2 and $a\cdot v$. If $a\cdot v = 0$, $\dot{a}\cdot v + a\cdot\dot{v} = 0$ so

$$\dot{a} + a\cdot(\dot{v}v) = \dot{a} - \dot{a}\cdot vv = \dot{a}\wedge vv$$

Projection of $\dot{a}$ perpendicular to v.

The Lorentz Force Law

Familiar with non-relativistic form

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

All relative vectors in γ_0 frame, $\boldsymbol{E} = E_i \boldsymbol{\sigma}_i$ *etc*. Want relativistic version of law. Have

$$\boldsymbol{p} = p \wedge \gamma_0, \quad \dot{t} = v \cdot \gamma_0$$

$\times$ through by $v \cdot \gamma_0$. Get $\dot{p} \wedge \gamma_0$ on left. On right need

$$\begin{aligned} v \cdot \gamma_0 \, \boldsymbol{E} &= (\boldsymbol{E} \cdot v) \wedge \gamma_0 \\ -v \cdot \gamma_0 \, \boldsymbol{v} \cdot (I\boldsymbol{B}) &= [(I\boldsymbol{B}) \cdot v] \wedge \gamma_0 \end{aligned}$$

(Proof in handouts.) Now have

$$\frac{d\boldsymbol{p}}{d\tau} = \dot{p} \wedge \gamma_0 = q[(\boldsymbol{E} + I\boldsymbol{B}) \cdot v] \wedge \gamma_0$$

Define Faraday bivector F

$$F = \boldsymbol{E} + I\boldsymbol{B}$$

The covariant electromagnetic field strength. More next time!

Now have

$$\dot{p} \wedge \gamma_0 = q(F \cdot v) \wedge \gamma_0$$

Rate of change of energy $p_0 = p\cdot\gamma_0$ in γ_0 frame

$$\frac{dp_0}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}$$

Multiply by $v\cdot\gamma_0$ again

$$\dot{p}\cdot\gamma_0 = q\boldsymbol{E}\cdot(v\wedge\gamma_0) = q(F\cdot v)\cdot\gamma_0$$

Used $(I\boldsymbol{B})\cdot(v\wedge\gamma_0) = 0$. Combine with force law

$$\dot{p}\cdot\gamma_0 + \dot{p}\wedge\gamma_0 = \dot{p}\gamma_0 = q(F\cdot v)\gamma_0$$

With $p = mv$, get relativistic form of Lorentz force law,

$$m\dot{v} = qF\cdot v$$

Manifestly Lorentz covariant. Acceleration bivector is

$$\dot{v}v = \frac{q}{m}F\cdot v\, v = \frac{q}{m}(F\cdot v)\wedge v = \frac{q}{m}\boldsymbol{E}_v$$

where $\boldsymbol{E}_v$ is relative electric field in the v frame.

Now use $v = R\gamma_0\tilde{R}$,

$$\dot{v} = 2(\dot{R}\tilde{R})\cdot v = \frac{q}{m}F\cdot v$$

Simplest possibility is to equate projected terms

$$\dot{R} = \frac{q}{2m}FR$$

Does not give Fermi transport of $\{R\gamma_i\tilde{R}\}$, but sufficient to determine v and trajectory.

Constant Field

Easy now! Integrate rotor equation

$$R = \exp(\frac{q}{2m} F\tau)$$

Now do invariant decomposition of F

$$F^2 = \langle F^2\rangle_0 + \langle F^2\rangle_4 = \rho\, \mathrm{e}^{I\beta}$$

so that

$$F = \rho^{1/2}\, \mathrm{e}^{I\beta/2}\, \hat{F} = \alpha\hat{F} + I\beta\hat{F}$$

where $\hat{F}^2 = 1$. (For null F use different procedure). Have

$$R = \exp(\frac{q}{2m}\alpha\hat{F}\tau)\exp(\frac{q}{2m}I\beta\hat{F}\tau)$$

Now decompose initial velocity $v_0 = \gamma_0$

$$v_0 = \hat{F}^2 v_0 = \hat{F}\,\hat{F}\cdot v_0 + \hat{F}\,\hat{F}\wedge v_0 = v_{0||} + v_{0\perp}$$

$v_{0||} = \hat{F}\,\hat{F}\cdot v_0$ anticommutes with $\hat{F}$, $v_{0\perp}$ commutes, so

$$\dot{x} = \exp(\frac{q}{m}\alpha\hat{F}\tau)v_{0||} + \exp(\frac{q}{m}I\beta\hat{F}\tau)v_{0\perp}$$

Now integrate to get the particle history

$$x - x_0 = \frac{\mathrm{e}^{q\alpha\hat{F}\tau/m} - 1}{q\alpha/m}\hat{F}\cdot v_0 - \frac{\mathrm{e}^{q\beta I\hat{F}\tau/m} - 1}{q\beta/m}(I\hat{F})\cdot v_0$$

$\hat{F} \Rightarrow$ linear acceleration, $I\hat{F} \Rightarrow$ rotation

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 10

SUMMARY

In this lecture we study the gyromagnetic ratio and Thomas precession. We then turn to the application of the STA to electromagnetism. This is one of the most compelling applications of geometric algebra. The geometric product unites all four Maxwell equations into a single equation, with many advantages.

- The gyromagnetic ratio.
- Thomas Precession.
- The electromagnetic field strength and observers in relative motion.
- The four Maxwell equations united into one.

The Gyromagnetic Ratio

In non-relativistic physics, spin vector $\boldsymbol{s}$ precesses in $\boldsymbol{B}$ by

$$\frac{d\boldsymbol{s}}{dt} = g\frac{q}{2m}\boldsymbol{s}\times\boldsymbol{B}$$

g is gyromagnetic ratio. Want to extend this to relativistic domain. Introduce spin vector s, with $s\cdot v = 0$. Spin interacts only with magnetic field in rest frame.

For particle at rest in γ_0, $\boldsymbol{s} = s\gamma_0$ and

$$\boldsymbol{s}\times\boldsymbol{B} = -\boldsymbol{s}\cdot(I\boldsymbol{B}) = -(s\gamma_0)\times(I\boldsymbol{B})$$

Now recall $F = \boldsymbol{E} + I\boldsymbol{B}$, so

$$I\boldsymbol{B} = \tfrac{1}{2}(F + \gamma_0 F\gamma_0)$$

Expanding out $-(s\gamma_0)\times(I\boldsymbol{B})$ get

$$\begin{aligned}
& \tfrac{1}{4}\big((F+\gamma_0 F\gamma_0)s\gamma_0 - s\gamma_0(F+\gamma_0 F\gamma_0)\big) \\
= \quad & \tfrac{1}{4}\big((Fs - sF)\gamma_0 - \gamma_0(Fs - sF)\big) \\
= \quad & \tfrac{1}{2}\big((F\cdot s)\gamma_0 - \gamma_0(F\cdot s)\big) \\
= \quad & (F\cdot s)\wedge\gamma_0
\end{aligned}$$

So our equation can be written

$$\dot{s} = g\frac{q}{2m}(F\cdot s)\wedge\gamma_0\,\gamma_0$$

COVARIANT FORM

For covariant form, first replace γ_0 by v. On left-hand side must replace $\dot{s}$ by covariant operation which maintains $s\cdot v = 0$. This is the Fermi derivative again, which has

$$(\dot{s} + s\cdot\dot{v}\,v)\cdot v = \dot{s}\cdot v + s\cdot\dot{v} = \partial_\tau(s\cdot v) = 0$$

Get relativistic, covariant form of the spin precession:

$$\dot{s} + s\cdot\dot{v}\,v = g\frac{q}{2m}(F\cdot s)\wedge v\,v$$

Eliminate $\dot{v}$ with Lorentz force law and use $s\cdot(F\cdot v) = -(F\cdot s)\cdot v$ to arrive at

$$\dot{s} = \frac{q}{m}F\cdot s + (g-2)\frac{q}{2m}(F\cdot s)\wedge vv$$

For $g = 2$ (value for electron!) get simple equation

$$\dot{s} = \frac{q}{m}F\cdot s$$

Same form as Lorentz force law. Set $s = R\gamma_3\tilde{R}$. Recover motion and spin precession from single rotor equation

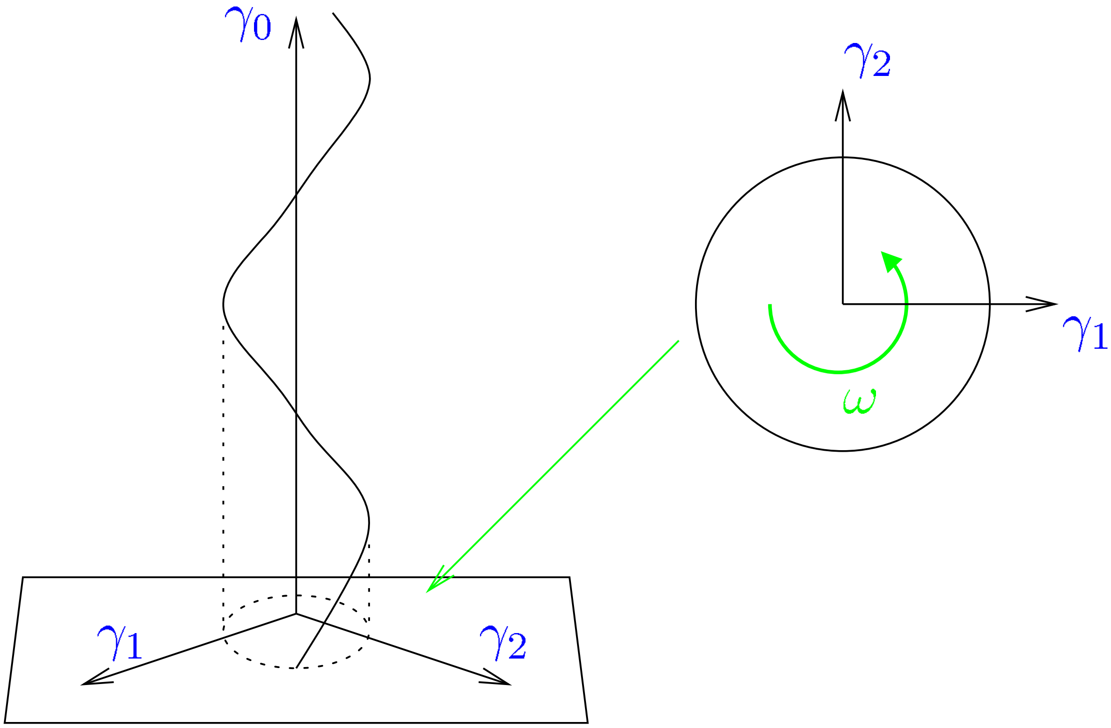
$$\dot{R} = \frac{q}{2m}FR$$

Extremely useful for point-particle relativistic models of electron behaviour.

