1 POTENTIAL FLOW THEORY – Formulation of the seakeeping problem

Objective of the Chapter:

Formulation of the potential flow around the hull of a ship advancing and oscillationg in waves

Results of the Chapter:

Potential flow problem, which is represented by a set of equations in the unknown Φ (velocity potential)

- First step is to develop the exact solution (within the ideal fluid assumption)
- Second step is to linearize the potential flow problem

1.1 - Ideal Fluid Hypothesis

Ideal Fluid Hypothesis

- Homogeneous fluid
- Incompressible fluid
- Inviscid fluid
- Surface tension may be neglected

Why?

To obtain a scalar function that satisfies the continuity equation within the fluid domain (except at singular points): the **velocity potential**

How?

How?

(1) If the fluid is **homogeneous** and **incompressible** then the equation of conservation of mass reduces to the equation of continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

where *u*, *v* and *w* are the components of the fluid velocity vector

(2) If the fluid id inviscid then it is also irrotational (there is no vorticity, or it remains constant) and the fluid velocity vector may be represented by a scalar function – the **velocity potential:**

$$\vec{V}=\nabla\Phi$$

(1) + **(2)** lead to the **LAPLACE** equation:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Among the various hypothesis the **inviscid flow assumption** may introduce some limitations. Examples:

- Horizontal motions
- Roll motion
- Vertical motions of Small Waterplane Area Twin Hulls

Overview

(1) Obtain the velocity potential

(2) With the velocity potential it is possible to calculate the fluid velocities

(3) Applying the Bernoulli equation one obtains the fluid pressures (on the hull)

(4) Integrating the fluid pressures results in the hydrodynamic forces acting on the ship advancing in waves

(5) Equating the fluid forces with the mass forces (Newton 2nd law) results on the equations of ship motions and structural loads

(6) Equations (5) are solved either in the frequency domain or in the time domain

Problem:

