DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

LEIC-A, LMAC, MECD, MEIC-A, MMA

2nd. Semester – 2020/2021
2021/07/14 – 8AM
Test 1 (*Recurso*)

Duration: 90 minutes

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 0 — Introduction to Stochastic Processes 2.5 points

Admit that a single particle moves on a line according to a simple symmetric random walk, that is, its position at time n ($n \in \mathbb{N}_0$) is given by $S_n = S_0 + \sum_{i=1}^n X_i$, where $S_0 = 0$ and the step sizes X_i are i.i.d. r.v. with common p.f. $P(X=+1) = P(X=-1) = \frac{1}{2}$.

(a) Derive the mean, variance, and autocovariance functions of the stochastic process $\{S_n : n \in \mathbb{N}_0\}$. (2.0)

- **Stochastic process**
  $\{S_n : n \in \mathbb{N}_0\}$
  $S_n = S_0 + \sum_{i=1}^n X_i$ = position of a single particle on a line at time n
  $S_0 = 0$
  $X_i \overset{i.i.d.}{\sim} X, \quad i \in \mathbb{N}$
  $P(X=+1) = P(X=-1) = \frac{1}{2}$
- **Requested mean, variance, and autocovariance functions**

$$\begin{aligned}
E(S_n) &= E\left(S_0 + \sum_{i=1}^n X_i\right) \\
&= S_0 + \sum_{i=1}^n E(X_i) \\
&\overset{X_i \sim X}{=} 0 + n \times E(X) \\
&= n \times \left[(-1) \times \frac{1}{2} + 1 \times \frac{1}{2}\right] \\
&= 0 \\
V(S_n) &= E\left(S_0 + \sum_{i=1}^n X_i\right) \\
&\overset{X_i\, indep}{=} \sum_{i=1}^n V(X_i) \\
&\overset{X_i \sim X}{=} n \times V(X) \\
&\overset{E(X)=0}{=} n \times E(X^2) \\
&= n \times \left[(-1)^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2}\right] \\
&= n \\
&= Cov(S_n, S_n) \\
Cov(S_n, S_{n+h}) &= Cov\left(S_n, S_n + \sum_{i=n+1}^{n+h} X_i\right), \quad h \in \mathbb{N} \\
&= Cov(S_n, S_n) + Cov(S_n, S_{n+h} - S_n) \\
&\overset{S_n \perp\!\!\!\perp S_{n+h} - S_n}{=} V(S_n) + 0 \\
&= n.
\end{aligned}$$

[Analogously, we get, for $h = -n+1, \ldots, -1$, $Cov(S_n, S_{n+h}) = Cov(S_{n+h}, S_{n+h} + S_n - S_{n+h}) = n+h$. In fact, $Cov(S_n, S_{n+h}) = \min\{n, n+h\}$, for $n \in \mathbb{N}$ and $h = -n+1, \ldots, -1, 0, 1, \ldots$.]

(b) Is $\{S_n : n \in \mathbb{N}_0\}$ a second order weakly stationary process? (0.5)

- **Investigating the second order weakly stationarity**
  The autocovariance function is time dependent, thus we not are dealing with a second order weakly stationary process.

Group 1 — Poisson Processes 9.5 points

1. Suppose customers arrive at a bank according to a Poisson process at a rate of 8 per hour.

(a) Obtain the mean and the variance of the number of customers who enter the bank during an 8-hour day — from 9AM (the time origin) to 5PM. (0.5)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda = 8)$
  $N(t)$ = number of arrivals of customers at the bank by time t
  (time in hours; $t = 0$ corresponds to 9AM)
- **Relevant distribution**
  $N(t) \sim \text{Poisson}(\lambda\, t = 8\, t), \quad t > 0$
- **Requested expected value and variance**

$$\begin{aligned}
E[N(8)] &= V[N(8)] \\
&= 8 \times 8 \\
&= 64.
\end{aligned}$$

(b) Find the probability than no customers arrive during the last 15 minutes of this 8-hour day. (0.5)

- **P.f. of** $N(t)$
  $P[N(t) = n] = e^{-\lambda\, t} \frac{(\lambda\, t)^n}{n!}, \quad n \in \mathbb{N}_0$
- **Requested probability**

$$\begin{aligned}
P[N(8) - N(7.45) = 0] &\overset{stationary\ inc.}{=} P[N(8 - 7.75) = 0] \\
&= P[N(0.25) = 0] \\
&\overset{N(0.25) \sim Poisson(\lambda \times 0.25)}{=} e^{-8 \times 0.25} \\
&= e^{-2} \\
&\simeq 0.135335.
\end{aligned}$$

(c) Determine the covariance between the number of customers who enter the bank between 9AM and 11AM, and those who enter between 10AM and noon. (1.5)

Hint: Firstly, derive the autocovariance function of a Poisson process with rate λ, $Cov(N(s), N(t))$, for $0 < s < t$.