Thomas Precession

Particle on circular orbit. Worldline

$$x(\tau) = t(\tau)\gamma_0 + a[\cos(\omega t)\gamma_1 + \sin(\omega t)\gamma_2]$$



Velocity is

$$v = \partial_\tau x = \dot{t}\,(\gamma_0 + a\omega[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2])$$

Relative velocity $\boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$ has $|\boldsymbol{v}| = a\omega$. Define

$$\tanh\alpha = a\omega, \qquad \dot{t} = \cosh\alpha$$

Velocity now

$$\begin{aligned} v &= \mathrm{ch}(\alpha)\gamma_0 + \mathrm{sh}(\alpha)[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2] \\ &= \mathrm{e}^{\alpha \boldsymbol{n}/2}\, \gamma_0 \, \mathrm{e}^{-\alpha \boldsymbol{n}/2} \end{aligned}$$

where

$$\boldsymbol{n} = -\sin(\omega t)\boldsymbol{\sigma}_1 + \cos(\omega t)\boldsymbol{\sigma}_2$$

A boost along $\boldsymbol{n}$ at each instant. Simplify with

$$\boldsymbol{n} = \mathrm{e}^{-\omega t I \boldsymbol{\sigma}_3}\, \boldsymbol{\sigma}_2 = R_\omega \boldsymbol{\sigma}_2 \tilde{R}_\omega, \quad R_\omega = \exp(-\omega t I \boldsymbol{\sigma}_3/2)$$

Gives

$$\mathrm{e}^{\alpha \boldsymbol{n}/2} = \exp(\alpha R_\omega \boldsymbol{\sigma}_2 \tilde{R}_\omega/2) = R_\omega \, \mathrm{e}^{\alpha \boldsymbol{\sigma}_2/2}\, \tilde{R}_\omega$$

Define $R_\alpha = \exp(\alpha \boldsymbol{\sigma}_2/2)$. Now have

$$v = R_\omega R_\alpha \tilde{R}_\omega \gamma_0 R_\omega \tilde{R}_\alpha \tilde{R}_\omega = R_\omega R_\alpha \gamma_0 \tilde{R}_\alpha \tilde{R}_\omega$$

Rotor for motion with Fermi transport must have form

$$R = R_\omega R_\alpha \Phi, \qquad \Phi = \exp(-\omega_T t I \boldsymbol{\sigma}_3/2)$$

Determine ω_T from $\dot{v}v$. Write

$$v = R_\omega v_\alpha \tilde{R}_\omega, \qquad v_\alpha = R_\alpha \gamma_0 \tilde{R}_\alpha$$

Get

$$\begin{aligned} \dot{v}v &= R_\omega [2(\tilde{R}_\omega \dot{R}_\omega)\cdot v_\alpha \, v_\alpha] \tilde{R}_\omega \\ &= \omega \,\mathrm{sh}(\alpha)\mathrm{ch}(\alpha) R_\omega [-\mathrm{ch}(\alpha)\boldsymbol{\sigma}_1 + \mathrm{sh}(\alpha) I \boldsymbol{\sigma}_3] \tilde{R}_\omega \end{aligned}$$

Also need $2\dot{R}\tilde{R}$, goes as

$$
\begin{aligned}
&2\dot{R}_\omega \tilde{R}_\omega + 2R_\omega R_\alpha \dot{\Phi}\tilde{\Phi}\tilde{R}_\alpha \tilde{R}_\omega \\
&= \mathrm{ch}(\alpha) R_\omega [-\omega I\boldsymbol{\sigma}_3 - \omega_T R_\alpha I\boldsymbol{\sigma}_3 \tilde{R}_\alpha]\tilde{R}_\omega \\
&= \mathrm{ch}(\alpha) R_\omega [-(\omega + \omega_T \mathrm{ch}(\alpha)) I\boldsymbol{\sigma}_3 + \omega_T \mathrm{sh}(\alpha)\boldsymbol{\sigma}_1]\tilde{R}_\omega
\end{aligned}
$$

So $\omega_T = -\mathrm{ch}(\alpha)\omega$. Full rotor is

$$
R = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2}\ \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\ \mathrm{e}^{\mathrm{ch}(\alpha)\omega t I\boldsymbol{\sigma}_3/2}
$$

$\omega_T \neq \omega \Rightarrow$ Thomas precession. Vector γ_1 Fermi transported round circle,

$$
e_1 = R\gamma_1\tilde{R}
$$

After time $t = 2\pi/\omega$, vector transformed to

$$
e_1(2\pi/\omega) = \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\ \mathrm{e}^{2\pi \mathrm{ch}(\alpha) I\boldsymbol{\sigma}_3}\ \gamma_1\, \mathrm{e}^{-\alpha\boldsymbol{\sigma}_2/2}
$$

Precessed through angle $\theta = 2\pi(\cosh\alpha - 1)$. Effect is order $|\boldsymbol{v}|^2/c^2$.

Maxwell's Equations

The four Maxwell equations are (natural units)

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B} \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{J} + \partial_t\boldsymbol{E}$$

These are Lorentz invariant. Want to make this apparent. Start by defining the vector derivative

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \frac{\partial}{\partial x_i} = \boldsymbol{\sigma}_i \partial_i$$

This inherits a geometric product as well! Write the $\boldsymbol{E}$ equations as

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho, \quad \boldsymbol{\nabla}\wedge\boldsymbol{E} = -\partial_t(I\boldsymbol{B})$$

and add them

$$\boldsymbol{\nabla}\boldsymbol{E} = \rho - \partial_t(I\boldsymbol{B})$$

The $\boldsymbol{\nabla}$ in $\boldsymbol{\nabla}\boldsymbol{E}$ has an inverse — a Green's function. The Green's function is (non-examinable)

$$\frac{\boldsymbol{x}-\boldsymbol{x}_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|^3}$$

You will find this all over your IB electromagnetism notes!

B-Field Equations

Similarly, have

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad \boldsymbol{\nabla}\wedge\boldsymbol{B} = I(\boldsymbol{J} + \partial_t \boldsymbol{E})$$

So have

$$\boldsymbol{\nabla}\boldsymbol{B} = I(\boldsymbol{J} + \partial_t \boldsymbol{E})$$

Convert to vector + trivector equations

$$\boldsymbol{\nabla}(I\boldsymbol{B}) = -(\boldsymbol{J} + \partial_t \boldsymbol{E})$$

Now combine all 8 equations into the single multivector equation

$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) + \partial_t(\boldsymbol{E} + I\boldsymbol{B}) = \rho - \boldsymbol{J}$$

No information lost in writing this! Introduce the electromagnetic field strength

$$F = \boldsymbol{E} + I\boldsymbol{B}$$

This is a relativistic bivector have

$$\boldsymbol{\nabla}F + \partial_t F = \rho - \boldsymbol{J}$$

Covariant Form

Introduce spacetime current J

$$\rho = J \cdot \gamma_0 \quad \boldsymbol{J} = J \wedge \gamma_0$$

so that

$$\rho - \boldsymbol{J} = \gamma_0 \cdot J + \gamma_0 \wedge J = \gamma_0 J$$

Now have

$$\gamma_0 (\partial_t + \boldsymbol{\nabla}) F = J$$

Differential operator is the spacetime vector derivative

$$\gamma_0 \partial_t + \gamma_0 \gamma_i \gamma_0 \partial_i = \gamma^0 \partial_t + \gamma^i \partial_i = \gamma^\mu \partial_\mu = \nabla$$

Be careful that spacetime split goes as

$$\nabla \gamma_0 = (\gamma^0 \partial_t + \gamma^i \partial_i) \gamma_0 = \partial_t - \boldsymbol{\sigma}_i \partial_i = \partial_t - \boldsymbol{\nabla}$$

Result is the manifestly covariant equation

$$\boxed{\nabla F = J}$$

Separate parts have tensor equivalents

$$\nabla \cdot F = J \quad \leftrightarrow \quad \partial_\mu F^{\mu\nu} = J^\nu$$

$$\nabla \wedge F = 0 \quad \leftrightarrow \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$$

Only GA unites these.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 11

SUMMARY

In this lecture we continue to study the application of the STA to electromagnetism. We look at the properties of the electromagnetic field strength and how it transforms under changes of observer frame. We will also look at some simple solutions to the Maxwell equations.

- The electromagnetic field strength and observers in relative motion.
- The field due to a point charge — Coulomb and radiation fields.
- Charges in uniform and circular motion.
- Field energy, the Poynting vector and the stress-energy tensor.
- Dirac theory.

The Electromagnetic Field Strength

F is the electromagnetic field strength, or Faraday bivector.

Can write in terms of vector potential A,

$$F = \nabla \wedge A$$

so

$$\nabla \wedge F = \nabla \wedge (\nabla \wedge A) = \gamma^\mu \wedge \gamma^\nu \wedge \left(\frac{\partial^2 A}{\partial x^\mu \partial x^\nu} \right) = 0,$$

Can add $\nabla \phi$ to A — a gauge freedom. One choice is Lorentz condition $\nabla \cdot A = 0$, so

$$F = \nabla A, \quad \nabla F = \nabla^2 A = J$$

Recovers wave equation.

Tensor version is rank-2 antisymmetric tensor $F^{\mu\nu}$

$$F^{\mu\nu} = (\gamma^\nu \wedge \gamma^\mu) \cdot F$$

As a matrix, has components

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

Often see this, but it hides the natural complex structure.

Lorentz Transformations

Since $\gamma_0 F \gamma_0 = (-\boldsymbol{E} + I\boldsymbol{B})$, get $\boldsymbol{E}$ and $I\boldsymbol{B}$ from

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \\ I\boldsymbol{B} &= \tfrac{1}{2}(F + \gamma_0 F \gamma_0) \end{aligned}$$

Split into $\boldsymbol{E}$ and $I\boldsymbol{B}$ depends on observer velocity (γ_0).

Different observers measure different fields.

Second observer, velocity $v = R\gamma_0\tilde{R}$, has comoving frame

$$\gamma'_\mu = R\gamma_\mu\tilde{R}$$

Observer measures components of electric field

$$E'_i = (\gamma'_i\gamma'_0)\cdot F = (R\boldsymbol{\sigma}_i\tilde{R})\cdot F = \boldsymbol{\sigma}_i\cdot(\tilde{R}FR)$$

Same transformation law as for vectors. Very efficient.

Example

Stationary charges in γ_0 frame set up field

$$F = \boldsymbol{E} = E_x\boldsymbol{\sigma}_1 + E_y\boldsymbol{\sigma}_2$$

Second observer, velocity $\tanh\alpha$ in γ_1 direction, so boost is

$$R = \mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2}$$

Second observer measures $\boldsymbol{\sigma}_i$ components of

$$\tilde{R}FR = \mathrm{e}^{-\alpha\boldsymbol{\sigma}_1/2}\,F\,\mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2} = E_x\boldsymbol{\sigma}_1 + E_y\,\mathrm{e}^{-\alpha\boldsymbol{\sigma}_1}\,\boldsymbol{\sigma}_2$$

Gives

$$E'_x = E_x, \quad E'_y = \mathrm{ch}(\alpha)E_y, \quad B'_z = -\mathrm{sh}(\alpha)E_y$$

INVARIANTS

Construct the scalar + pseudoscalar

$$F^2 = \langle FF\rangle + \langle FF\rangle_4 = \alpha + I\beta$$

But

$$(\tilde{R}FR)(\tilde{R}FR) = \tilde{R}(\alpha + I\beta)R = \alpha + I\beta$$

Both are Lorentz invariant — independent of observer frame.

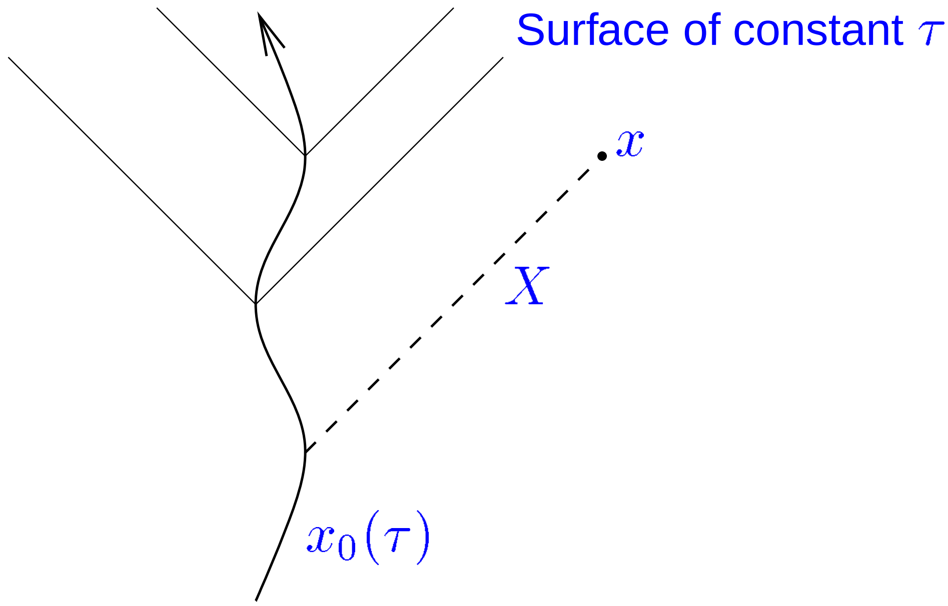
In γ_0 frame

$$\alpha = \langle(\boldsymbol{E} + I\boldsymbol{B})(\boldsymbol{E} + I\boldsymbol{B})\rangle = \boldsymbol{E}^2 - \boldsymbol{B}^2$$

$$\beta = -\langle I(\boldsymbol{E} + I\boldsymbol{B})(\boldsymbol{E} + I\boldsymbol{B})\rangle = 2\boldsymbol{E}\cdot\boldsymbol{B}$$

First is Lagrangian density. Second less common.

