# Exercises on categories and functors

§1. Exercise. Prove that every section is a monomorphism, and that every retraction is an epimorphism.

§2. Exercise. Prove that, in any category, any morphism which is both a section and an epimorphism is an isomorphism. Dualize to obtain the analogous statement about retractions that are monomorphisms.

§3. Exercise. Let $(P, \leq)$ be a poset, regarded as a small category. Prove that every morphism in $P$ is both a monomorphism and an epimorphism. Is every morphism in $P$ an isomorphism?

§4. Exercise. Prove that the inclusion of monoids $i : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is both a monomorphism and an epimorphism in *Mon*, but that it is not an isomorphism.

§5. Exercise. Prove that the inclusion of unital rings $i : \mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism in *Rng*, but that it is not an isomorphism.

§6. Exercise. Let $F : C \to D$. By *image* of $F$ we mean the classes $F(C_0) \subset D_0$ and $F(C_1) \subset D_1$ equipped with the structure obtained from $D$ (domain and codomain maps, etc.). Is the image of $F$ necessarily a subcategory of $D$? (Hint: the image can contain composable pairs of morphisms $(f, g)$ in the image whose composition $fg$ is not in the image.) Give a condition on $F$ that ensures that its image is a subcategory of $D$.

§7. Exercise. Let $F : C \to D$ be a functor. Prove that if $f \in C_1$ is a retraction then so is $Ff$, and that if $f$ is a section then so is $Ff$. Conclude that $Ff$ is an isomorphism whenever $f$ is.

§8. Exercise. Let $F : C \to D$ be a faithful functor. We say that $F$ *reflects* a property $X$ if, for all arrows $f$ of $C$, whenever $F(f)$ has the property $X$ then so does $f$.

1. Prove that $F$ reflects monomorphisms and that it reflects epimorphisms.

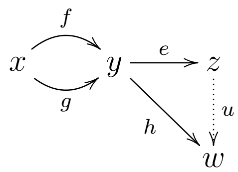
2. Assuming that $F$ is also full, prove that it reflects retractions and that it reflects isomorphisms.

§9. Exercise. Is the forgetful functor $U : \mathbb{C}\text{-}\mathit{Vect} \to \mathit{Set}$ faithful? Is it full?

§10. Exercise. A *concrete category* is a pair $(C, U)$ consisting of a category $C$ and a faithful functor $U : C \to \mathit{Set}$. Give three examples of concrete categories which have already been mentioned in the lectures.

§11. Exercise. Characterize the monomorphisms and the epimorphisms in *Set* and *Ab*.

§12. Exercise. Let $f, g : x \to y$ in a category $C$. An arrow $e : y \to z$ is a *coequalizer* of $f$ and $g$ if $ef = eg$ and for all $h : y \to w$ such that $hf = hg$ there is a unique arrow $u : z \to w$ such that the triangle commutes:




1. Describe the coequalizers in *Set*. Hint: notice that a pair of maps $f, g : X \to Y$ defines a binary relation on $Y$, consisting of all the pairs $(f(x), g(x))$ with $x \in X$.

2. An arrow is said to be a *regular epimorphism* if it is the coequalizer of some pair of arrows. Prove that every regular epimorphism is an epimorphism.

3. Prove that every retraction is a regular epimorphism.

4. Prove that if an arrow is both a monomorphism and a regular epimorphism then it is an isomorphism.

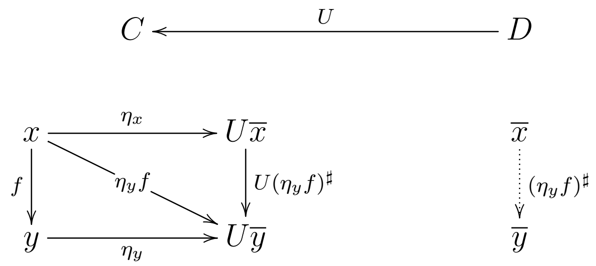
5. Give an example of an epimorphism which is not regular. Hint: recall some of the above exercises.

6. (*This exercise requires the material of section 0.3 of Dummit&Foote, and also the group theory of sections 1.1 and 2.3.*) Give an example of a regular epimorphism which is not a retraction. Hint: consider the surjective homomorphism of abelian groups (with $n \in \mathbb{Z}_{\geq 2}$)

$$q : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

which to each $k \in \mathbb{Z}$ assigns the class $\overline{k} = k + n\mathbb{Z} = \{k + nm \mid m \in \mathbb{Z}\}$.

§13. Exercise. (This was not addressed in the lectures.) Let $C$ and $D$ be categories, and $U : D \to C$ a functor. Assuming that for every $x \in C_0$ there is a universal arrow from $x$ to $U$, prove that there is a functor $F : C \to D$. Hint: for each universal arrow $(\overline{x}, \eta_x)$ from $x$ to $U$ define $F(x)$ to be $\overline{x}$, and for each $f : x \to y$ in $C_1$ define $F(f)$ to be the extension $(\eta_y \circ f)^\sharp$:




§14. Exercise. Let $f : x \to y$ in a category $C$. A pair of arrows $h_1, h_2 : z \to x$ is called a *kernel pair* of $f$ if $fh_1 = fh_2$ and for all other pairs of arrows $k_1, k_2 : w \to x$ such that $fk_1 = fk_2$ there is a unique arrow $u : w \to z$ such that $h_1 u = k_1$ and $h_2 u = k_2$.

1. Adopting the view that a pair $h_1, h_2$ defines a "binary relation" on $x$, describe the kernel pair of a function $f : X \to Y$ in *Set*.

2. Describe the kernel pair of a homomorphism $f : A \to B$ of abelian groups in *Ab*.

3. How does the above kernel pair relate to the usual kernel of a group homomorphism?

4. Prove that if a regular epimorphism $f$ has a kernel pair $h_1, h_2$, then $f$ is a coequalizer of $h_1, h_2$.

5. Give an example in *Set* of a function $f : X \to Y$ which is not the coequalizer of its kernel pair.

§15. Exercise. Let $Matr_{\mathbb{C}}$ be the category whose objects are the natural numbers and whose arrows are the complex valued matrices, where an arrow $A : n \to m$ consists of an $m \times n$ matrix, and composition of arrows is given by matrix multiplication.

1. Verify that $Matr_{\mathbb{C}}$ is indeed a category.
2. Define a functor from $Matr_{\mathbb{C}}$ to the category of complex vector spaces that sends each object $n$ to the space $\mathbb{C}^n$ and each matrix $A : n \to m$ to a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^m$.