- **Relevant facts**
  $N(t) \sim \text{Poisson}(\lambda\, t = 8\, t), \quad t > 0$
  $\{N(t) : t \geq 0\}$ has independent and stationary increments
- **Autocovariance function**
  Consider $0 < s < t$. Then

$$\begin{aligned}
Cov(N(s), N(t)) &= E[N(s) \times N(t)] - E[N(s)] \times E[N(t)] \\
&= E\{N(s) \times [N(t) - N(s) + N(s)]\} - E[N(s)] \times E[N(t)] \\
&= E\{N(s) \times [N(t) - N(s)]\} + E[N^2(s)]\} - E[N(s)] \times E[N(t)] \\
&\overset{indep.\ incr.}{=} E[N(s)] \times E[N(t) - N(s)] + \{V[N(s)] + E^2[N(s)]\} \\
&\quad - E[N(s)] \times E[N(t)]
\end{aligned}$$

$$
\begin{aligned}
Cov(N(s),N(t)) &\overset{station.incr.}{=} E[N(s)]\times E[N(t-s)]+\{V[N(s)]+E^2[N(s)]\} \\
&\quad -E[N(s)]\times E[N(t)] \\
&= \lambda s\times\lambda(t-s)+[\lambda s+(\lambda s)^2]-\lambda s\times\lambda t \\
&= \lambda^2 s t-\lambda^2 s^2+\lambda s+\lambda^2 s^2-\lambda^2 s t \\
&= \lambda s.
\end{aligned}
$$

[Capitalizing on the symmetry of the covariance operator, we obtain, for $s,t \geq 0$, $Cov(N(s),N(t)) = \min\{s,t\}$.]

- **Requested covariance**

  Given the fact that the covariance is a bilinear operator and the autocovariance function above, we can successively write

$$
\begin{aligned}
Cov(N(11-9)-N(9-9),N(12-9)-N(10-9)) &\overset{N(0)=0}{=} Cov(N(2),N(3)-N(1)) \\
&= Cov(N(2),N(3))-Cov(N(1),N(2)) \\
&= 2\lambda-\lambda \\
&= \lambda.
\end{aligned}
$$

2. Let $S_{N(t)}$ be the time of the last renewal prior to (or at) time t ($t>0$) in a Poisson process with rate λ, $\{N(t): t\geq 0\}$. (2.0)

Derive $E\left[S_{N(t)}\right]$ by conditioning on $N(t)$.

- **Stochastic process**

  $N(t)$ = number of events up to time t

  $N(t) \sim \text{Poisson}(\lambda t)$

  $S_{N(t)}$ = last renewal prior to (or at) time t

- **Requested expected value**

  By conditioning on $N(t)$, we get $E\left[S_{N(t)}\right] = E\left\{E\left[S_{N(t)} \mid N(t)\right]\right\}$, where the r.v. $E\left[S_{N(t)} \mid N(t)\right]$ takes the value $E\left[S_{N(t)} \mid N(t)=n\right]$ with probability $P[N(t)=n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, $n\in\mathbb{N}$.

  Under the condition that n events have occurred in $(0,t]$, the event times $S_1,\ldots,S_n$ behave as the order statistics $Y_{(1)},\ldots,Y_{(n)}$ associated with $Y_i \sim_{i.i.d.} Y \sim \text{uniform}(0,t)$, $i=1,\ldots,n$. Moreover,

$$
\begin{aligned}
\frac{Y_{(i)}}{t} &\sim \text{beta}(i,n-i+1), \quad i=1,\ldots,n \\
\left(\frac{S_n}{t} \mid N(t)=n\right) &\sim \frac{Y_{(n)}}{t} \\
&\sim \text{beta}(n,n-n+1) \equiv \text{beta}(n,1)
\end{aligned}
$$