Point Charges



Point charge q, world-line $x_0(\tau)$. Observer at x. Influence from intersection of past light-cone with charge's worldline. Define

$$X \equiv x - x_0(\tau), \quad X^2 = 0$$

View τ as a field, value extended over the forward light cone.

Need Liénard-Wiechert potential. In rest frame of charge at retarded position

$$A \cdot v = \frac{q}{4\pi |X \wedge v|}, \quad A \wedge v = 0$$

where $v = \dot{x}_0$. As $|X \wedge v| = X \cdot v$, get

$$A = \frac{q}{4\pi} \frac{v}{X \cdot v}$$

FIELD STRENGTH

Differentiate equation $X^2 = 0$

$$\gamma^\mu(\partial_\mu X)\cdot X = \gamma^\mu\,(\gamma_\mu - \partial_\mu\tau\partial_\tau x_0)\cdot X$$
$$= X - \nabla\tau(v\cdot X) = 0$$

So

$$\nabla\tau = \frac{X}{X\cdot v}$$

Gradient of τ points in direction of constant τ! A peculiarity of null surfaces. Confirmed that τ is an adjunct field.

Also need

$$\nabla(X\cdot v) = \gamma^\mu(\partial_\mu X)\cdot v + \nabla\tau\, X\cdot(\partial_\tau v)$$
$$= v - \nabla\tau + \nabla\tau\, X\cdot\dot{v}$$

Find that (handout)

$$\nabla A = \frac{q}{4\pi}\left(\frac{X\wedge\dot{v}}{(X\cdot v)^2} + \frac{X\wedge v - X\cdot\dot{v}\, X\wedge v}{(X\cdot v)^3}\right)$$

Confirms that $\nabla\cdot A = 0$. Can write

$$F = \frac{q}{4\pi}\,\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}$$

where acceleration bivector $\Omega_v = \dot{v}\wedge v$.

First term is Coulomb field in rest frame. Second is radiation term,

$$F_{\text{rad}} = \frac{q}{4\pi} \frac{\frac{1}{2} X \Omega_v X}{(X \cdot v)^3}$$

The rest-frame acceleration projected down the null-vector X.

Uniform Motion

Charge with constant velocity v. Trajectory

$$x_0(\tau) = v\tau$$

passes through origin at $\tau = 0$. $X^2 = 0$ gives

$$\tau = v \cdot x - [(v \cdot x)^2 - x^2]^{1/2}$$

Gives intersection on past lightcone. Form $X \cdot v$,

$$X \cdot v = (x - v\tau) \cdot v = [(v \cdot x)^2 - x^2]^{1/2}$$

Can write this as $|x \wedge v|$ since

$$(x \wedge v)^2 = (X \wedge v)^2 = (X \cdot v)^2 - X^2 v^2 = (X \cdot v)^2$$

Acceleration bivector vanishes since v is constant. Gives Faraday bivector

$$F = \frac{q}{4\pi} \frac{x \wedge v}{|x \wedge v|^3}$$

F decomposes in γ_0 frame into $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. Need

$$\begin{aligned} x\wedge v &= \langle x\gamma_0\gamma_0 v\rangle_2 \\ &= \gamma\langle(t+\boldsymbol{x})(1-\boldsymbol{v})\rangle_2 = \gamma(\boldsymbol{x}-\boldsymbol{v}t) - \gamma\boldsymbol{x}\wedge\boldsymbol{v} \end{aligned}$$

With $F = \boldsymbol{E} + I\boldsymbol{B}$ get

$$\begin{aligned} \boldsymbol{E} &= \frac{q\gamma}{4\pi d^3}(\boldsymbol{x}-\boldsymbol{v}t) \\ \boldsymbol{B} &= \frac{q\gamma}{4\pi d^3} I\boldsymbol{x}\wedge\boldsymbol{v} \end{aligned}$$

where effective distance d is

$$d^2 = \gamma^2(|\boldsymbol{v}|t - \boldsymbol{v}\cdot\boldsymbol{x}/|\boldsymbol{v}|)^2 + \boldsymbol{x}^2 - (\boldsymbol{x}\cdot\boldsymbol{v})^2/\boldsymbol{v}^2$$

$\boldsymbol{E}$ points to current position!

Circular Motion

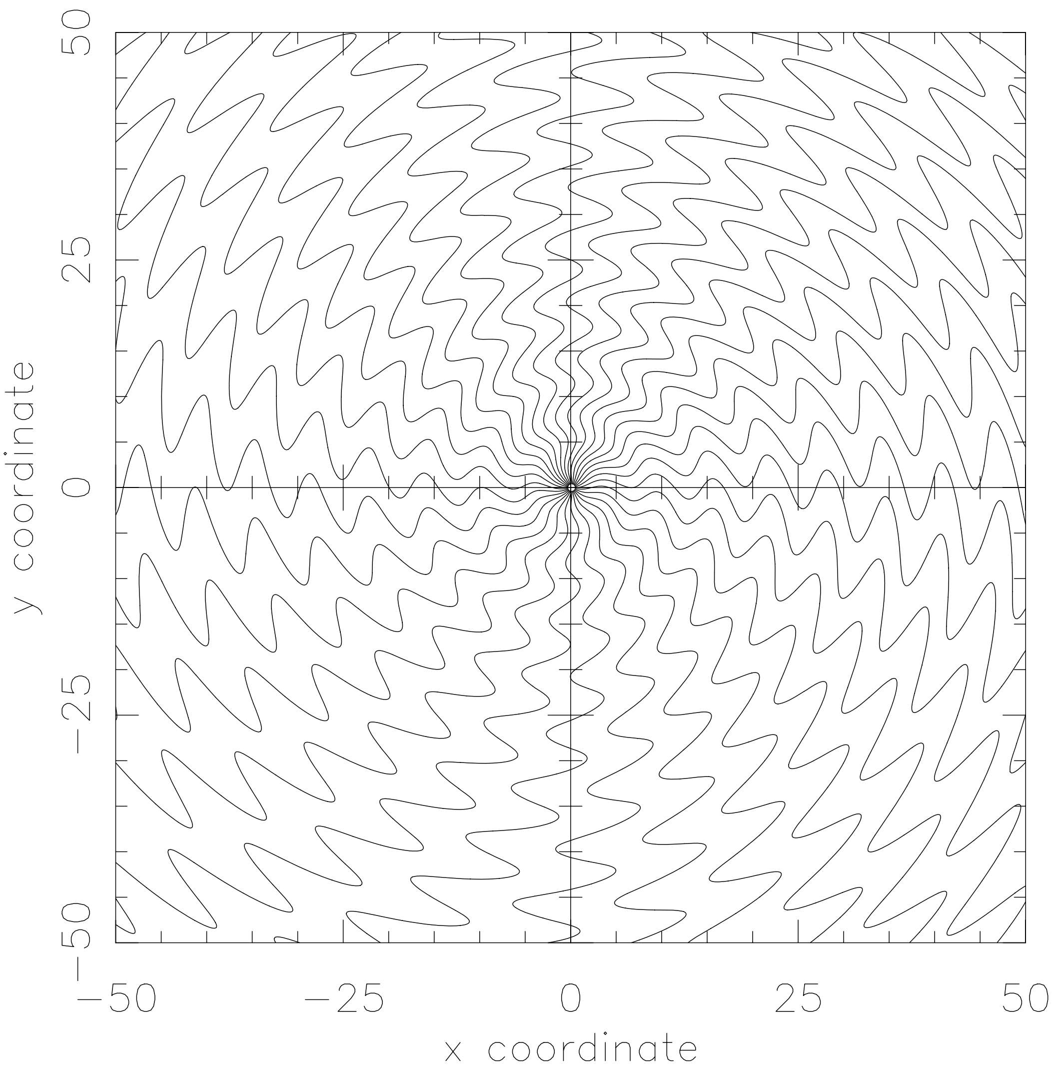
(Non-Examinable). Write trajectory

$$x_0(\tau) = \mathrm{ch}(\alpha)\tau\gamma_0 + a[\cos(\omega\tau)\gamma_1 + \sin(\omega\tau)\gamma_2]$$

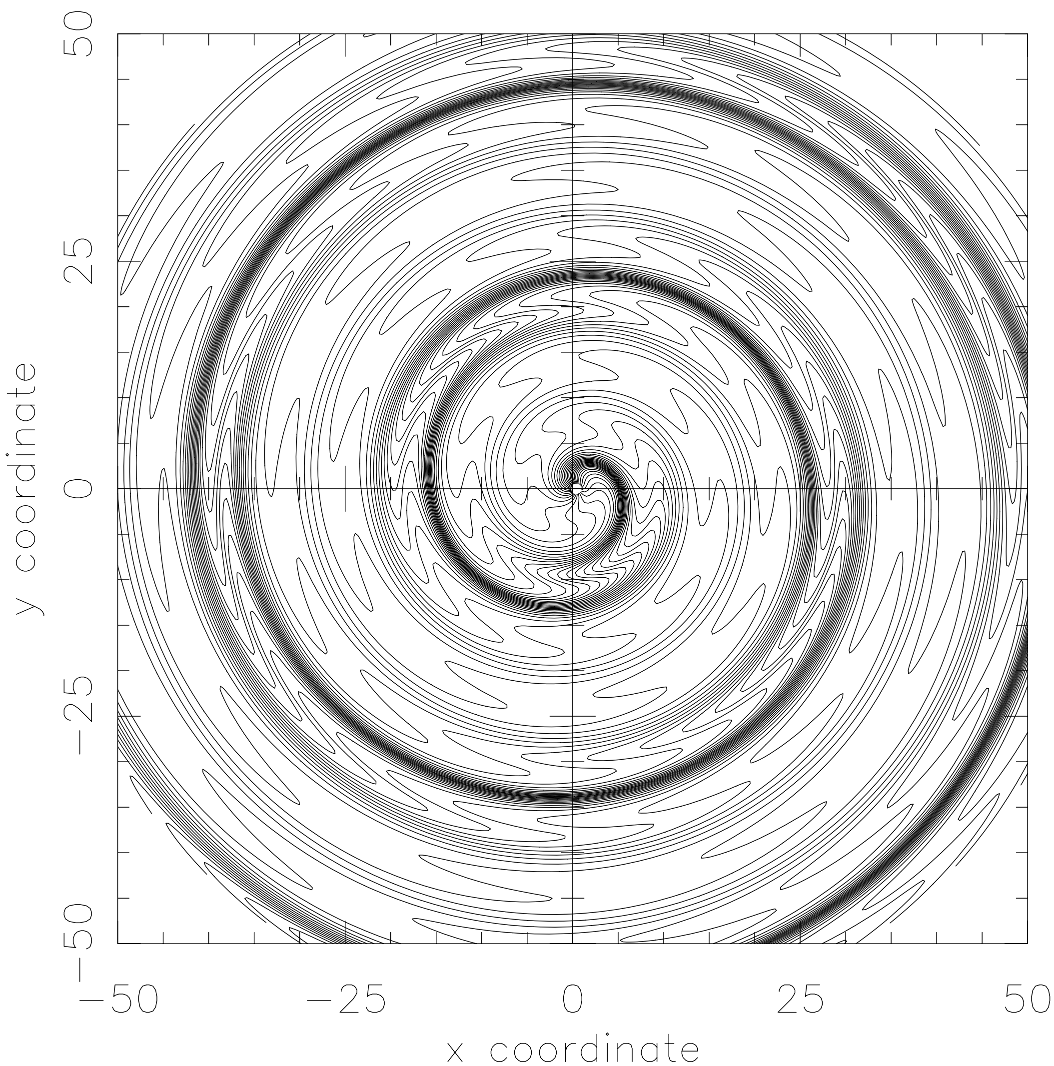
Angular speed ω refers to proper time. Find τ from $X^2 = 0$. Gives

$$t = \tau\mathrm{ch}(\alpha) + \surd\{\boldsymbol{x}^2 + a^2 - 2a[\mathrm{x}\cos(\omega\tau) + \mathrm{y}\sin(\omega\tau)]\}$$

