Information and Communication Theory

2023

Problem Set 5

Department of Electrical and Computer Engineering,

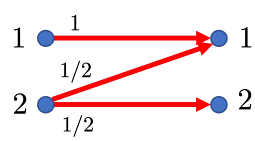
Instituto Superior Técnico, Lisboa, Portugal

1. Compute the capacity of a series connection of two binary symmetric channels.

2. Consider the parallel of two independent channels $(\mathcal{X}_1, p_1(y|x), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y|x), \mathcal{Y}_2)$, *i.e.*, the channel
$$(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1|x_1)p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2).$$
what is the capacity of this channel?

3. Consider N channels with $|\mathcal{X}| = |\mathcal{Y}|$ and non-maximal capacity, *i.e.*, $C < \log |\mathcal{X}|$, connected in series. Show that the capacity of the resulting channel converges to zero as N goes to infinity. Hint: use the data processing inequality.

4. Compute the capacity and the maximizing $p(x)$ for the Z channel.



5. Consider a channel obtained by taking two conditional independent looks at the output of a channel of capacity C, for each input: Y_1 and Y_2. Show that the resulting capacity $C' \leq 2C$. Hint: begin by showing that $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

6. A symmetric channel is one in which every row of the channel matrix is a permutation of every other row and every column is a permutation of every other column. Show that in this case, the capacity is
$$C = \log |\mathcal{Y}| - H(\text{any row of the channel matrix}).$$

7. Show that the same result applies to weakly symmetric channels, where the columns are only required to sum to he same number.

8. Show that the repetition code of $R = 1/3$ is a Hamming(n, k) code. Find r, n, k, and the matrices $\mathbf{H}$ and $\mathbf{G}$.

9. Show that for a Hamming$(7, 4)$ code, $d_{\min} = 3$, thus it corrects 1 error and detects up to 2 errors.

10. Consider a Hamming$(7, 4)$ code in systematic form. Decode the word (1011011).

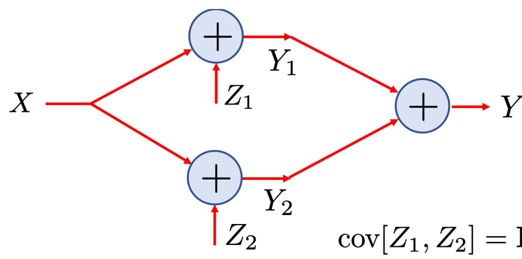
11. A Hamming code is a particular case of the more general family of linear codes, i.e., where the code words are generated as $\mathbf{x} = \mathbf{mG}$. Show that for any binary linear code,

    a) the zero word is a valid codeword;

    b) d_{min} is the weight (number of 1s) is the minimum-weight code word.

12. Assuming a Hamming$(7, 4)$ code is used on a BSC with probability of error α, what is the probability of an erroneous decoding?

13. Consider the multi-path channel where the noises Z_1 an Z_2 follow a Gaussian joint probability density function with zero mean and covariance $\mathbf{K}$



$$\text{cov}[Z_1, Z_2] = \mathbf{K} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

where σ^2 is the noise variance and ρ the correlation coefficient. Find the capacity of the channel. What is the capacity for $\rho = 1$, $\rho = 0$, $\rho = -1$; interpret the results.

14. Continuous channel with discrete input: consider a channel with input $X \in \{0, 1\}$ and ouput $Y = X + Z$, where $Z \in [0, a]$ with uniform density. Assuming $a > 1$, find the capacity of the channel. Repeat for $a < 1$ and interpret the result.