# Tensor products of modules. II

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**Abstract**

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Although categorical coproducts of modules coincide with products, this is only true for finitary products and coproducts. Given a set $I$, finite or finite, the categorical product of a family of modules $(M_i)_{i\in I}$ is their cartesian product equipped with componentwise action and group structure. Each element of $\prod_{i\in I} M_i$ is a family $(m_i)_{i\in I}$ where for each $i \in I$ we have $m_i \in M_i$. Such a family can be regarded as a function $f : I \to \prod_{i\in I} M_i$ that to each $i \in I$ assigns an element of $M_i$. If $f$ if finitely supported we say that the family $(m_i)_{i\in I}$ is *finitely supported*. Then we define:

§1. Definition. The coproduct $\bigoplus_{i\in I} M_i$ is the submodule of $\prod_{i\in I} M_i$ that consists of all the finitely supported elements.

§2. Example. $\bigoplus_{i\in I} R$ is isomorphic to the free module $F_R(I)$, whereas $\prod_{i\in I} R$ is isomorphic to the function module $R^I$ that consists of all the $R$-valued functions on $I$ with pointwise module operations. In particular,

$$\underbrace{R \oplus \cdots \oplus R}_{n \text{ times}} \cong R^n.$$

§3. Theorem. *The tensor product of modules distributes over direct sums, up to isomorphisms. More precisely, given an $(R,S)$-bimodule $M$ and a family $(N_i)_{i\in I}$ of $(S,T)$-bimodules, $\bigoplus_{i\in I} N_i$ is an $(S,T)$-bimodule and we have an isomorphism of $(R,T)$-bimodules*

$$M \otimes_S \bigoplus_{i\in I} N_i \cong \bigoplus_{i\in I} M \otimes_S N_i$$

*which to each simple tensor $m \otimes (n_i)$ assigns $\sum_{i\in I} m \otimes n_i$.*

*Proof.* Sketched in the lecture. ∎

§4. Corollary. $M \otimes_S (N \oplus P) \cong (M \otimes N) \oplus (M \otimes P)$.

§5. Exercise. Describe the universal properties of $\prod_{i\in I} M_i$ and $\bigoplus_{i\in I} M_i$.

§6. Exercises. Let $R$ and $S$ be rings.

1. Let $M$ be an $(S,R)$-bimodule. Prove that there is a an isomorphism of $(S,R)$-bimodules
$$M \otimes_R R^n \cong M^n.$$
Show that this generalizes to an isomorphism of $(S,R)$-bimodules
$$M \otimes F_R(I) \cong \oplus_{i\in I} M.$$

2. Prove that there is an isomorphism of $(R,R)$-bimodules
$$R^n \otimes_R R^m \cong R^{nm}$$
and, more generally, an isomorphism
$$F_R(A) \otimes_R F_R(B) \cong F_R(A \times B).$$

§7. Exercises. Prove the following isomorphisms:

1. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ (isomorphism of complex vector spaces)
2. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ (isomorphism of rational vector spaces)
3. $\mathbb{Q} \otimes_{\mathbb{Z}} A \cong 0$ for any finite abelian group $A$
4. Give an example showing that we may have $N \subset M$ but
$$N \otimes_R P \not\subset M \otimes_R P.$$
5. $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong 0$
6. $(R/I) \otimes_R N \cong N/IN$ (isomorphism of abelian groups) for all ideals $I$ of $R$, where $N$ is a left $R$-module and
$$IN = \left\{ \sum_{\text{finite}} a_i n_i \mid a_i \in A,\ n_i \in N \right\}.$$