  Consequently,

$$
\begin{aligned}
E\left[S_{N(t)} \mid N(t)=n\right] &= t\times E\left[\frac{S_{N(t)}}{t} \mid N(t)=n\right] \\
&= t\times E[\text{beta}(n,1)] \\
&\overset{form.}{=} t\times\frac{n}{n+1} \\
E\left[S_{N(t)}\right] &= E\left\{E\left[S_{N(t)} \mid N(t)\right]\right\} \\
&= E\left[t\times\frac{N(t)}{N(t)+1}\right] \\
&= t\times E\left[1-\frac{1}{N(t)+1}\right] \\
&= t-t\times\sum_{n=0}^{+\infty}\frac{1}{n+1}\times e^{-\lambda t}\frac{(\lambda t)^n}{n!}
\end{aligned}
$$

$$
\begin{aligned}
E\left[S_{N(t)}\right] &= t-\frac{e^{-\lambda t}}{\lambda}\times\sum_{n=1}^{+\infty}\frac{(\lambda t)^n}{n!} \\
&= t-\frac{e^{-\lambda t}}{\lambda}\times\left(e^{\lambda t}-1\right) \\
&= t-\frac{1-e^{-\lambda t}}{\lambda}.
\end{aligned}
$$

3. Admit that requests arrive to a server according to a non-homogeneous Poisson process with intensity function $\lambda(t) = 1+e^{-t}$, $t\geq 0$ (time in hours).

(a) Obtain the probability that the number of requests in the first 24 hours exceeds its expected value. (1.0)

- **Stochastic process**

  $\{N(t): t>0\} \sim NHPP(\lambda(t))$

  $N(t)$ = number of requests arrived to the server until time t

- **Intensity and mean value functions**

$$
\lambda(t) = 1+e^{-t}, \quad t\geq 0
$$

$$
\begin{aligned}
m(t) &= E[N(t)] \\
&= \int_0^t \lambda(s)\,ds \\
&= \int_0^t \left(1+e^{-s}\right)ds \\
&= t+1-e^{-t}, \quad t\geq 0
\end{aligned}
$$

  Relevant r.v.

  $N(24)$ = number of requests in the first 24 hours

  $N(24) \sim \text{Poisson}(m(24))$, where $m(24) = 24+1-e^{-24} = 25-e^{-24} \in (24,25)$

- **Requested probability**

$$
\begin{aligned}
P[N(24)>m(24)] &= 1-F_{Poisson(25-e^{-24})}(25-e^{-24}) \\
&= 1-F_{Poisson(25-e^{-24})}([25-e^{-24}]) \\
&= 1-F_{Poisson(25-e^{-24})}(24) \\
&\simeq 1-F_{Poisson(25)}(24) \\
&\overset{tables}{=} 1-0.4734 \\
&= 0.5266.
\end{aligned}
$$

(b) Suppose a request arrived at time s is considered a priority job with probability $p(s) = \frac{e^s}{1+e^s}$, $s>0$, independently of everything else. (1.5)

Let $N_{PR}(t)$ be the number of priority requests up to time t. After briefly explaining why $N_{PR}(t) \sim \text{Poisson}\left(\int_0^t \lambda(s)\times p(s)\,ds\right)$, obtain the probability that the number of priority requests in the first 24 hours exceeds its expected value.

- **Relevant r.v. and its distribution**

  $N_{PR}(t)$ = number of priority requests up to time t

  $N_{PR}(t) \sim \text{Poisson}\left(\int_0^t \lambda(s)\times p(s)\,ds\right)$

- **Brief explanation**

  If $N^*_{PR}(t)$ resulted from a non-homogenous Bernoulli splitting of a $\{N^*(t): t\geq 0\} \sim PP(\lambda)$, then we would get from the formulae $N^*_{PR}(t) \sim \text{Poisson}\left(\int_0^t \lambda\times p(s)\,ds\right)$.

However, $N_{PR}(t)$ results from a non-homogenous Bernoulli splitting of $\{N(t) : t \geq 0\} \sim NHPP(\lambda(t))$, thus the extension of the previous result would be obtained by replacing the constant rate λ by the time-dependent intensity function $\lambda(s)$.

- **Expected value of $N_{PR}(t)$**

$$E[N_{PR}(t)] = \int_0^t \lambda(s) \times p(s)\, ds = \int_0^t (1+e^{-s}) \times \frac{e^s}{1+e^s}\, ds = \int_0^t ds = t$$

- **Requested probability**

$$\begin{aligned} P\{N_{PR}(24) > E[N_{PR}(24)]\} &= P[N_{PR}(24) > 24] \\ &= 1 - F_{Poisson(24)}(24) \\ &\overset{tables}{=} 1 - 0.5540 \\ &= 0.4460. \end{aligned}$$

4. An insurance company admits that the number of claims made by a policyholder (chosen at random) is governed by a conditional Poisson process $\{N(t) : t \geq 0\}$ with random rate Λ uniformly distributed over $(0,1)$.

