Information and Communication Theory

2023

Problem Set 3

Department of Electrical and Computer Engineering,

Instituto Superior Técnico, Lisboa, Portugal

1. For each of the following codes

x	$p(x)$	C_1	C_2	C_3	C_4	C_5
x_1	1/2	00	0	0	0	0
x_2	1/4	01	0	1	01	10
x_3	1/8	10	1	01	011	110
x_4	1/8	11	11	10	0111	111

say if it is non-singular, uniquely decodable, or instantaneous. Compute the expected length of each of the codes.

2. Converse of the KMI: given a collection of N positive integers, $l_1, ..., l_N$, that satisfy

$$\sum_{x=1}^{N} D^{-l_x} \leq 1,$$

is it necessarily possible to construct an instantaneous code such that $l_C(x) = l_x$? Give an example.

3. Given the collection of numbers $(2, 2, 2, 3, 4)$, is it possible to build an instantaneous code with these lengths? Why? If yes, give an example. Could the code be optimal, or can we build another instantaneous code with necessarily shorted expected code-length?

4. Repeat the previous question for the set of numbers $(1, 2, 2, 3, 4)$.

5. List all possible distributions of lengths for instantaneous codes with the following numbers of words: $N = 3$, $N = 4$, and $N = 5$.

6. For the source $X \in \mathcal{X} = \{1, 2, 3, 4, 5\}$ with $\mathbf{f}_X = (1/3, 1/3, 1/9, 1/9, 1/9)$, obtain a binary Shannon-Fano code and compute its expected length and efficiency. Is it an optimal code?

7. For the source $X \in \mathcal{X} = \{1, 2, 3, 4\}$ with $\mathbf{f}_X = (1/2, 1/4, 1/8, 1/8)$, obtain an optimal binary code and compute its expected length and efficiency. Is it an ideal code?

8. Repeat the two previous exercise, but now for ternary codes.

9. Show that there are sources and corresponding optimal codes such that $L(C^{\text{optimal}})$ is arbitrarily close to $H(X) + 1$.

10. For the source $X \in \mathcal{X} = \{1, 2\}$ with $\mathbf{f}_X = (7/8, 1/8)$, obtain a binary optimal code and compute its expected length and efficiency. Find de optimal code for the order-2 extension of this source.

11. Repeat the previous question for $\mathbf{f}_X = (1/2, 1/2)$.

12. Consider a stationary Markov source $X_t \in \mathcal{X} = \{1, 2, 3\}$, with the following probability transition matrix:

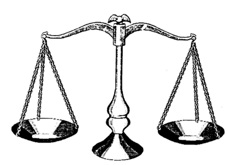
$$\mathbf{P} = \begin{bmatrix} 0 & 1/8 & 7/8 \\ 7/8 & 0 & 1/8 \\ 1/8 & 7/8 & 0 \end{bmatrix}.$$

Obtain the optimal coding for this source and for its order-2 extension.

13. Find a binary Huffman code for the source $X \in \mathcal{X} = \{1, 2, 3, 4\}$ with $\mathbf{f}_X = (0.27, 0.26, 0.24, 0.23)$. Are there optimal codes for this source that are not Huffman codes? If yes, how many?

14. Find a ternary Huffman code for the source in the previous question. Are there optimal ternary codes for this source that are not Huffman codes? If yes, how many?

15. Consider 9 apparently equal spheres, 8 of which weight 1Kg and one weights 1.05Kg. Suppose you have a balance scale.



What is the minimum number of weightings needed to identify the heavier sphere? Propose an non-sequential procedure (*i.e.*, each weighting does not depend on the results of previous ones) to find the sphere in the minimum number of weightings.

16. You have 6 bottles of wine, but you know that one has gone bad (tastes like vinegar). By inspection, you determine that the probabilities of bottle i being the bad one are $(8, 6, 4, 2, 2, 1)/23$. In what order should you taste the bottles to find the bad one in the minimal number of tastings? What is the expected number of tastings? Can you do better if you are allowed to mix wines and taste the mixture?