Step 1 – obtain the velocity potential

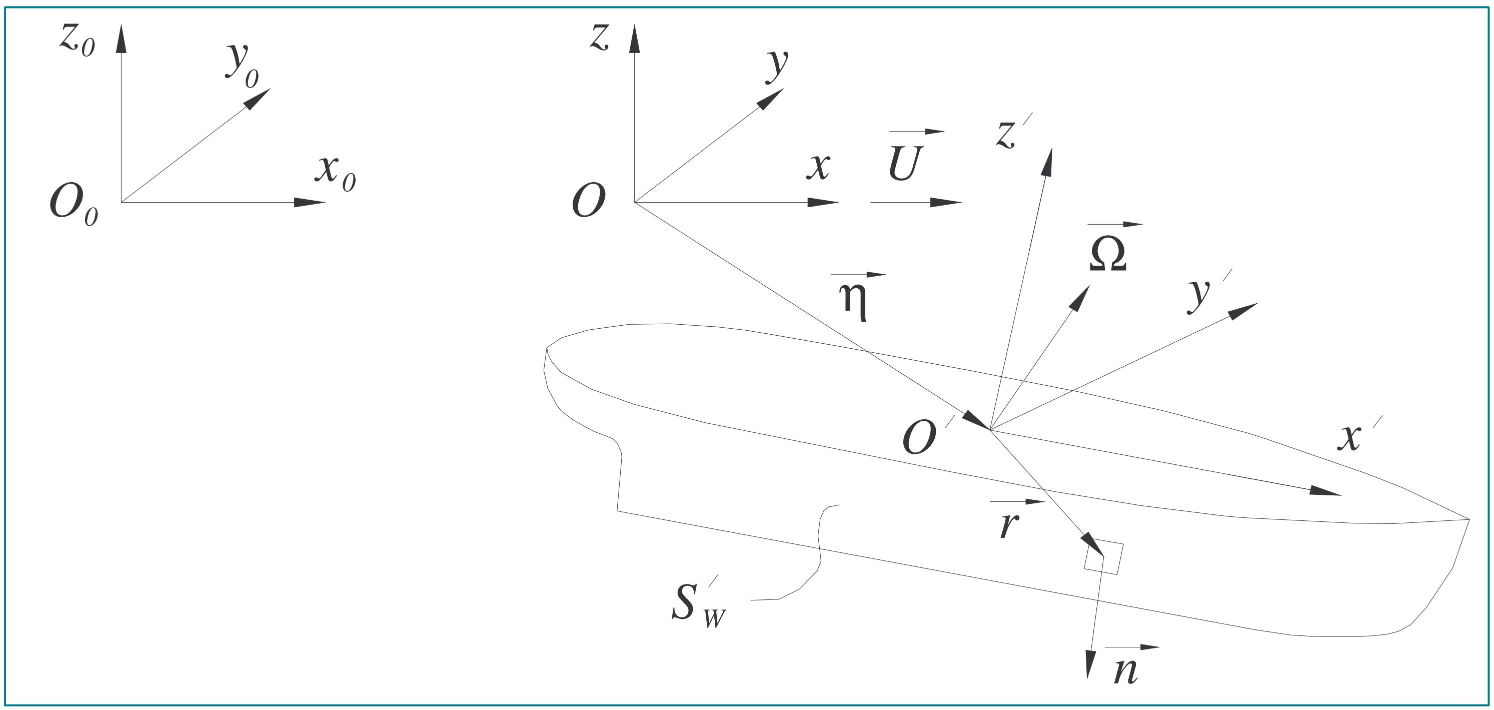
1.2 – Coordinate Systems

Three orthogonal and right handed coordinate systems will be used:

$X_0 = (x_0, y_0, z_0)$ ⟹ fixed in space

$X = (x, y, z)$ ⟹ advances with the ship forward speed

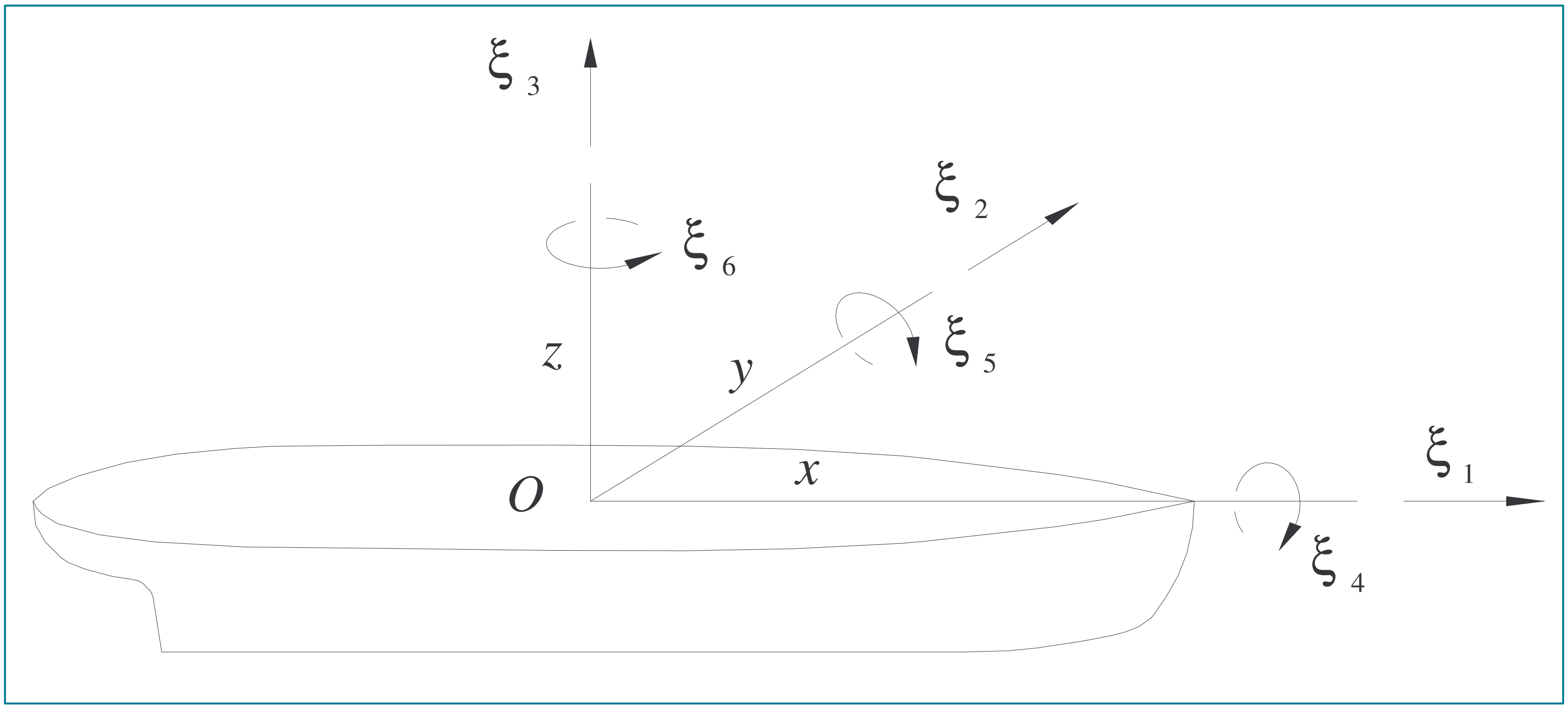
$X' = (x', y', z')$ ⟹ body fixed



The rigid body motions in the forward speed reference system consist of:

➢ Three translations in the directions of x, y and z ; respectively surge (ξ_1), sway (ξ_2) and heave (ξ_3)

➢ Three rotations around x, y and z ; respectively roll (ξ_4), pitch (ξ_5) and yaw (ξ_6)



1.3 – Hydrodynamic Problem

Assuming that the fluid is homogeneous, incompressible and inviscid

then

the hydrodynamic problem may be formulated in terms of the **potential flow theory**

This means that the velocity vector of the fluid particles may be represented by the gradient of a velocity potential:

$$\vec{V} = \nabla\Phi$$

Velocity vector $\vec{\mathrm{V}}(\vec{\mathrm{x}}_0, t)$

Velocity potential $\Phi(\vec{x}_0, t)$

Gradient operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

The velocity potential satisfies the Laplace equation:

$$\nabla^2 \Phi = 0 \tag{1.3}$$

With the velocity potential known the fluid pressure may be determined by Bernoulli equation:

$$p = -\rho\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left|\nabla \Phi\right|^2 + g z_0\right) \tag{1.4}$$

where:

$p(\vec{x}_0, t)$	represents the fluid pressure
ρ	is the fluid specific mass
g	is the acceleration of gravity

Integration of the fluid pressure over the hull wetted surface results on the hydrodynamic forces acting on the ship

Problem: obtain solution of the Laplace equation

Uniqueness of the solution depends of the appropriate boundary conditions

We consider a rigid body advancing through the free surface which is infinite in all horizontal directions. In this case the boundary conditions are:

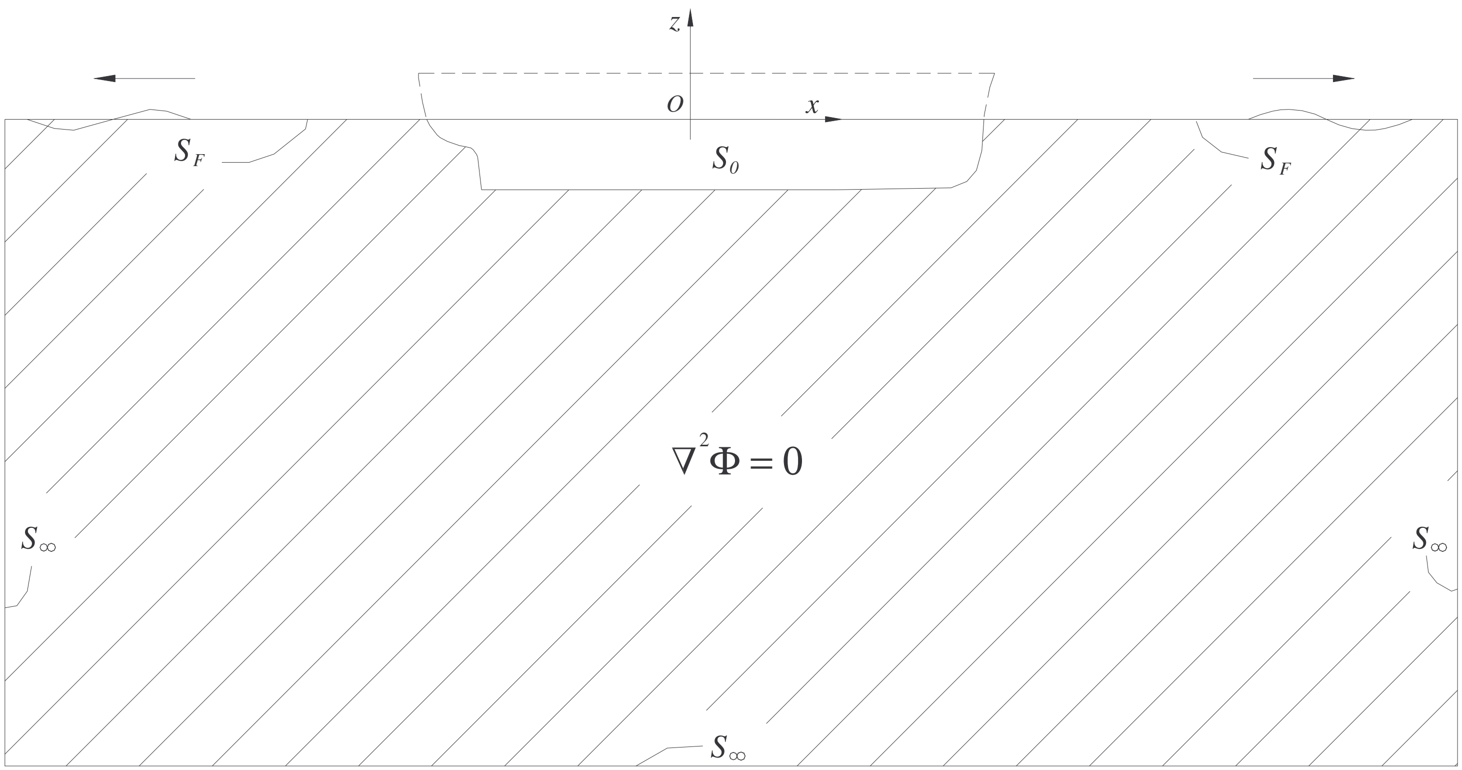
(1) Body wetted surface

(2) Free surface

(3) Sea bottom

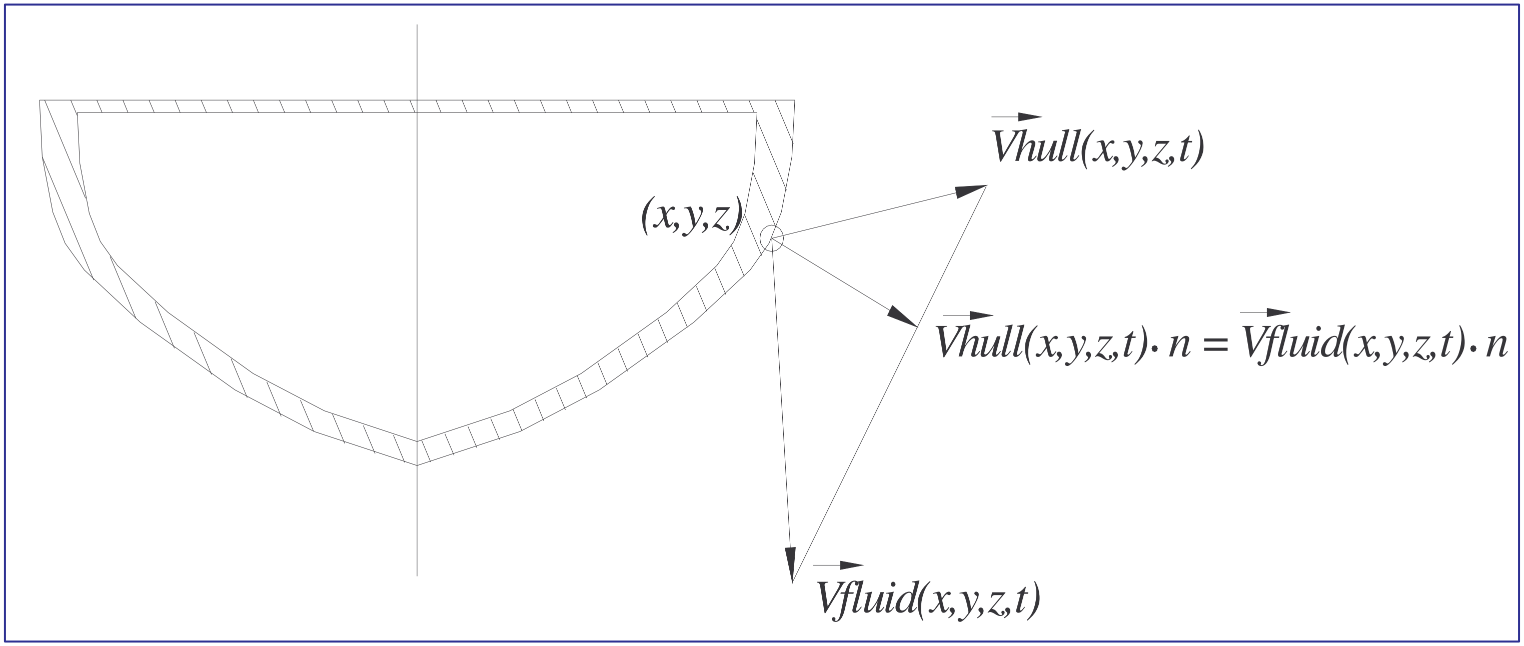
(4) Control surface between the sea bottom and the free surface and far way from the source of disturbance

To obtain the desired solution of the Laplace equation it is necessary to impose conditions on all boundaries surrounding the fluid domain:



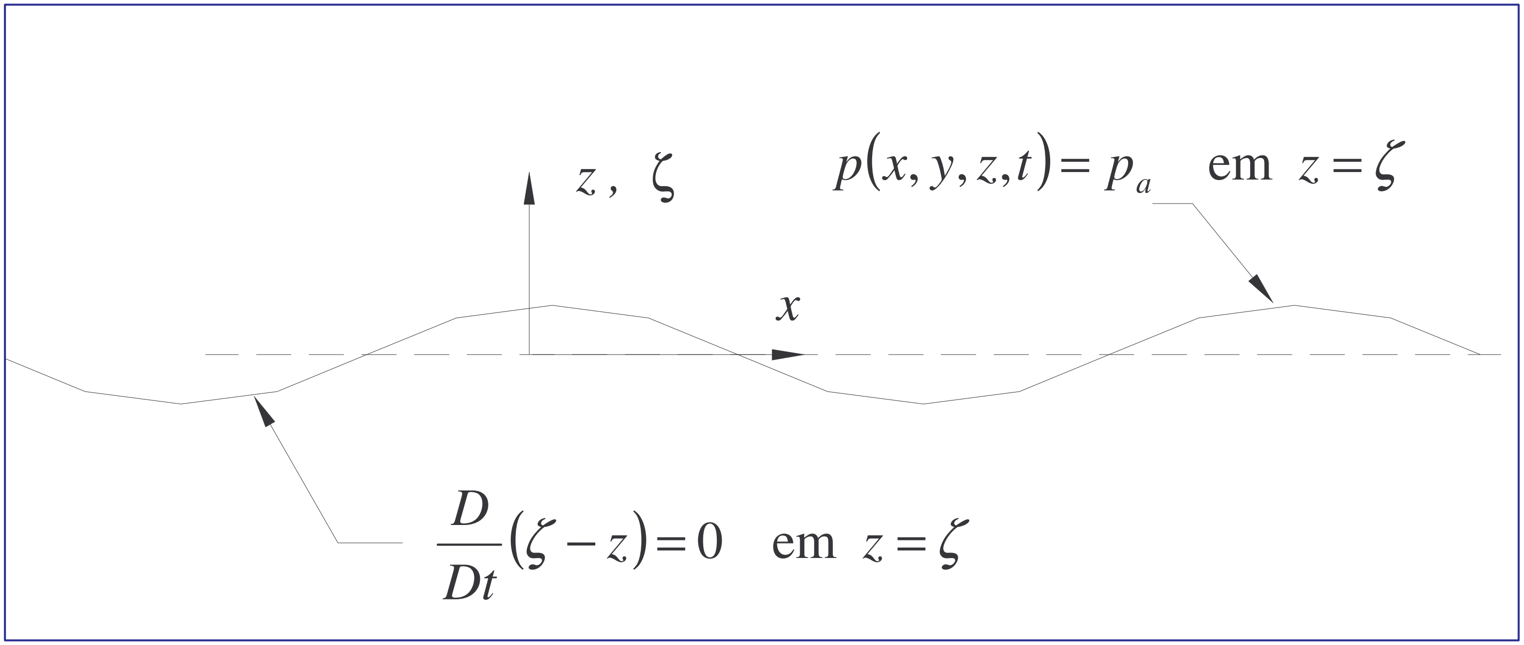
Body boundary condition

$$\vec{V}_S \cdot \vec{n}_w = \vec{V} \cdot \vec{n}_w \qquad \text{em } S_W \tag{1.5}$$



- Position of the boundary is not known a priori

Free surface boundary condition



- Nonlinear boundary condition
- Position of the boundary is not known a priori

(3) Bottom Boundary Condition

A - If deep water is assumed then the fluid perturbations are not felt near the bottom and the fluid particles are at rest

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z} = 0 \qquad \text{on } \; z \to \infty \tag{1.12a}$$

B – If the bottom is at a finite distance from the free surface, h, then the impermeability condition implies that the normal velocity is zero at the bottom:

$$\frac{\partial \Phi}{\partial z} = 0 \qquad \text{on } \; z = -h \tag{1.12b}$$

(4) Radiation Condition

Time Domain:

For an initial value problem the disturbance generated by the ship tend to vanish at large horizontal distances (r) from the body:

$$\nabla\Phi \to 0, \qquad \mathrm{r} \to \infty, \qquad \text{for } t < \infty \tag{1.13}$$

It is not necessary to satisfy a radiation condition at infinite

Frequency domain:

Waves due to body disturbance must be outgoing to infinity. It is necessary to establish a condition on a boundary extending from the free surface to the sea bottom at a distance r tending to infinite.

Example for cylindrical waves outgoing to infinite (Sommerfeld condition):

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial\Phi}{\partial r} - ik\Phi\right) = 0 \qquad r \to \infty \tag{1.14}$$

Exact Boundary Value Problem (within ideal fluid assumption)

(1) Body boundary condition

$$\vec{V}_S \cdot \vec{n}_w = \vec{V} \cdot \vec{n}_w \qquad \text{em } S_W$$

$on \quad S_W$

(2) Free Surface Boundary Condition

$$\Phi_{tt} + g\Phi_{z_0} + 2\nabla\Phi \cdot \nabla\Phi_t + \frac{1}{2}\nabla\Phi \cdot \nabla(\nabla\Phi \cdot \nabla\Phi) = 0$$

$on \quad z_0 = \zeta$

(3) Bottom Boundary Condition

$$\frac{\partial \Phi}{\partial z} = 0$$

$on \;\; z = -h$

(4) Radiation Condition

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial \Phi}{\partial r} - ik\Phi\right) = 0$$

$r \to \infty$

1.4 – Linearization of the Hydrodynamic Problem

The numerical solution of the fully nonlinear boundary value problem is possible but very complex.

