Information and Communication Theory

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Problem Set 4

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1. Compute $h(X)$, for a source $X \in \mathbb{R}_+$, with $f_X(x) = \lambda e^{-\lambda x}$ (exponential density). Useful fact: $\mathbb{E}(X) = 1/\lambda$.

2. Compute $h(X)$, for a source $X \in \mathbb{R}$, with $f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}$ (Laplacian density).

3. Let $X \in [-1, 1]$. Consider $Y = 1$, if $X < 0$, and $Y = 2$, if $X \geq 0$. Compute $I(X;Y)$ from the definition and from the master definition in slide 21 of Lecture 4.

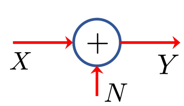
4. Let $X \in \{1, 2, 3\}$, with pmf $\mathbf{f}_X = (1/3, 1/3, 1/3)$, and $Y = 1$, if $X = 1$ or $X = 2$, and $Y = 2$, if $X = 3$. Compute $I(X;Y)$ from the definition $I(X;Y) = H(Y) - H(Y|X)$ and from the master definition in slide 21 of Lecture 4.

5. Check that the differential entropy satisfies the following form of grouping. Let $X \in \mathbb{R}$ with pdf f_X. Let A, B be a partition of $\mathbb{R}$ such that $p_A = \mathbb{P}[X \in A] = \int_A f_X(x)\, dx$ and $p_B = \mathbb{P}[X \in B] = \int_B f_X(x)\, dx$. Then,

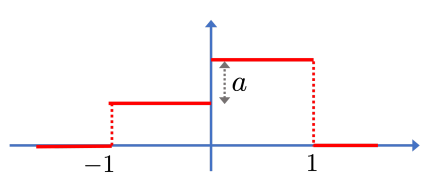
$$h(X) = H(p_A, p_B) + p_A\, h(X|X \in A) + p_B\, h(X|X \in B).$$

6. Choose the parameters of the exponential, Laplacian, and uniform densities to have the same variance and confirm that the Gaussian density has larger entropy for that same variance.

7. Let X have pdf $\mathcal{N}(0, \tau^2)$ and Y be a noisy version of X, that is $Y = X + N$, where N has pdf $\mathcal{N}(0, \sigma^2)$ and is independent of X. Compute $I(X;Y)$. Find the limits of $I(X;Y)$ as the noise variance σ^2 goes to 0 and ∞ and interpret the results.



8. Using the grouping property, compute $h(X)$ for the following pdf, as a function of $a \in [-1, 1]$:



9. Consider a Gaussian source X with pdf $\mathcal{N}(0, \tau^2)$ and two functions of X: $Y_1 = |X|$ and $Y_2 = \text{sign}(X) \in \{-1, 1\}$. Compute $I(X; Y_1)$, $I(X; Y_2)$, and $I(X; Y_1, Y_2)$.

10. Consider the problem of estimating X from Y_1 or from Y_2. Find the corresponding lower bounds for the squared error of the estimates.

11. Consider a source X with uniform density $U(x; 0, 1)$ and the two following functions of X:

$$Z_1(X) = \begin{cases} 1, & \text{if } X \leq 1/2 \\ 2, & \text{if } X > 1/2, \end{cases} \qquad Z_2(X) = \begin{cases} 1 & \text{if } X \leq 1/4 \\ 2 & \text{if } X \in]1/4,\ 1/2] \\ 3 & \text{if } X \in]1/2,\ 3/4] \\ 4 & \text{if } X > 3/4 \end{cases}$$

Consider the problem of estimating X from Z_1 or from Z_2. Find the corresponding lower bounds for the squared error of the estimates.