(a) Derive a simplified expression for $P[\Lambda \leq x \mid N(t) = n]$, the probability that the rate of this policyholder does not exceed x $(0 < x < 1)$, given that he/she has made n $(n \in \mathbb{N}_0)$ claims in his/her first t $(t > 0)$ years. (2.0)

- **Stochastic process**

  $\{N(t) : t \geq 0\} \sim CdPP(\Lambda)$

  $N(t)$ = number of claims made by the policyholder (chosen at random) until time t

- **Random arrival rate**

  $\Lambda \sim \text{uniform}(0,1)$

$$f_\Lambda(\lambda) = \begin{cases} 1, & 0 < \lambda < 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Requested probability**

  We are supposed to calculate

$$P[\Lambda \leq x \mid N(t) = n] = \frac{P[N(t) = n, \Lambda \leq x]}{P[N(t) = n]}.$$

By applying the total probability law, we obtain the numerator of this conditional probability:

$$\begin{aligned} P[N(t) = n, \Lambda \leq x] &= \int_0^x P[N(t) = n \mid \Lambda = \lambda] \times f_\Lambda(\lambda)\, d\lambda, \quad 0 < x < 1 \\ &= \int_0^x \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, d\lambda \\ &= \frac{1}{t}\int_0^x \frac{t^{n+1}}{n!} \lambda^{(n+1)-1} e^{-t\lambda}\, d\lambda \\ &= \frac{1}{t}\int_0^x f_{gamma(n+1,t)}(\lambda)\, d\lambda \\ &= \frac{1}{t} F_{gamma(n+1,t)}(x) \\ &= \frac{1}{t}[1 - F_{Poisson(x\,t)}(n)] \end{aligned}$$

The denominator of $P[\Lambda \leq x \mid N(t) = n]$ is given by

$$\begin{aligned} P[N(t) = n] &= \int_0^1 P[N(t) = n \mid \Lambda = \lambda]\, dF_\Lambda(\lambda) \\ &\overset{form.}{=} \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, dF_\Lambda(\lambda) \\ &= \int_0^1 \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, d\lambda \end{aligned}$$

$$\begin{aligned} P[N(t) = n] &= P[N(t) = n, \Lambda \leq 1] \\ &= \frac{1}{t}[1 - F_{Poisson(t)}(n)], \end{aligned}$$

yielding

$$\begin{aligned} P[\Lambda \leq x \mid N(t) = n] &= \frac{P[N(t) = n, \Lambda \leq x]}{P[N(t) = n]} \\ &= \frac{\int_0^x \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, d\lambda}{\int_0^1 \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, d\lambda} \\ &= \frac{\int_0^x e^{-\lambda t} \lambda^n\, d\lambda}{\int_0^1 e^{-\lambda t} \lambda^n\, d\lambda} \\ &= \frac{\frac{1}{t}[1 - F_{Poisson(x\,t)}(n)]}{\frac{1}{t}[1 - F_{Poisson(t)}(n)]} \\ &= \frac{1 - F_{Poisson(x\,t)}(n)}{1 - F_{Poisson(t)}(n)}, \quad 0 < x < 1. \end{aligned}$$

(b) Obtain the probability in (a), when $n = 1$, $t = 1$, and $x = 0.5$. (0.5)

Hint: If you have not solved (a), consider $P[\Lambda \leq x \mid N(t) = n] = \frac{\int_0^x e^{-\lambda t} \lambda^n\, d\lambda}{\int_0^1 e^{-\lambda t} \lambda^n\, d\lambda}$.

- **Requested probability**

$$\begin{aligned} P[\Lambda \leq x \mid N(t) = n] &= \frac{1 - F_{Poisson(x\,t)}(n)}{1 - F_{Poisson(t)}(n)} \\ &\overset{n=1,t=1,x=0.5}{=} \frac{1 - F_{Poisson(0.5)}(1)}{1 - F_{Poisson(1)}(1)} \\ &\overset{tables}{=} \frac{1 - 0.9098}{1 - 0.7358} \\ &\simeq 0.341408. \end{aligned}$$

Group 2 — Renewal Processes 8.0 points

1. Consider a renewal process $\{N(t) : t \geq 0\}$, with inter-renewal times $X_i \overset{i.i.d.}{\sim} \text{gamma}(3,1)$, for $i \in \mathbb{N}$. Admit that $N(t)$ represents the cumulative number of covid-19 cases in a region in the interval $(0,t]$.