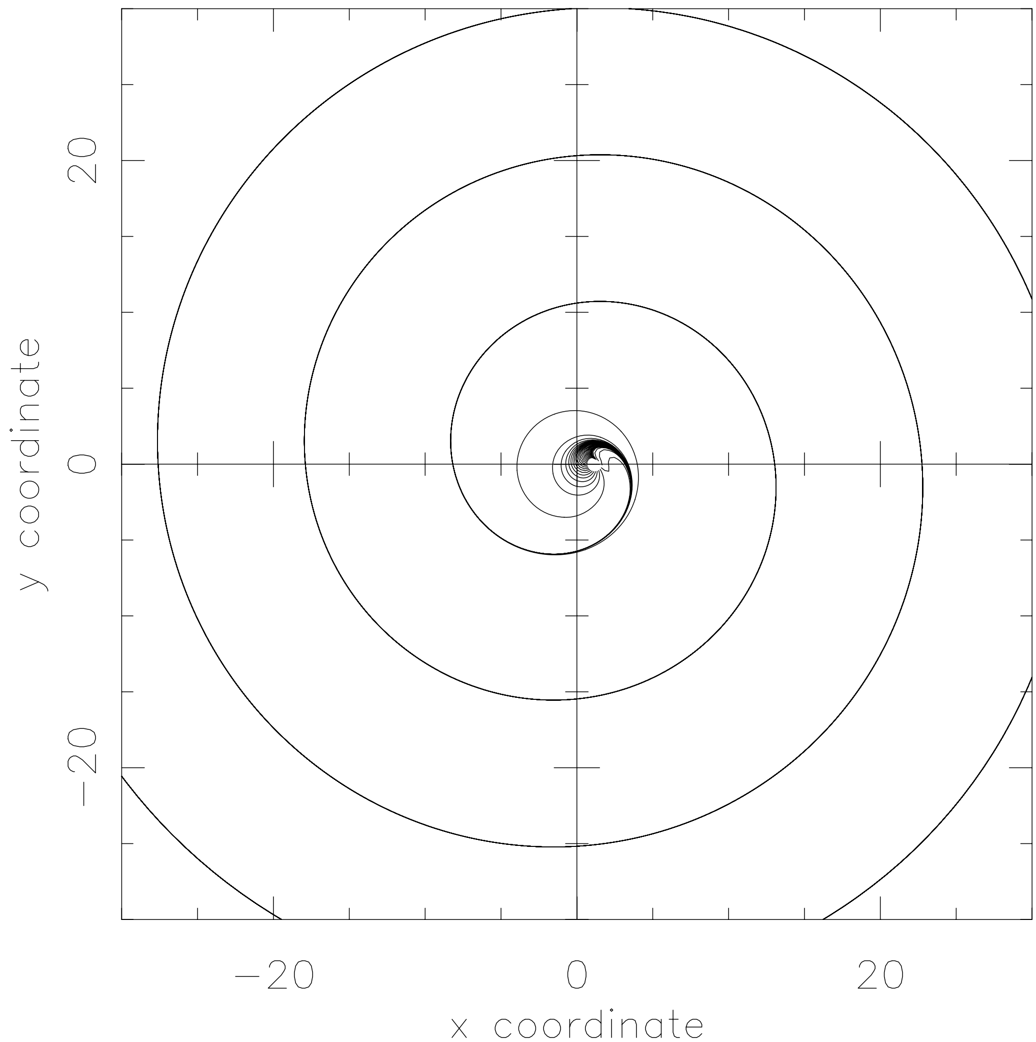
where $\boldsymbol{x} = \mathrm{x}\boldsymbol{\sigma}_1 + \mathrm{y}\boldsymbol{\sigma}_2 + \mathrm{z}\boldsymbol{\sigma}_3$. Implicit equation for τ. Solve numerically and plot field lines of $\boldsymbol{E}$.



$\alpha = 0.1$, velocity $\tanh(\alpha)$ is low. Get gentle wavy pattern.



Intermediate velocities, $\alpha = 0.4$. Complicated structure emerging. Field lines start to concentrate together.



Synchrotron Radiation. By $\alpha = 1$ field lines concentrate into pure synchrotron pulses. Radiation focussed in charge's direction of motion.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 12

SUMMARY

In this lecture we briefly study the concept of field energy and the stress-energy tensor with particular application to electromagnetism. We then turn to relativistic quantum theory and the Dirac equation. The rotor description of boosts and rotations makes it a simple matter to construct plane wave states.

- Field energy, the Poynting vector and the stress-energy tensor.
- Dirac spinors and the Dirac equation.
- The analogy with complex variables.
- Dirac observables.
- Plane wave states.

Field Momentum

The energy density in F is

$$\mathcal{E} = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$$

The momentum density is the Poynting vector

$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{E}\cdot(I\boldsymbol{B})$$

Should form a spacetime 4-vector

$$\begin{aligned}(\mathcal{E} + \boldsymbol{P})\gamma_0 &= \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\gamma_0 + \frac{1}{2}(I\boldsymbol{B}\boldsymbol{E} - \boldsymbol{E}I\boldsymbol{B})\gamma_0 \\ &= \frac{1}{2}(\boldsymbol{E} + I\boldsymbol{B})(\boldsymbol{E} - I\boldsymbol{B})\gamma_0 \\ &= \frac{1}{2}F(-\gamma_0 F\gamma_0)\gamma_0 = -\frac{1}{2}F\gamma_0 F\end{aligned}$$

Integrating this over space we form

$$P_{\text{tot}} = -\frac{1}{2}\int |d^3x|\, F\gamma_0 F$$

This is the total field momentum. For free fields it is independent of the choice of observer (or hypersurface) and so is constant.

To prove this write

$$P_{\text{tot}} = -\frac{1}{2}\int dA\, F n F$$

Integral over 3-space perpendicular to n.

Difference for two observers is

$$\Delta P_{\rm tot} = -\frac{1}{2}\int_{\partial V} dA\, F n F$$

A surface integral in 4D. For each component

$$\begin{aligned}\gamma_\mu\cdot(\Delta P_{\rm tot}) &= -\tfrac{1}{2}\int_{\partial V} dA\,\langle \gamma_\mu F n F\rangle \\ &= -\tfrac{1}{2}\int_{\partial V} dA\, n\cdot(F\gamma_\mu F)\end{aligned}$$

Use divergence theorem to convert to volume integral.

Integrand is

$$\nabla\cdot(F\gamma_\mu F) = \langle \nabla F\gamma_\mu F\rangle - \langle F\gamma_\mu(\nabla F)^\sim\rangle = 0$$

Used $\nabla F = 0$ for free fields.

Call $-\frac{1}{2}FaF$ the *stress-energy tensor* of the electromagnetic field. Write as

$$\mathsf{T}(a) = -\tfrac{1}{2}FaF$$

compare with the tensor form

$$\mathsf{T}^\mu{}_\nu = \tfrac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\alpha}F_{\alpha\nu}$$

STA version is far neater.

Properties of the Stress-Energy Tensor

- All relativistic fields, classical and quantum, have a stress-energy tensor $\mathsf{T}(a)$.
- $\mathsf{T}(a)$ gives the flux of 4-momentum across the hypersurface perpendicular to a.
- This flux is a conserved current

$$\nabla\cdot\mathsf{T}(\gamma_\mu) = 0 \qquad \mu = 0\ldots 3.$$

  Proved this for the electromagnetic case.
- Generalises the stress tensor of elasticity.
- $\mathsf{T}(a)$ is usually symmetric, *e.g.*

$$a\cdot\mathsf{T}(b) = -\tfrac{1}{2}\langle aFbF\rangle = -\tfrac{1}{2}\langle FaFb\rangle = \mathsf{T}(a)\cdot b$$

- $\mathsf{T}(a)$ has a non-symmetric contribution from quantum spin.
- $v\cdot\mathsf{T}(v)$ is positive for any timelike vector v. Matter not obeying this is exotic.
- The stress-energy tensor acts as a source of gravity.

Relativistic Quantum Spin

Relativistic quantum mechanics of spin-$1/2$ particles described by Dirac theory. The Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \hat{\gamma}_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $\hat{\gamma}_5 = -i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ and I is the 2×2 identity matrix.

Act on Dirac spinors. 4 complex components (8 real degrees of freedom).

Follow same procedure as Pauli case. Map spinors onto elements of the 8-dimensional even subalgebra of the STA. First write

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}$$

where $|\phi\rangle$ and $|\eta\rangle$ are 2-component spinors. Know how to represent the latter. Full map is simply

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\boldsymbol{\sigma}_3$$

Uses both Pauli-even and Pauli-odd terms.

Action of Dirac matrix operators become

$$
\begin{aligned}
\hat{\gamma}_\mu|\psi\rangle \quad &\leftrightarrow \quad \gamma_\mu\psi\gamma_0 \quad (\mu = 0, \dots, 3) \\
i|\psi\rangle \quad &\leftrightarrow \quad \psi\, I\boldsymbol{\sigma}_3 \\
\hat{\gamma}_5|\psi\rangle \quad &\leftrightarrow \quad \psi\boldsymbol{\sigma}_3
\end{aligned}
$$

Verification is routine computation. E.g.

$$
\begin{aligned}
\hat{\gamma}_k|\psi\rangle \quad &= \quad \begin{pmatrix} -\hat{\sigma}_k|\eta\rangle \\ \hat{\sigma}_k|\phi\rangle \end{pmatrix} \leftrightarrow -\sigma_k\eta\boldsymbol{\sigma}_3 + \sigma_k\phi\boldsymbol{\sigma}_3\boldsymbol{\sigma}_3 \\
&= \quad \gamma_k(\phi + \eta\boldsymbol{\sigma}_3)\gamma_0 = \gamma_k\psi\gamma_0
\end{aligned}
$$

Multiplication on right by $I\boldsymbol{\sigma}_3$ plays role of multiplication by i. This is all consistent:

$$
i\hat{\gamma}_\mu|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0 I\boldsymbol{\sigma}_3 = \gamma_\mu\psi I\boldsymbol{\sigma}_3\gamma_0 \leftrightarrow \hat{\gamma}_\mu i|\psi\rangle
$$

The Dirac Equation

Wavefunction is the spinor field $\psi(x)$. On left can only place scalars and pseudoscalars and ∇. Simplest possible equation is

$$
\boxed{\nabla\psi = 0}
$$

This is the wave equation for a neutrino.

Solutions decompose as

$$\psi = \psi \tfrac{1}{2}(1 + \boldsymbol{\sigma}_3) + \psi \tfrac{1}{2}(1 - \boldsymbol{\sigma}_3) = \psi_+ + \psi_-$$

Since post-multiplying $\nabla\psi = 0$ by $\frac{1}{2}(1 \pm \boldsymbol{\sigma}_3)$ gives

$$\nabla\psi = 0 \quad \implies \nabla\psi_\pm = 0$$

ψ_+ and ψ_- are right and left-handed helicity eigenstates. Important in electroweak theory.

Analytic Functions

In two dimensions have

$$\boldsymbol{\nabla} = \boldsymbol{e}_1 \partial_x + \boldsymbol{e}_2 \partial_y.$$

Act on the 'complex' field $\psi = u + \boldsymbol{e}_1 \boldsymbol{e}_2 v$,

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\boldsymbol{e}_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\boldsymbol{e}_2.$$

Terms appear in the Cauchy-Riemann equations. If u, v part of an analytic function get

$$\boldsymbol{\nabla}\psi = 0$$

This is the equation to generalise to higher dimensions. Includes the Neutrino equation in spacetime!

