CEE - Exame 1 - 2020/2021 

30 Junho 2020

P1 a)

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^2 - 1 = 0$$

$\lambda_1 = 1$ $\lambda_2 = -1$

$\lambda - v_2^i = 0 \longrightarrow v_2^i = \lambda$

$\lambda_1 = 1$ $v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $v^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$x(t) = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + K_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + K_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} K_1 + K_2 = 1 \\ K_1 - K_2 = 0 \end{cases}$$

$K_1 = K_2 \longrightarrow 2K_1 = 1$

$K_1 = 1/2$ $K_2 = 1/2$

2/

$x_1(t) = \frac{1}{2}\left(e^{t} + e^{-t}\right)$

$x_2(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$

b) $k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$k_2 = -k_1$

$k_1 - k_2 = 1 \longrightarrow 2k_1 = 1$

$k_1 = \frac{1}{2} \quad k_2 = -\frac{1}{2}$

$x_1(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$

$x_2(t) = \frac{1}{2}\left(e^{t} + e^{-t}\right)$

c) $e^{At} = \left[x^{a}(t) \,\vdots\, x^{b}(t)\right] =$

$= \frac{1}{2}\begin{bmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}$

d) $x(0) = e^{-A}\, x(1)$

$\det e^{A} = \frac{1}{4}\left[\left(e+e^{-1}\right)^2 - \left(e-e^{-1}\right)^2\right] =$

$= \left[\cancel{e^2} + 2e\,e^{-1} + \cancel{e^{-2}} - \left(\cancel{e^2} - 2e\,e^{-1} + \cancel{e^{-2}}\right)\right]\frac{1}{4} = 1$

3/

$$e^{-A} = \begin{bmatrix} e+e^{-1} & e^{-1}-e \\ e^{-1}-e & e+e^{-1} \end{bmatrix}^T =$$

$$= \begin{bmatrix} e+e^{-1} & e^{-1}-e \\ e^{-1}-e & e+e^{-1} \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2e^{-1} \\ 2e^{-1} \end{bmatrix} = 2e^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\boxed{\diagup}$

<u>P2 a)</u>

$$\mathcal{C} = [b \,\vdots\, Ab] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

car $\mathcal{C} = 2 = \dim x \Rightarrow$ controlável

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

car $\mathcal{O} = 2 = \dim x \Rightarrow$ observável

b) $$A - bk = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \;\; k_2] =$$

$$= \begin{bmatrix} -k_1 & 1-k_2 \\ 1 & 0 \end{bmatrix}$$

$$\det(sI - A + bk) =$$

$$= \begin{vmatrix} s+k_1 & k_2-1 \\ -1 & s \end{vmatrix} =$$

$$= s^2 + k_1 s + k_2 - 1$$

$$\alpha_c(s) = (s+2)(s+3) = s^2 + 5s + 6$$

$$k_1 = 5 \qquad k_2 = 7$$

c) $$A - LC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -L_1 & 1 \\ 1-L_2 & 0 \end{bmatrix}$$

$$\det(sI - A + LC) = \begin{vmatrix} s+L_1 & -1 \\ L_2-1 & s \end{vmatrix} =$$

$$= s^2 + L_1 s + L_2 - 1$$

$$\alpha_o(s) = (s+5)(s+6) = s^2 + 11s + 30$$

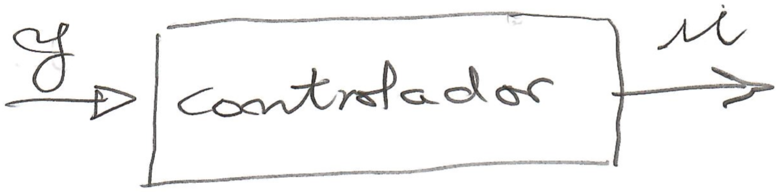
$$L_1 = 11 \qquad L_2 = 31$$

d) $$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$u = -K\hat{x}$$

$$\dot{\hat{x}} = (A - bk - LC)\hat{x} + Ly$$

$$u = -k\hat{x}$$




$$G_c(s) = -k(sI - A + bk + LC)^{-1} L$$

Denominador:

$$\det(sI - A + bk + LC)$$

$$-A + bk + LC = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + L_1 & k_2 - 1 \\ L_2 - 1 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 6 \\ 30 & 0 \end{bmatrix}$$

Denominador:

$$\begin{vmatrix} s+16 & 6 \\ 30 & s \end{vmatrix} = s^2 + 16s - 180$$

P3 a) condições de equilíbrio



$$x_2 = 0$$

$$x_1 + x_1^2 = x_1(1 + x_1) = 0$$

Equilibrium points

$P_1(0,0)$ $\quad$ $P_2(-1,0)$

b) Linearization

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -(1+2x_1) & 0 \end{bmatrix}$$

P1 (0,0)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(\lambda I - A) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm j$$

Eigenvalues on the imaginary axis. Nothing can be said about the stability of the corresponding points in the nonlinear system.