Usually one imposes restrictions on the parameters governing the solution in order to simplify the B.B.C and F.S.B.C.

Restrictions on:

- Slenderness of the hull
- Speed of the ship
- Amplitude of oscillation of the boundaries (free surface and hull)
- Frequency of oscillation of the boundaries

Objective:

- Remove nonlinearities
- Remove/simplify 3D interactions between steady and unsteady flows

Different combinations of restrictions results on different seakeeping formulations, which may be or not be adequate depending on the physical problem one intends to represent.

Examples:

- Thin Ship Theory
- Slender Body Theory
- Strip Theories
- 2 1/2D Theory
- Panel Methods

Flow around the hull

The flow around the hull is decomposed in two parts:

- ✓ The **steady flow** associated with the hull advancing through the free surface in calm water, $\overline{\Phi}$
- ✓ The **oscillatory flow** associated with the incoming waves, diffracted waves and radiated (ship motions) waves, $\tilde{\Phi}$

The velocity potential becomes:

$$\Phi(\vec{x}_0, t) = \Phi(x + Ut, y, z, t) = \overline{\Phi}(\vec{x}) + \tilde{\Phi}(\vec{x}, t) \quad (1.17)$$

Linear Free Surface Boundary Condition

Procedure:

- ✓ Apply the small perturbations method to linearise the boundary condition to the first order
- ✓ Expand the boundary condition around the mean free surface which is known a priori (Taylor expansion)

The linear free surface boundary condition is:

$$\tilde{\Phi}_{tt} - 2U\tilde{\Phi}_{xt} + U^2\tilde{\Phi}_{xx} + g\tilde{\Phi}_z = 0 \quad \text{on} \quad z = 0 \qquad (1.37a)$$

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 \tilde{\Phi} + g\tilde{\Phi}_z = 0 \quad \text{on} \quad z = 0 \qquad (1.37b)$$

Simplified Body Boundary Condition

Objective:

Obtain a body boundary condition represented in terms of the mean wetted surface but including the interferences between steady and unsteady flows up to the first order

Since the steady potential is $\overline{\Phi} = -Ux + \Phi_S$ and $\vec{V}_0 = \nabla\overline{\Phi}$ it becomes:

$$\frac{\partial \Phi_S}{\partial n} = U n_1 \quad on \quad S_0 \qquad (1.46)$$

Boundary condition to be satisfied by the steady potential

For the unsteady potential the body boundary condition is:

$$\frac{\partial \tilde{\Phi}}{\partial n} = \left[\dot{\vec{\alpha}} + \left(\vec{V}_0 \cdot \nabla\right)\vec{\alpha} - \left(\vec{\alpha} \cdot \nabla\right)\vec{V}_0\right] \cdot \vec{n} \quad on \quad S_0 \qquad (1.47)$$

Boundary condition to be satisfied by the unsteady potential

Decomposition of the Velocity Potential

To derive the linear boundary conditions it was assumed that the oscillatory ship motions as well as the related potential are of small amplitude.

For this reason the oscillatory potential, $\tilde{\Phi}$, may be further decomposed into independent components related to:

- ✓ Incident / incoming waves Φ^I
- ✓ Diffracted waves Φ^D
- ✓ Radiated waves Φ^R

The result is:

$$\tilde{\Phi} = \Phi^I + \Phi^D + \Phi^R \qquad (1.48)$$

The radiation potential is further decomposed into components related with each of the six oscillatory ship motions:

$$\Phi^R = \sum_{j=1}^{6} \Phi_j^R \quad , \quad j = 1,\ldots,6 \qquad (1.49)$$

Linear Boundary Value Problem

In order to linearise the boundary conditions it was necessary to impose some restriction on the basic parameters that govern the solution of the hydrodynamic problem:

- The hull must be slender, B/L must be $O(\varepsilon)$
- The amplitude of the incoming waves must be small
- The oscillatory ship motions must be of small amplitude

Given the former simplifications, the **linear boundary value problem** consists of determining the velocity potential that satisfies the following conditions:

Laplace equation

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

Linear free surface b. c.

$$\Phi_{tt} - 2U\Phi_{xt} + U^2\Phi_{xx} + g\Phi_z = 0 \quad on \quad z = 0$$

Linear body b.c. (radiation)

$$\frac{\partial \Phi_j^R}{\partial n} = \dot{\xi}_j n_j + \xi_j U m_j \quad , \quad j = 1,\ldots,6 \qquad on \quad S_0$$

Linear body b.c. (diffraction)

$$\frac{\partial \Phi^D}{\partial n} = -\frac{\partial \Phi^I}{\partial n} \qquad on \quad S_0$$

Bottom boundary condition

$$\nabla\Phi \to 0 \quad \text{para} \quad z \to -\infty$$

Appropriate radiation condition at infinite

Linear Hydrodynamic Forces

The **Bernoulli equation** represented in the forward speed reference system is:

$$p - p_a = -\rho\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left|\nabla\Phi\right|^2 + gz_0 + \frac{1}{2}U^2\right) \quad (1.57)$$

where p is the fluid pressure, p_a is the atmospheric pressure, ρ is the fluid spec. mass

Substituting the potential decomposition (1.17) in the Bernoulli equation results in two groups of terms:

Constant pressure related to the steady flow

$$\frac{p - p_a}{\rho} = -\frac{1}{2}\left(\left|\nabla\overline{\Phi}\right|^2 + U^2\right) \quad (1.58)$$

Oscillatory pressure related to the unsteady flow

$$\frac{p - p_a}{\rho} = -\left(\frac{\partial\tilde{\Phi}}{\partial t} + \nabla\overline{\Phi}\cdot\nabla\tilde{\Phi} + zg\right) \quad (1.59)$$

Higher order terms in $\tilde{\Phi}$ are neglected in the former equations

The hydrodynamic force acting on the hull results from pressure integration over the mean wetted surface S_0 (according to the linear b. v. problem formulated):

Steady force ⇨

$$\overline{F} = -\frac{1}{2}\iint_{S_0}\left(\left|\nabla\tilde{\Phi}\right|^2 + U^2\right)\tilde{n}\,ds \qquad (1.60)$$

Unsteady force ⇨

$$\tilde{F} = -\iint_{S_0}\left(\frac{\partial\Phi_1}{\partial t} + \nabla\overline{\Phi}\cdot\nabla\tilde{\Phi}\right)\tilde{n}\,ds - \rho g\iint_{S}\left(z\tilde{n}\right)ds \qquad (1.61)$$

From now on we will consider only the unsteady forces

The first integral in equation (1.61) may be simplified applying a variation of the Stokes theorem.

The objective is to convert the surface integral which includes spatial derivatives of the oscillatory potential $\tilde{\Phi}$, on a surface integral that simple values of $\tilde{\Phi}$

Substituting (1.62), without the line integral which was neglected, into (1.61):

$$\tilde{F} = -\rho \iint_{S_0} \left(\frac{\partial \tilde{\Phi}}{\partial t} \tilde{n} - \tilde{\Phi} U \tilde{m} \right) ds - \rho g \iint_{S} (z \tilde{n}) ds \qquad (1.63)$$

Applying the potential decomposition of (1.48) and (1.49) results in three groups of hydrodynamic forces:

Exciting Forces ⇨

$$F^{E} = -\rho \iint_{S_0} \left(\frac{\partial \left(\Phi^{I} + \Phi^{D} \right)}{\partial t} \tilde{n} - \left(\Phi^{I} + \Phi^{D} \right) U \tilde{m} \right) ds \qquad (1.64)$$

Radiation Forces ⇨

$$F^{R} = -\rho \iint_{S_0} \left(\frac{\partial \Phi^{R}}{\partial t} \tilde{n} - \Phi^{R} U \tilde{m} \right) ds \qquad (1.65)$$

Hydrostatic Forces ⇨

$$F^{H} = -\rho g \iint_{S} (z \tilde{n}) ds \qquad (1.66)$$