Massive Dirac Equation

Operator identification $i\partial_\mu = p_\mu$. A massive, free particle should have $\nabla^2\psi = -m^2\psi$. Put term on right, linear in m. For plane-wave states, momentum p, get

$$p\psi = m\psi a_0$$

where a_0 to be determined. Must be odd grade and square to $+1$. Obvious choice: $a_0 = \gamma_0$. Gives

$$\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

Dirac equation in STA form. Convert back to matrix form with

$$\nabla\psi\gamma_0 \; \leftrightarrow \; \hat{\gamma}_\mu\partial_\mu|\psi\rangle$$

to get

$$(i\hat{\gamma}_\mu\partial_\mu + m)|\psi\rangle = 0$$

Two other forms are useful

$$\nabla\psi = -m\psi I\gamma_3$$
$$(\nabla\psi)^{\sim} = mI\gamma_3\tilde{\psi}$$

Conserved Current

From previous two equations find

$$(\nabla\psi)\gamma_0\tilde{\psi} + \psi\gamma_0(\nabla\psi)^{\sim} = 0$$

Follows that

$$\begin{aligned}\nabla\cdot\langle\psi\gamma_0\tilde{\psi}\rangle_1 &= \gamma^\mu\cdot\langle\partial_\mu\psi\gamma_0\tilde{\psi} + \psi\gamma_0\partial_\mu\tilde{\psi}\rangle_1 \\ &= \langle\nabla\psi\gamma_0\tilde{\psi} + \psi\gamma_0(\nabla\psi)^{\sim}\rangle = 0\end{aligned}$$

Defines the Dirac Current $J = \psi\gamma_0\tilde{\psi}$. This is conserved,

$$\nabla\cdot J = 0$$

Timelike component in γ_0 frame is (cf. Section 1.3.2)

$$J_0 = \gamma_0\cdot J = \langle\gamma_0\tilde{\psi}\gamma_0\psi\rangle > 0$$

This is positive definite. The probability density for locating the electron. Met Dirac's original goal, but theory cannot account for pair production.

Integral over space of J_0 is constant. For localised states normalise to

$$\int |d^3x|\, J_0 = 1$$

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 13

SUMMARY

In this lecture we look at the two most important sets of solutions to the Dirac equation. These are the plane wave states, and the bound state Hydrogen atom solutions. Finally, we start to discuss gauge fields, which lead to the inclusion of gravity into the Dirac equation.

1. Observables and the Dirac current.
2. Plane wave states.
3. Particle and anti-particle solutions.
4. The Hydrogen atom ground state and energy spectrum.
5. Gauge fields.

Observables

The current $J = \psi\gamma_0\tilde{\psi}$ is one of a number of observables that can be formed from a Dirac spinor. To simplify these first write

$$\psi\tilde{\psi} = \rho\, e^{I\beta}$$

$\rho > 0$ and ρ, β are scalars. Assuming $\rho \neq 0$ set

$$R = \psi\rho^{-1/2}\, e^{-I\beta/2}, \qquad R\tilde{R} = 1$$

This decomposes ψ into

$$\psi = \rho^{1/2}\, e^{I\beta/2}\, R$$

Density ρ and rotor R. β connected to particle / antiparticle wavefunctions. Current now

$$J = \psi\gamma_0\tilde{\psi} = \rho\, e^{I\beta/2}\, R\gamma_0\tilde{R}\, e^{I\beta/2} = \rho R\gamma_0\tilde{R}$$

Rotor takes γ_0 to direction of the current. Density ρ dilates $R\gamma_0\tilde{R}$ to return the current. $J = \rho R\gamma_0\tilde{R}$ confirms that J is timelike and future-pointing.

A second observable is the spin vector

$$s = \psi\gamma_3\tilde{\psi} = \rho R\gamma_3\tilde{R}$$

Plane Wave States

Positive energy plane-wave state defined by

$$\psi = \psi_0 \, \mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x}$$

Dirac equation gives

$$p\psi_0 = m\psi_0\gamma_0$$

so

$$p\psi_0\tilde{\psi}_0 = mJ$$

But can write

$$\psi_0\tilde{\psi}_0 = \rho \, \mathrm{e}^{i\beta}$$

So $\exp(i\beta) = \pm 1$ (no pseudoscalar on right).

Know that $J\cdot\gamma_0 > 0$, and positive energy states have $p\cdot\gamma_0 > 0$. Must have $\beta = 0$.

For +ve energy ψ_0 is rotor + normalisation constant.

Decompose the rotor into rotation (in γ_0 frame) and boost.

$$R = LU$$

Boost L takes $m\gamma_0$ to p.

$$p = mL\gamma_0\tilde{L} = mL^2\gamma_0$$

Know how to solve this:

$$L(p) = \frac{m + p\gamma_0}{[2m(m + p\cdot\gamma_0)]^{1/2}} = \frac{E + m + \boldsymbol{p}}{[2m(E + m)]^{1/2}}$$

where $p\gamma_0 = E + \boldsymbol{p}$.

Negative energy solutions can be written

$$\psi = \psi_0 \, \mathrm{e}^{+I\boldsymbol{\sigma}_3 p\cdot x}$$

With $E = \gamma_0 \cdot p > 0$. For these

$$-p\psi\tilde{\psi} = mJ$$

Need $\beta = \pi$. Summarise by

positive energy	$\psi^{(+)}(x) = L(p)U \, \mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x}$
negative energy	$\psi^{(-)}(x) = L(p)UI \, \mathrm{e}^{I\boldsymbol{\sigma}_3 p\cdot x}$

Important in scattering theory.

HAMILTONIAN FORM

Take $\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$ and pre-multiply by γ_0:

$$\partial_t\psi I\boldsymbol{\sigma}_3 = -\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3 + m\gamma_0\psi\gamma_0 = -i\boldsymbol{\nabla}\psi + m\bar{\psi}$$

Here $i\psi = \psi I\boldsymbol{\sigma}_3$ and $\bar{\psi} = \gamma_0\psi\gamma_0$. Right-hand side is the free field Dirac Hamiltonian. Stationary states of energy E satisfy

$$E\psi = -i\boldsymbol{\nabla}\psi + m\bar{\psi}$$

This form of the equation is observer-dependent.

THE HYDROGEN ATOM

Electron in an attractive Coulomb potential, energy

$$eV(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Z\alpha}{r}$$

$\alpha \approx 1/137$ is the fine structure constant and Z is atomic number. Add this to the free-field Hamiltonian to get coupled equation

$$E\psi = -i\boldsymbol{\nabla}\psi - \frac{Z\alpha}{r}\psi + m\bar{\psi}$$

H-ATOM GROUND STATE

Equation to solve is

$$E\psi = -i\boldsymbol{\nabla}\psi - \frac{Z\alpha}{r}\psi + m\bar{\psi}$$

ψ is a Dirac spinor — an even element. Expect that the ground-state is spherically-symmetric. Build from real and imaginary combinations of $1, \boldsymbol{x}$.

Introduce polar coordinates (r, θ, ϕ) and frame vectors

$$\begin{aligned}
\boldsymbol{\sigma}_r &= \sin\theta(\cos\phi\,\boldsymbol{\sigma}_1 + \sin\phi\,\boldsymbol{\sigma}_2) + \cos\theta\,\boldsymbol{\sigma}_3 \\
\boldsymbol{\sigma}_\theta &= \cos\theta(\cos\phi\,\boldsymbol{\sigma}_1 + \sin\phi\,\boldsymbol{\sigma}_2) - \sin\theta\,\boldsymbol{\sigma}_3 \\
\boldsymbol{\sigma}_\phi &= -\sin\phi\,\boldsymbol{\sigma}_1 + \cos\phi\,\boldsymbol{\sigma}_2
\end{aligned}$$

In terms of this frame the vector derivative is

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_r\frac{\partial}{\partial r} + \frac{1}{r}\boldsymbol{\sigma}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\boldsymbol{\sigma}_\phi\frac{\partial}{\partial\phi}$$

So

$$\boldsymbol{\nabla}\boldsymbol{\sigma}_r = \frac{1}{r}\boldsymbol{\sigma}_\theta^2 + \frac{1}{r\sin\theta}\boldsymbol{\sigma}_\phi\sin\theta\boldsymbol{\sigma}_\phi = \frac{2}{r}$$

Trial Wavefunction

For spinor wavefunction try

$$\psi = u(r) + \boldsymbol{\sigma}_r v(r) I\boldsymbol{\sigma}_3$$

where u and v are "complex": $\{1, I\boldsymbol{\sigma}_3\}$. Dirac equation becomes

$$E(u + \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3) = -\boldsymbol{\sigma}_r(u' + \boldsymbol{\sigma}_r v' I\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_3$$
$$-\tfrac{2}{r} v I\boldsymbol{\sigma}_3 I\boldsymbol{\sigma}_3 - \tfrac{Z\alpha}{r}(u + \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3) + m(u - \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3)$$

Equating terms get

$$Eu = v' + \frac{2}{r}v - \frac{Z\alpha}{r}u + mu$$
$$Ev = -u' - \frac{Z\alpha}{r}v - mv$$

A pair of real equations. Can choose u and v as real functions. Look at asymptotic behaviour.

Large r

$$v' \approx (E-m)u, \qquad u' \approx -(E+m)v$$

so have $v'' \approx (m^2 - E^2)v$. Bound states have $E < m$, so define

$$\delta = \sqrt{m^2 - E^2}$$

u and v then go as $\mathrm{e}^{-\delta r}$.

Small r

$$ru' \approx -Z\alpha v, \qquad rv' \approx Z\alpha u - 2v.$$

Set $u = u_0 r^\beta$ and $v = v_0 r^\beta$. Equations reduce to

$$\begin{pmatrix} \beta & Z\alpha \\ -Z\alpha & \beta+2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix must have zero determinant. Implies that

$$\beta^2 + 2\beta + (Z\alpha)^2 = 0$$

hence

$$\beta = -1 \pm \sqrt{1-(Z\alpha)^2}$$

Need to look at the current

$$\psi\gamma_0\tilde{\psi} = (u^2+v^2)\gamma_0 + 2uv\,\sin\theta\boldsymbol{\sigma}_\phi\gamma_0$$

Density in γ_0-frame is u^2+v^2. Integral of this must be finite. Near origin

$$\int d^3x J\cdot\gamma_0 \sim 4\pi \int r^2 r^{2\beta}\,dr$$

so $2\beta > -3$. For small $Z < 118$ need positive root,

$$\beta = \beta_+ = -1 + \sqrt{1-(Z\alpha)^2}$$

Full Solution

So far have

$$u = U(r) r^{\beta_+} \, \mathrm{e}^{-\delta r}\,, \qquad v = V(r) r^{\beta_+} \, \mathrm{e}^{-\delta r}$$

Solves equations with constant $U = u_0$, $V = v_0$. Simplest solution, so ground state. Separate asymptotics require

$$v_0 = -\frac{\beta_+}{Z\alpha} u_0, \quad \delta u_0 = (E+m) v_0$$

Satisfying both fixes the energy via

$$\frac{E+m}{\delta} = -\frac{Z\alpha}{\beta_+} = \frac{Z\alpha}{1 - \sqrt{1-(Z\alpha)^2}}$$

Left-hand side is

$$\frac{E+m}{\delta} = \frac{m^2 - E^2}{(m-E)\delta} = \frac{\delta/m}{1 - E/m} = \frac{\delta/m}{1 - \sqrt{1-(\delta/m)^2}}$$

So have $\delta = mZ\alpha$. Energy is

$$E = m\sqrt{1-(Z\alpha)^2}$$

Lowest energy of more general formula

$$E^2 = m^2 \left(1 - \frac{(Z\alpha)^2}{n^2 + 2n\nu + \kappa^2} \right).$$

n is a non-negative integer, $\kappa = l+1$ is positive integer and $\nu = [(l+1)^2 - (Z\alpha)^2]^{1/2}$.