(a) Derive a simplified expression for $P[N(t) = 0]$. (1.5)

Identify the distribution of the event time $S_n = \sum_{i=1}^n X_i$ and use it to derive a simplified expression $P[N(t) = n]$, for $n \in \mathbb{N}$.

- **Renewal process**

  $\{N(t) : t \geq 0\}$

  $N(t)$ = cumulative number of covid-19 cases until and at time t

- **Inter-renewal times**

  $X_i \overset{i.i.d.}{\sim} X \sim \text{gamma}(3,1), \quad i \in \mathbb{N}$ (*)

- **Requested probability**

$$\begin{aligned} P[N(t) = 0] &= P(X_1 > t) \\ &= 1 - F_{gamma(3,1)}(t) \\ &\overset{form.}{=} F_{Poisson(t)}(3-1) \\ &= \sum_{i=0}^{2} \frac{e^{-t} t^i}{i!} \\ &= e^{-t}\left(1 + t + \frac{t^2}{2}\right) \end{aligned}$$

- **Requested distribution**

  $S_n = \sum_{i=1}^{n} X_i \overset{(*)}{\sim} \text{gamma}(3n, 1), \quad n \in \mathbb{N}$

  $F_n(t) = P(S_n \le t)$

- **Requested p.f.**

  For $n \in \mathbb{N}$,

$$\begin{aligned}
P[N(t)=n] &\overset{form.}{=} F_n(t) - F_{n+1}(t) \\
&= F_{gamma(3n,1)}(t) - F_{gamma(3(n+1),1)}(t) \\
&= [1 - F_{Poisson(t)}(3n-1)] - [1 - F_{Poisson(t)}(3n+3-1)]
\end{aligned}$$

$$\begin{aligned}
P[N(t)=n] &= F_{Poisson(t)}(3n+2) - F_{Poisson(t)}(3n-1) \\
&= \sum_{i=3n}^{3n+2} \frac{e^{-t} t^i}{i!} \\
&= e^{-t}\left[\frac{t^{3n}}{(3n)!} + \frac{t^{3n+1}}{(3n+1)!} + \frac{t^{3n+2}}{(3n+2)!}\right], \\
&= e^{-t}\frac{t^{3n}}{(3n)!}\left[1 + \frac{t}{3n+1} + \frac{t^2}{(3n+2)(3n+1)}\right], \quad n \in \mathbb{N}.
\end{aligned}$$

(b) Calculate the exact and an approximate value to $P[N(t) < n]$, where $t = 24$ and $n = 8$. (1.5)

- **Requested exact probability**

$$\begin{aligned}
P[N(t) < n] &= 1 - P[N(t) \ge n] \\
&= 1 - F_n(t) \\
&= 1 - F_{gamma(3n,1)}(t) \\
&\overset{form.}{=} F_{Poisson(t)}(3n-1) \\
&\overset{t=24, n=8, (a)}{=} F_{Poisson(24)}(23) \\
&\overset{tables}{=} 0.4728.
\end{aligned}$$

- **Requested approximate probability**

  $\mu = E(X) \overset{form.}{=} \frac{3}{1} = 3$

  $\sigma^2 = V(X) \overset{form.}{=} \frac{3}{1^2} = 3$ [In this problem, we do not need to know σ^2 to obtain the prob.]

  Now, applying the CLT for RP, we get

$$\begin{aligned}
P[N(t) < n] &\overset{form.}{\simeq} \Phi\left(\frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right) \\
&\overset{t=24, n=8}{=} \Phi\left(\frac{8 - 24/3}{\sqrt{24 \times 3/3^3}}\right) \\
&= \Phi(0) \\
&= 0.5.
\end{aligned}$$

- **[Comment**

  The approximate value overestimates the exact one; the relative error is $(0.5-0.4728)/0.4728 \times 100\% \simeq 5.75\%$. This slight deviation is probably due to the fact that $t = 24$ is not sufficiently large to achieve a very good approximate value by applying the CLT for RP.]

(c) Consider time in hours and the time origin the beginning of 2020. Now, admit an official inspected the region at the beginning of 2021. (1.5)

Provide an approximation to the expected value of the time until the first covid-19 case after this inspection.