P2 $(-1, 0)$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^2 - 1 = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

Sadle point (ponto de sela).
there is a positive eigenvalue
and therefore the equilibrium point $(-1, 0)$ of the nonlinear system is <u>unstable</u>.
around P2

c) Local behaviour

$$\lambda_1 = 1 \qquad v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \qquad v^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$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P4

a) $\dot{V} = x\dot{x} + \tilde{\alpha}\dot{\tilde{\alpha}}$

$\dot{x} = ax - (\hat{a} - a_m)x$

$\dot{x} = -\tilde{a}x + a_m x$

$\dot{V} = a_m x^2 - \tilde{a}x^2 + \tilde{a}\dot{\tilde{\alpha}}$

$\dot{V} = a_m x^2 + \tilde{a}\underbrace{(\dot{\tilde{\alpha}} - x^2)}_{=0}$

$\dot{\tilde{\alpha}} = x^2$

$\dot{\hat{\alpha}} = x^2$

$\hat{\alpha}(t) = \hat{\alpha}(0) + \int_0^t x^2(\tau)\,d\tau$

with ~~this~~ adaptation law

$\dot{V} = a_m x^2 < 0$ for $a_m < 0$

corresponds to specify the nominal close loop system asymptotically stable

the adaptive control system is ~~stable~~ but not necessarily asymptotically stable.

b) According to the invariant set theorem all the trajectories converge to the set in which $\dot{V} = 0$ this set is $\{(x, \tilde{\alpha}); x^2 = 0\}$ and hence $x \to 0$.

Nothing can be said about $\tilde{\alpha} \to 0$. Hence, we can't ensure that the adaptive system is asymptotically stable (only that $x \to 0$; nothing can be said about $\tilde{\alpha} \to 0$).

P5 a)

$$L = -\frac{1}{2}\left[(r-x)^2 + \rho u^2\right], \quad L_x = r - x$$

$$\dot{\lambda} = -\lambda - r + x$$

$$H = \lambda(x+u) - \frac{1}{2}\left[(r-x)^2 + \rho u^2\right]$$

$$\frac{\partial H}{\partial u} = \lambda - \rho u = 0 \longrightarrow u = \frac{1}{\rho}\lambda$$

Assume

$\lambda = -\phi x + g$ $\quad \phi, g$ const.

↑ this term is needed to allow $x$ to converge to a nonzero value.

$$u = -\frac{\phi}{\rho}x + \frac{1}{\rho}g$$

$$\dot{\lambda} = -\phi\dot{x}$$

$$-\lambda - r + x = -\phi(x+u)$$

$$\left[\frac{\phi^2}{\rho} - 2\phi - 1\right]x = \left(\frac{\phi}{\rho} - 1\right)g - r$$

$$\phi^2 - 2\phi\rho - \rho = 0 \quad ; \quad g = \frac{\rho}{\phi - \rho}$$

$$u^*(t) = -\frac{p}{\rho}\, x(t) + \frac{1}{p-\rho}\, r$$ 11/

$$p = \rho + \sqrt{\rho^2 + \rho}$$

b) closed-loop

$$\dot{x} = -\sqrt{1+\frac{1}{\rho}}\, x + \frac{1}{\sqrt{1+\frac{1}{\rho}}}\, r$$

Equilibrium

$$\bar{x} = \frac{1}{1+\rho}\, r$$

Relative error

$$\frac{r-\bar{x}}{r} = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} \leq \frac{M}{100}$$

$$\rho \leq \frac{\frac{M}{100}}{1-\frac{M}{100}}$$

$$M = 5\% \longrightarrow \rho \leq \frac{5}{95} \simeq 0{,}05263$$

$$\rho \leq 0{,}05263 \Longrightarrow M \leq 5\%$$

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