Energy Spectrum

$\alpha \approx 1/137$ is small. For low Z approximate to

$$E \approx m \left[1 - \frac{(Z\alpha)^2}{2} \frac{1}{n^2 + 2n(l+1) + (l+1)^2} \right]$$

Subtract off rest-mass to recover non-relativistic formula

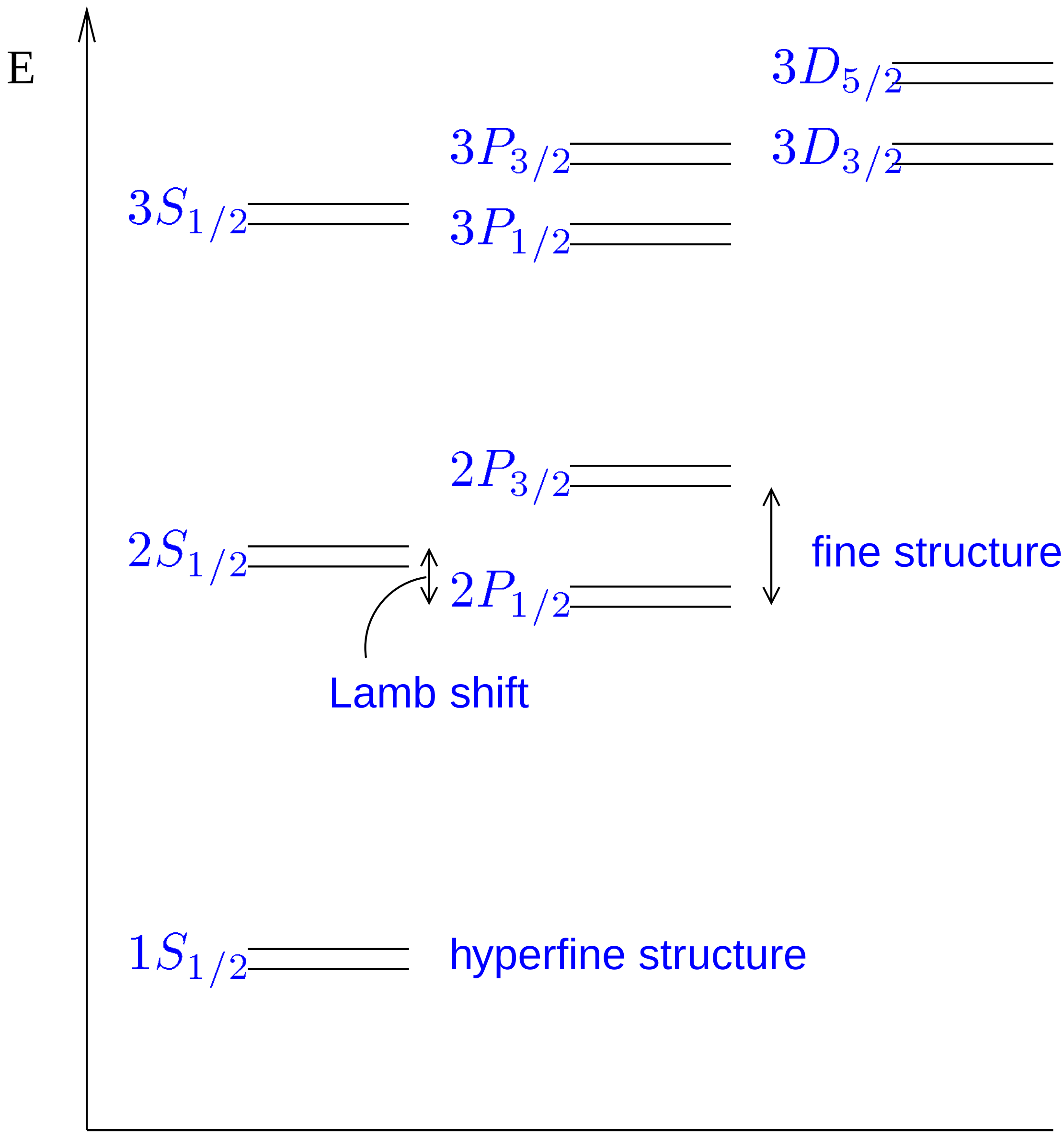
$$E_{NR} = -m \frac{(Z\alpha)^2}{2} \frac{1}{(n+l+1)^2} = -\frac{mZ^2e^4}{32\pi^2\epsilon_0^2\hbar^2} \frac{1}{n'^2}$$

where $n' = n + l + 1$. To next order

$$E_{NR} = -m \frac{(Z\alpha)^2}{2n'^2} - m \frac{(Z\alpha)^4}{2n'^4} \left(\frac{n'}{l+1} - \frac{3}{4} \right)$$

- Dependence on l gives fine structure.
- Hyperfine structure due to interaction with the magnetic moment of nucleus.
- Lamb shift is explained by quantum field theory. Lifts degeneracy between $S_{1/2}$ and $P_{1/2}$

H-Atom Energy Levels



PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 14

SUMMARY

In this lecture we study gauge theories. These describe all of the fundamental interactions known in physics, including gravity.

1. Phase Invariance and electromagnetism
2. Covariant Derivatives and connections
3. The gauge field strength
4. Electroweak interactions
5. Gauge fields for gravity

Phase Invariance

Free-particle Dirac equation,

$$\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

has global symmetry

$$\psi \mapsto \psi' = \psi\,\mathrm{e}^{I\boldsymbol{\sigma}_3\theta}$$

If ψ satisfies equation, so does ψ'. Global because θ is constant.

Write $R = \exp(I\boldsymbol{\sigma}_3\theta)$ and $\psi' = \psi R$. Useful for more general discussion. If θ not constant, get

$$\nabla\psi' = (\nabla\psi)R + (\nabla\theta)\psi RI\boldsymbol{\sigma}_3$$

So why do want to maintain symmetry under local phase changes?. Answer is from nature of physical statements. Involve

1. Observables, $J = \psi\gamma_0\tilde{\psi}$ or $s = \psi\gamma_3\tilde{\psi}$.
2. Statements of equality, *e.g.* $\psi = \psi_1 + \psi_2$.

Both are locally phase invariant.

Covariant Derivatives

Must modify Dirac equation to make local phase changes a symmetry. Write $\nabla = \gamma^\mu \partial_\mu$. Equation includes term

$$\nabla\psi' = \gamma^\mu \left(\partial_\mu \psi R + \psi \partial_\mu R\right)$$

so ψ' satisfies

$$\nabla\psi' I\boldsymbol{\sigma}_3 - \gamma^\mu \psi' (\tilde{R}\partial_\mu R) I\boldsymbol{\sigma}_3 = m\psi'\gamma_0$$

$\tilde{R}\partial_\mu R$ is a bivector. Remove by adding a new bivector-valued term. This is the connection, Ω_μ. Define covariant derivative D_μ,

$$D_\mu \psi = \partial_\mu \psi + \tfrac{1}{2}\psi\Omega_\mu$$

So modified equation is

$$\gamma^\mu D_\mu \psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

Now transform the pair ψ and Ω_μ to give local symmetry. With $\psi' = \psi R$,

$$\begin{aligned}\gamma^\mu D'_\mu \psi' I\boldsymbol{\sigma}_3 &= \gamma^\mu(\partial_\mu \psi' + \tfrac{1}{2}\psi'\Omega'_\mu) I\boldsymbol{\sigma}_3 \\ &= \gamma^\mu(\partial_\mu \psi R + \psi\partial_\mu R + \tfrac{1}{2}\psi R\Omega'_\mu) I\boldsymbol{\sigma}_3 = m\psi'\gamma_0\end{aligned}$$

But $m\psi\gamma_0 = \gamma^\mu D_\mu \psi I\boldsymbol{\sigma}_3$, so

$$\tfrac{1}{2}R\Omega'_\mu + \partial_\mu R = \tfrac{1}{2}\Omega_\mu R$$

Gives the transformation law for a connection field,

$$\Omega'_\mu = \tilde{R}\Omega_\mu R - 2\tilde{R}\partial_\mu R$$

Note that

- Ω_μ is an arbitrary bivector field.
- Cannot write $\Omega_\mu = -2\tilde{R}\partial_\mu R$. This would have no new physics.
- The difference between Ω_μ and derivatives of rotors gives new physics.
- New dynamical degrees of freedom require new equations.
- These are also gauge invariant.

Minimally Coupled Dirac Equation

Return to electromagnetism. Restrict to $R = \exp(I\boldsymbol{\sigma}_3\theta)$, so

$$-2\tilde{R}\partial_\mu R = -2\,\mathrm{e}^{-I\boldsymbol{\sigma}_3\theta}\,\partial_\mu\theta\,\mathrm{e}^{+I\boldsymbol{\sigma}_3\theta}\,I\boldsymbol{\sigma}_3 = -2\partial_\mu\theta I\boldsymbol{\sigma}_3$$

Connection restricted to

$$\Omega_\mu = \alpha A_\mu I\boldsymbol{\sigma}_3$$

A_μ are components of a vector, α is coupling constant. Under gauge transformations,

$$\alpha A_\mu \mapsto \alpha A'_\mu = \alpha A_\mu - 2\partial_\mu\theta,$$

or $\alpha A \mapsto \alpha A' = \alpha A - 2\nabla\theta$. Should be familiar. A is electromagnetic vector potential. Physics in $F = \nabla\wedge A$ unchanged by $A \mapsto A'$. Have

$$\gamma^\mu D_\mu\psi = \gamma^\mu(\partial_\mu\psi + \tfrac{1}{2}\alpha A_\mu\psi I\boldsymbol{\sigma}_3) = \nabla\psi + \tfrac{1}{2}\alpha A\psi I\boldsymbol{\sigma}_3$$

Hamiltonian form sets $\alpha = 2e$ (e -ve for electron). Recover 'minimally coupled' Dirac equation

$$\nabla\psi I\boldsymbol{\sigma}_3 - eA\psi = m\psi\gamma_0$$

The simplest modification.

FIELD STRENGTH

Need to encode the part of a gauge field which is not pure gauge. Look at commutator of covariant derivatives.

$$\begin{aligned}[D_\mu, D_\nu]\psi &= D_\mu(\partial_\nu\psi + \tfrac{1}{2}\psi\Omega_\nu) - D_\nu(\partial_\mu\psi + \tfrac{1}{2}\psi\Omega_\mu) \\ &= \tfrac{1}{2}\psi(\partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - \Omega_\mu\times\Omega_\nu)\end{aligned}$$

Defines the field strength $\mathsf{F}_{\mu\nu}$,

$$\mathsf{F}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - \Omega_\mu\times\Omega_\nu$$

Bivector-valued, and antisymmetric on μ, ν. Find transformation properties from

$$[D'_\mu, D'_\nu]\psi' = \tfrac{1}{2}\psi'\mathsf{F}'_{\mu\nu}$$

But

$$D'_\mu(D'_\nu\psi') = D'_\mu(D_\nu\psi R) = (D_\mu D_\nu\psi)R$$

so find covariant transformation law

$$\mathsf{F}'_{\mu\nu} = \tilde{R}\mathsf{F}_{\mu\nu}R$$

Transformation law for the field strength in a general Yang-Mills gauge theory.

Electromagnetic Field Strength

Electromagnetic case has some special features.

- Have $\Omega_\mu = 2eA_\mu I\boldsymbol{\sigma}_3$, so form

$$e(\partial_\mu A_\nu I\boldsymbol{\sigma}_3 - \partial_\nu A_\mu I\boldsymbol{\sigma}_3) - 2e^2 A_\mu A_\nu I\boldsymbol{\sigma}_3 \times I\boldsymbol{\sigma}_3$$
$$= e(\partial_\mu A_\nu - \partial_\nu A_\mu) I\boldsymbol{\sigma}_3 = e(\gamma_\nu \wedge \gamma_\mu)\cdot(\nabla\wedge A) I\boldsymbol{\sigma}_3$$

  Maps bivector $\gamma_\nu \wedge \gamma_\mu$ onto pure phase term.

- Commutator term $\Omega_\mu \times \Omega_\nu$ vanishes. (All bivectors in same plane).