- **Recurrence time**

  $Y(t) \overset{form.}{=} S_{N(t)+1} - t$ = time until the first occurrence of a covid-19 case after the inspection at t

- **Requested approximate expected value**

  Since the value of $t = 365 \times 24 = 8760$ hours is rather large,

$$\begin{aligned}
E[Y(t)] &\simeq \lim_{z \to +\infty} E[Y(z)] \\
&\overset{form.}{=} \frac{E(X^2)}{2E(X)} \\
&= \frac{V(X) + E^2(X)}{2E(X)} \\
&\overset{(a)}{=} \frac{3 + 3^2}{2 \times 3} \\
&= 2.
\end{aligned}$$

2. The duration of the consecutive phone calls made by a seller are independent r.v. with common c.d.f.

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ 3x(1-x), & 0 < x < \tau \\ 1, & x \ge \tau, \end{cases}$$

where τ is a constant in $(0, \frac{1}{2}]$. If the seller manages to persuade a customer to acquire a product before time τ, then the seller get a reward of one monetary unit.

(a) Derive the expected reward per time unit in the long-run. (2.0)

Hint: Recall that for any non-negative r.v. X, $E(X) = \int_0^{+\infty} [1 - F_X(x)]\, dx$.

- **Renewal process**

  $\{N(t) : t \ge 0\}$

  $N(t)$ = number of phone calls by time t

- **IRT**

  $X_n \overset{i.i.d.}{\sim} X, \quad n \in \mathbb{N}$

$$F(x) = F_X(x) = \begin{cases} 0, & x \le 0 \\ 3x(1-x), & 0 < x < \tau \\ 1, & x \ge \tau \end{cases}$$

- **Reward renewal process**

  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \ge 0\}$

  $R(t)$ = total amount of rewards got by the seller until time t

$$R_n = \begin{cases} 1, & \text{if } X_n < \tau \text{ (i.e., during the } n^{th} \text{ phone call the seller persuaded the customer to acquire a product before time } \tau) \\ 0, & \text{otherwise} \end{cases}$$

  $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R),\ n \in \mathbb{N}$

- **Expected IRT**

$$\begin{aligned}
E(X) &\overset{X \ge 0}{=} \int_0^{+\infty} [1 - F_X(x)]\, dx \\
&= \int_0^{\tau} [1 - 3x(1-x)]\, dx \\
&= \int_0^{\tau} (1 - 3x + 3x^2)\, dx \\
&= \left(x - \frac{3x^2}{2} + x^3\right)\Big|_0^{\tau} \\
&= \tau - \frac{3\tau^2}{2} + \tau^3 \\
&= \tau(1 - \frac{3\tau}{2} + \tau^2)
\end{aligned}$$

- **Expected reward per time unit in the long-run**
  Since $E(X), E(R) < +\infty$, we can invoke the ERT for renewal reward processes and add that the expected reward per time unit in the long-run is given by

$$\begin{aligned}
\lim_{t\to+\infty} \frac{E[R(t)]}{t} &= \frac{E(R)}{E(X)} \\
&= \frac{3\tau(1-\tau)}{\tau(1-\frac{3\tau}{2}+\tau^2)} \\
&= \frac{6(1-\tau)}{2-3\tau+2\tau^2} \equiv h(\tau).
\end{aligned}$$

- **Expected reward per phone call**

$$\begin{aligned}
E(R) &= 1\times P(X<\tau)+0\times P(X\geq\tau) \\
&= \lim_{h\to 0^+} F_X(\tau-h) \\
&= 3\tau(1-\tau)
\end{aligned}$$

(b) Find the value of τ ($\tau \in (0, \frac{1}{2}]$) that maximizes the expected reward per time unit in the long-run. (1.5)

- **Maximizing $h(\tau)$**

$$\tau \in (0, \frac{1}{2}] \quad : \quad \frac{dh(\tau)}{d\tau} = 0 \quad \left(\text{and } \frac{d^2h(\tau)}{d\tau^2} < 0\right)$$

$$-6(2-3\tau+2\tau^2)-(6-6\tau)(-3+4\tau)=0$$

$$-12+18\tau-12\tau^2+18-24\tau-18\tau+24\tau^2=0$$

$$6(1-4\tau+2\tau^2)=0$$

$$2\tau^2-4\tau+1=0$$

$$\tau = \frac{4\pm\sqrt{16-8}}{4}$$

$$\tau = 1\pm\frac{\sqrt{2}}{2}$$

$$\tau = 1-\frac{\sqrt{2}}{2} \simeq 0.292893.$$