- The field strength is linear in A. Makes electromagnetism much simpler than other interactions.

- Transformation law $\mathsf{F}'_{\mu\nu} = \tilde{R}\mathsf{F}_{\mu\nu}R$ has no effect. Bivector $I\boldsymbol{\sigma}_3$ unaffected by rotations is same plane. Ignore the $I\boldsymbol{\sigma}_3$ term and work directly with $F = \nabla\wedge A$.

- Extract desired field strength, $F = \nabla\wedge A$. Encodes physically measurable content of the electromagnetic field.

- Express field equations in terms of F, $\nabla F = J$.

Electroweak Interactions

Beyond the scope of this course, but can make some observations

- Main observables are $J = \psi\gamma_0\tilde{\psi}$ and $s = \psi\gamma_3\tilde{\psi}$
- To leave both invariant need

$$R\gamma_0\tilde{R} = \gamma_0, \qquad R\gamma_3\tilde{R} = \gamma_3$$

  Only have rotations in $I\boldsymbol{\sigma}_3$ plane $\implies$ electromagnetism.

- If relax spin requirement, just have $R\gamma_0\tilde{R} = \gamma_0$ — spatial rotors in γ_0-frame.
- 3D rotors from group SU(2) — of 2×2 complex unitary matrices, determinant $+1$. One of main building blocks of electroweak.
- Need a further U(1) group. Can incorporate as exponentials of pseudoscalar I. Still leave J invariant

$$\psi\, \mathrm{e}^{I\beta/2}\, \gamma_0\, \mathrm{e}^{I\beta/2}\, \tilde{\psi} = \psi\gamma_0\tilde{\psi}$$

  But only symmetries of the massless equation.

- Electroweak theory starts with massless. Gives mass through interaction with Higgs field.
- Structure arises naturally in STA approach.

Gauge Principles for Gravitation

Aim: to model gravitational interactions in terms of (gauge) fields in the STA. Main principles:

- Absolute position and orientation of particles or fields in the STA is not measurable.
- Only extract relative relations between fields.
- These relations must be entirely local.

Satisfying these ensures no conflict with principles of GR.

Also need more mathematical encoding. Physical statements can be of form

$$\psi_1(x) = \psi_2(x)$$

Independent of where we place the fields in the STA. Could introduce two new fields $\psi_1'(x) = \psi_1(x')$, $\psi_2'(x) = \psi_2(x')$. Statement $\psi_1'(x) = \psi_2'(x)$ has same physical content.

Same true for rotations

$$\psi_1' = R\psi_1, \quad \psi_2' = R\psi_2$$

Do not alter physical consequences. Also applies to observables $J = \psi\gamma_0\tilde{\psi}$, $J' = RJ\tilde{R}$. Rotate all other fields with J, no observable consequences. Absolute direction in STA is irrelevant.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 15

SUMMARY

In this lecture show how gravity is constructed as a gauge theory based on the Dirac equation. The resulting theory reproduces the predictions of General Relativity (GR), but only in the absence of quantum spin.

1. Gauge fields for gravity
2. Spinor and observable covariant derivatives
3. Field strength and the field equations
4. Point particle trajectories.

Gauge Principles for Gravitation

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Position Invariance

Write $x' = f(x)$ — map between points in STA. Use this to move field around,

$$\psi'(x) = \psi(x')$$

Physical equations should be unaffected by this. Call this a displacement. Again, problems arise if ψ is differentiated. Have new coordinates

$$x^{\mu\prime} = \gamma^\mu \cdot x', \qquad \psi'(x^\mu) = \psi(x^{\mu\prime})$$

Derivatives go as

$$\partial_\mu \psi' = \partial_\mu \psi(x^{\nu\prime}) = (\partial_\mu x^{\nu\prime}) \partial'_\nu \psi(x^{\lambda\prime})$$

Term $\partial_\mu x^{\nu\prime}$ is the Jacobian. Want to remove this with a gauge field. But acts on ∂_μ directly. Cannot add a bivector connection.

Instead, replace γ^μ with vector gauge fields $g^\mu(x)$,

$$\gamma^\mu \mapsto g^\mu(x)$$

Defined to transform in required way (slightly messy).

Replacing the γ^μ frame vectors with gauge fields means they are no longer observable. Precisely one of our goals!

Price is the introduction of $4 \times 4 = 16$ degrees of freedom.

Rotations

So far have

$$g^\mu(x)\partial_\mu\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

Want symmetry $\psi \mapsto \psi' = R\psi$. Impose transformation law for g^μ:

$$g^\mu(x) \mapsto g^\mu(x)' = Rg^\mu(x)\tilde{R}$$

Ensures equation has symmetry for global R. Make this local with

$$\partial_\mu\psi \mapsto D_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}\Omega_\mu\psi$$

Ω_μ gauge field has transformation law

$$\Omega_\mu \mapsto \Omega'_\mu = R\Omega_\mu\tilde{R} - 2\partial_\mu R\tilde{R}$$

R arbitrary so Ω_μ contains all blades. Has $6 \times 4 = 24$ degrees of freedom.

Final equation is

$$g^\mu D_\mu\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$$

Invariant under displacements (not discussed here) and

$$\psi \mapsto R\psi, \quad g^\mu \mapsto Rg^\mu\tilde{R}, \quad \Omega_\mu \mapsto R\Omega_\mu\tilde{R} - 2\partial_\mu R\tilde{R}$$

Covariant Derivatives for Observables

Observables have general form

$$M = \psi \Gamma \tilde{\psi}$$

Γ a combination of γ_0, γ_3, $I\boldsymbol{\sigma}_3$. M inherits transformation properties from ψ

$$M(x) \mapsto M'(x) = M(x'), \quad M \mapsto M' = RM\tilde{R}$$

Call these covariant multivectors.

Form derivatives of M:

$$\partial_\mu M = (\partial_\mu \psi)\Gamma\tilde{\psi} + \psi\Gamma(\partial_\mu \psi)^\sim$$

Need to replace spinor directional derivatives with covariant derivatives:

$$\begin{aligned}
&(D_\mu \psi)\Gamma\tilde{\psi} + \psi\Gamma(D_\mu\psi)^\sim \\
&\quad = \partial_\mu \psi\Gamma\tilde{\psi} + \psi\Gamma(\partial_\mu\psi)^\sim + \tfrac{1}{2}\Omega_\mu\psi\Gamma\tilde{\psi} - \tfrac{1}{2}\psi\Gamma\tilde{\psi}\Omega_\mu \\
&\quad = \partial_\mu(\psi\Gamma\tilde{\psi}) + \Omega_\mu \times (\psi\Gamma\tilde{\psi})
\end{aligned}$$

Get covariant derivative for observables

$$D_\mu M = \partial_\mu M + \Omega_\mu \times M$$

Use the same universal symbol D_μ. Object being differentiated dictates explicit form.

Properties

- Bivector commutator is grade preserving
- Commutator satisfies

$$\Omega_\mu \times (MN) = (\Omega_\mu \times M)N + M(\Omega_\mu \times N)$$

Ensures that D_μ is a derivation:

$$D_\mu(MN) = (D_\mu M)N + M(D_\mu N)$$

THE FIELD EQUATIONS

Gauge fields introduce new degrees of freedom:

$$4 \times 4(g^\mu) + 4 \times 6(\Omega_\mu) = 40$$

So need to find 40 equations! The g^μ generalise a coordinate frame. These satisfy

$$e^\mu = \nabla x^\mu, \quad e_\mu = \partial_\mu x, \quad e^\mu \cdot e_\nu = \delta^\mu_\nu$$

The e_μ satisfy

$$\partial_\mu e_\nu - \partial_\nu e_\mu = [\partial_\mu, \partial_\nu] x = 0$$

Some version must survive the gauging process.

Introduce the reciprocal frame

$$g^{\mu} \cdot g_{\nu} = \delta^{\mu}_{\nu}$$

Since g^{μ} transform as $g^{\mu} \mapsto R g^{\mu} \tilde{R}$, must also have

$$g_{\mu} \mapsto R g_{\mu} \tilde{R}$$

Suitable covariant equation is

$$D_{\mu} g_{\nu} - D_{\nu} g_{\mu} = 0$$

Vector valued, and antisymmetric on indices, so $4 \times 6 = 24$ equations. Need a further 16

Comments

- In fact, Lagrangian derivation tells us that right-hand side is the matter spin density.
- Spin density usually irrelevant on gravitational scales.
- Shows that GR only recovered when quantum effects are ignored!
- Has implications for attempts to construct a quantum theory.

Gravitational Field Strength

Commute covariant derivatives to form

$$[D_\mu, D_\nu]\psi = \tfrac{1}{2}\mathsf{R}_{\mu\nu}\psi$$

$$\mathsf{R}_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + \Omega_\mu \times \Omega_\nu$$

Bivector-valued function, and antisymmetric on μ, ν so $6 \times 6 = 36$ degrees of freedom. Gauge theory version of the Riemann tensor.

Only need 16 equations. Introduce contraction

$$\mathsf{R}_\nu = g^\mu \cdot \mathsf{R}_{\mu\nu}$$

Vector-valued with 1 index so 16 degrees of freedom. Analog of the Ricci tensor. Further contraction gives Ricci scalar

$$R = g^\mu \cdot \mathsf{R}_\mu$$

Lagrangian principle gives second field equation

$$\mathsf{R}_\mu - \tfrac{1}{2} R g_\mu = 8\pi G \mathsf{T}_\mu$$

T_μ is matter stress-energy tensor. Energy is the source of gravitation, not just mass. Vacuum equations simply

$$\mathsf{R}_\mu = 0$$

Point Particle Trajectories

Point particles follow paths $x(\lambda)$. Absolute position is irrelevant, so tangents cannot be physical either. Form

$$\frac{dx}{d\lambda} = \frac{dx^\mu}{d\lambda}\gamma_\mu$$

Form covariant vector by $\gamma_\mu \mapsto g_\mu(x)$. Form

$$v = \frac{dx^\mu}{d\lambda} g_\mu(x)$$

v carried from point to point under displacements. Under rotations $v \mapsto Rv\tilde{R}$. Absolute value of v is gauge-dependent. No physical meaning.

Inertial Observers

In special relativity inertial observers have constant velocity. Write this

$$\partial_\lambda(\partial_\lambda x(\lambda)) = 0$$

Covariant version describes free fall. Paths taken by observers in absence of other forces. Generalise the notion of inertial observers.

Geodesic Equation

First map $\partial_\lambda x(\lambda) \mapsto v$. Next write

$$\partial_\lambda v = \frac{\partial x^\mu}{\partial \lambda} \partial_\mu v$$

Replace ∂_μ by covariant derivative D_μ. Get equation

$$\frac{\partial x^\mu}{\partial \lambda}(\partial_\mu v + \Omega_\mu \cdot v) = \partial_\lambda v + \frac{\partial x^\mu}{\partial \lambda}\Omega_\mu \cdot v = 0$$

Contract with v. Since $(\Omega_\mu \cdot v) \cdot v = 0$ get $v^2 =$ constant. Choose parameter so that

$$v^2 = 1$$

Defines observer proper time. Trajectory equation

$$\dot{v} + \dot{x}^\mu \Omega_\mu \cdot v = 0, \qquad v^2 = 1$$

- Gauge equivalent of the geodesic equation.
- Independent of particle mass — the equivalence principle.
- In gauge theory can derive this from the gauged Dirac equation.

THE METRIC

Proper time (or distance) is

$$s = \int_{\tau_1}^{\tau_2} \sqrt{|v^2|}\, d\tau = \int_{\tau_1}^{\tau_2} |\dot{x}^\mu \dot{x}^\nu g_\mu \cdot g_\nu|^{1/2}\, d\tau$$
$$= \int_{x_1}^{x_2} |g_\mu \cdot g_\nu dx^\mu\, dx^\nu|^{1/2}$$

Recover the metric

$$g_{\mu\nu} = g_\mu \cdot g_\nu$$

Can get geodesic equation from extremising proper distance. Same as in GR.

In gauge theory a solution is consists of specifying the g_μ. Find Ω_μ from

$$D_\mu g_\nu - D_\nu g_\mu = 0$$

Then construct field strength, R_μ and R. Equate to matter Stress-Energy Tensor.

If g_μ solve the gauge field equations, the metric solves the Einstein equations for same matter fields. Two approaches are equivalent if quantum effects ignored.

Proof is slightly technical, but not too complicated.

PHYSICAL APPLICATIONS OF GEOMETRIC ALGEBRA

LECTURE 16

SUMMARY

In this final lecture we discuss some of the physical properties of the fields outside a spherically-symmetric source. The solution contains a horizon and represents a (non-rotating) black hole. Matter and photon trajectories are easily computed using the gauge theory treatment and rotors.

- Spherically symmetric Fields.
- The field strength and its contraction.
- Point particle trajectories.
- The horizon.
- Photon Paths.
- The Dirac Equation

SPHERICALLY-SYMMETRIC FIELDS

Considerable gauge freedom when writing a solution. Employ this to make as simple as possible. For spherical symmetry, can write Schwarzschild solution as

$$g_0 = \gamma_0 + \sqrt{(2GM/r)}e_r, \quad g_i = \gamma_i, \quad i = 1\ldots 3$$

where e_r is radial vector

$$e_r = \frac{1}{r} x^i \gamma_i$$

Work in natural units $G = c = \hbar = 1$. Reciprocal frame is

$$g^0 = \gamma^0, \quad g^i = \gamma^i - \sqrt{(2M/r)}(x^i/r)\gamma^0$$

Connection bivectors found to be

$$\Omega_0 = \frac{M}{r^2}\boldsymbol{\sigma}_r$$

$$\Omega_i = -\frac{1}{2r}\left(\frac{2M}{r}\right)^{1/2}(2\boldsymbol{\sigma}_i - 3\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_r\, \boldsymbol{\sigma}_r),$$

where

$$\boldsymbol{\sigma}_r = e_r \gamma_0$$

Acceleration term Ω_0, equal to Newtonian value GM/r^2!

Solution

Can also write

$$\Omega_\mu = \frac{-1}{2r}\left(\frac{2M}{r}\right)^{1/2}(2g_\mu\wedge\gamma_0 + 3g_\mu\cdot e_r\,\boldsymbol{\sigma}_r)$$

useful in calculations. Find field strength

$$\mathsf{R}_{\mu\nu} = -\frac{M}{2r^3}(g_\mu\wedge g_\nu + 3\boldsymbol{\sigma}_r g_\mu\wedge g_\nu\boldsymbol{\sigma}_r)$$

Vacuum equations are $g^\mu\cdot\mathsf{R}_{\mu\nu} = 0$. Construct

$$g^\mu\cdot(g_\mu\wedge g_\nu) = 4g_\nu - \delta^\mu_\nu g_\mu = 3g_\nu$$

and

$$g^\mu\boldsymbol{\sigma}_r g_\mu = 0$$

Combining these, find that

$$\begin{aligned} g^\mu\cdot(g_\mu\wedge g_\nu + 3\boldsymbol{\sigma}_r g_\mu\wedge g_\nu\boldsymbol{\sigma}_r) \\ = 3g_\nu + 3g^\mu\boldsymbol{\sigma}_r(g_\mu g_\nu - g_\mu\cdot g_\nu)\boldsymbol{\sigma}_r \\ = 3g_\nu - 3g^\mu g_\mu\cdot g_\nu\boldsymbol{\sigma}_r\boldsymbol{\sigma}_r = 0 \end{aligned}$$

Confirms that we have a vacuum solution.

Point Particle Trajectories

Radial motion has $x(\tau) = t(\tau)\gamma_0 + r(\tau)e_r$, so

$$\dot{x} = \dot{t}\gamma_0 + \dot{r}e_r = \dot{t}\gamma_0 + \dot{r}\frac{x^i}{r}\gamma_i$$

Follows that ($\gamma_\mu \mapsto g_\mu$)

$$v = \dot{t}\gamma_0 + (\dot{r} + \surd(2M/r)\dot{t})e_r$$

Constraint $v^2 = 1$ implies

$$v = \cosh(\alpha)\gamma_0 + \sinh(\alpha)e_r = \mathrm{e}^{\alpha\boldsymbol{\sigma}_r}\gamma_0$$

so

$$\dot{v} = \dot{\alpha}\boldsymbol{\sigma}_r \cdot v.$$

Geodesic equation is $\dot{v} + \dot{x}^\mu \Omega_\mu \cdot v = 0$. Need

$$\begin{aligned}\dot{x}^\mu \Omega_\mu &= \frac{-1}{2r}\left(\frac{2M}{r}\right)^{1/2}(2v\wedge\gamma_0 + 3v\cdot e_r\,\boldsymbol{\sigma}_r) \\ &= \sinh(\alpha)\frac{1}{2r}\left(\frac{2M}{r}\right)^{1/2}\boldsymbol{\sigma}_r\end{aligned}$$

Must cancel the $\dot{\alpha}\boldsymbol{\sigma}_r$ term.

Radial Free Fall

Radial free fall equations reduce to

$$\begin{aligned}\dot{t} &= \cosh(\alpha) \\ \dot{r} &= \sinh(\alpha) - \surd(2M/r)\cosh(\alpha) \\ \dot{\alpha} &= -\sinh(\alpha)\surd(2M/r)/(2r)\end{aligned}$$

First order. Give unique trajectory given initial t, r, $\tanh(\alpha)$. Simplest solution is

$$\alpha = 0, \quad \dot{r} = -\surd(2M/r), \quad \dot{t} = 1$$

Free fall from rest at infinity. t coordinate is proper time for these observers. Also have

$$v = \gamma_0$$

Extremely simple! Gives this gauge many attractive features.

Differentiate $\dot{r}$ equation to find

$$\ddot{r} = -\frac{GM}{r^2}$$

Newtonian force law. But derivatives are proper time, and r defined by $\mathsf{R}_{\mu\nu}$. Get local equations, as required.

The Horizon

Can write $\dot{r}$ equation as

$$\dot{r}/\cosh(\alpha) = \tanh(\alpha) - \surd(2M/r)$$

If $2M/r > 1$ only motion is inwards. This defines the horizon $r = 2M = 2GM/c^2$. Escape velocity is c. Predicted by John Michell ($\sim$ 1782).

Need to look at photon paths more carefully to understand horizon. Same geodesic equation, but $v^2 = 0$. Since

$$v = \dot{t}\gamma_0 + (\dot{r} + \surd(2M/r)\dot{t})e_r$$

must have

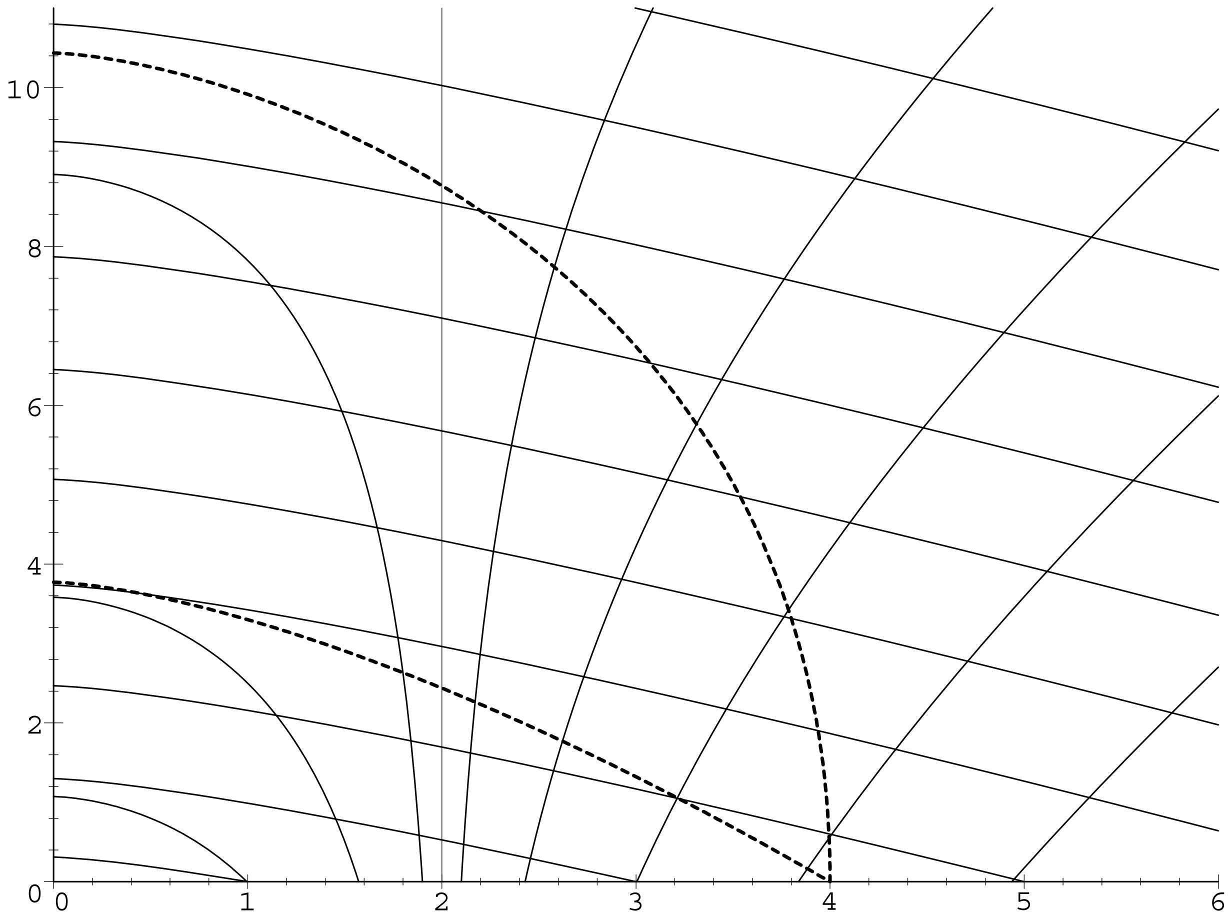
$$\dot{t} = \pm(\dot{r} + \surd(2M/r)\dot{t})$$

Summarise equations as

$$\frac{dr}{dt} = -\surd(2M/r) + \begin{cases} +1 & \text{outgoing} \\ -1 & \text{ingoing,} \end{cases}$$

Straightforward to integrate. Confirm that inside horizon, all photon paths point inwards. Not even light can escape. Call this a black hole

Summary of Radial Paths



- $GM = 1$, horizon at $r = 2$
- Solid lines are photon trajectories
- I: Particle released from rest at $r = 4$
- II: Particle released from rest at $r = \infty$

The Dirac Equation

Final application — Dirac equation in a black hole background. Need

$$
\begin{aligned}
g^\mu \partial_\mu &= \nabla - \gamma_0 \sqrt{(2GM/r)}(x^i/r)\partial_{x^i} \\
&= \nabla - \gamma_0 \sqrt{(2GM/r)}\partial_r
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac{1}{2} g^\mu \Omega_\mu &= \frac{-1}{4r}\left(\frac{2M}{r}\right)^{1/2} g^\mu \left(2 g_\mu \wedge \gamma_0 + 3 g_\mu \cdot e_r\, \boldsymbol{\sigma}_r\right) \\
&= \frac{-3}{4r}\left(\frac{2M}{r}\right)^{1/2} \gamma_0
\end{aligned}
$$

Dirac equation reduces to

$$
\nabla \psi I \boldsymbol{\sigma}_3 - \gamma_0 \left(\frac{2M}{r}\right)^{1/2} \left(\partial_r \psi + 3/(4r)\psi\right) I \boldsymbol{\sigma}_3 = m\psi\gamma_0
$$

All gravitational effects in a single interaction Hamiltonian. Multiply through by γ_0, extract

$$
\hat{H}_I = i\hbar \left(\frac{2M}{r}\right)^{1/2} \left(\partial_r + 3/(4r)\right)
$$

Describes all of relativistic quantum mechanics in a black hole background!