Introduction to Stochastic Processes

MMA, LMAC, MECD

2nd. Semester – 2019/2020
2020/09/08 – 9AM, Room **P9**

Duration: 180 minutes **Exam** (*Época Especial*)

- Please justify all your answers.
- This test has THREE PAGES and SIX GROUPS. The total of points is 40.0.

Group 1 — Introduction to Stochastic Processes (2.5 points)

Consider the stochastic process $\{Y(t) = (-1)^{N(t)} : t \geq 0\}$, where $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Thus, $Y(t)$ starts at $Y(0) = 1$ and switches back and forth from $+1$ to -1 at random times.[1]

(a) Find the mean function and the variance function of $\{Y(t) : t \geq 0\}$.[2] (1.0)

- **Stochastic processes**
  $\{N(t) : t \geq 0\} \sim PP(\lambda)$
  $\{Y(t) = (-1)^{N(t)} : t \geq 0\}$ (semi-random telegraph signal process)
- **R.v.**
  $N(t) \sim \text{Poisson}(\lambda t)$
- **Requested mean function**
  The mean function can be written in terms of the p.g.f. of $N(t)$

$$\begin{aligned} E[Y(t)] &= E\left[(-1)^{N(t)}\right] \\ &= P_{N(t)}(-1) \\ &\overset{form.}{=} e^{-(\lambda t)\times[1-(-1)]} \\ &= e^{-2\lambda t} \end{aligned}$$

- **Requested variance function**

$$\begin{aligned} V[Y(t)] &= E[Y^2(t)] - E^2[Y(t)] \\ &= E\left[(-1)^{2N(t)}\right] - E^2[Y(t)] \\ &\overset{N(t)\,is\,even}{=} E(1) - \left(e^{-2\lambda t}\right)^2 \\ &= 1 - e^{-4\lambda t}. \end{aligned}$$

(b) Derive the autocorrelation function of $\{Y(t) : t \geq 0\}$. Are we dealing with a strictly stationary process? (1.5)

- **Requested autocorrelation function**
  Taking advantage once more of the p.g.f. of $N(t)$, we get

$$\begin{aligned} E[Y(t) \times Y(t+h)] &= E\left[(-1)^{N(t)+N(t+h)}\right] \\ &= E\left[(-1)^{2N(t)+[N(t+h)-N(t)]}\right] \\ &\overset{indep.\,inc.}{=} E\left[(-1)^{2N(t)}\right] \times E\left[(-1)^{N(t+h)-N(t)}\right] \\ &\overset{station.\,inc.}{=} E(1) \times E\left[(-1)^{N(h)}\right] \\ &= 1 \times P_{N(h)}(-1) \\ &= e^{-2\lambda h}, \quad t,h \geq 0. \end{aligned}$$

  Coonsequently:

$$\begin{aligned} cov(Y(t), Y(t+h)) &= E[Y(t) \times Y(t+h)] - E[Y(t)] \times E[Y(t+h)] \\ &= e^{-2\lambda h} - e^{-2\lambda t} \times e^{-2\lambda(t+h)} \\ &= e^{-2\lambda h} \times \left(1 - e^{-4\lambda t}\right), \quad t,h \geq 0; \end{aligned}$$

[1] $\{Y(t) : t \geq 0\}$ is known as the *semi-random telegraph signal* process because its initial value is not random.

[2] **Hint**: You may need to use a p.g.f. or recall that: $\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$; $\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$.

$$\begin{aligned} corr(Y(t), Y(t+s)) &= \frac{cov(Y(t), Y(t+h))}{\sqrt{V[Y(t)] \times V[Y(t+h)]}} \\ &= \frac{e^{-2\lambda h} \times \left(1 - e^{-4\lambda t}\right)}{\sqrt{\left(1 - e^{-4\lambda t}\right) \times \left[1 - e^{-4\lambda(t+h)}\right]}} \\ &= e^{-2\lambda h} \times \sqrt{\frac{1 - e^{-4\lambda t}}{1 - e^{-4\lambda(t+h)}}}, \quad t,h \geq 0. \end{aligned}$$

- **Investigating the strict stationarity**
  The mean function depends on t (and the autocovariance function is also time dependent), thus we not are dealing with a (second order weakly) stationary or a strict stationary process.

Group 2 — Poisson Processes (8.5 points)

1. Patients arrive at a small clinic according to a Poisson process with rate $\lambda = 0.1$ (patients per minute).

(a) The doctor will not see a patient until at least three patients are in the waiting room. (1.5)
Find the expected waiting time until the first patient is admitted to see the doctor.
What is the probability that nobody is admitted to see the doctor in the first hour?

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda = 0.1)$
  $N(t)$ = number of patients that arrived to the clinic by time t
- **Relevant r.v. and its distribution**
  Since the doctor will not see a patient until at least three patients are in the waiting room, the waiting time until the first patient is admitted to see the doctor coincides with the event time

$$\begin{aligned} S_3 &= \text{arrival time of the third patient} \\ &\sim \text{gamma}(3, \lambda). \end{aligned}$$

- **Requested expected value**

$$\begin{aligned} E(S_3) &= \frac{3}{\lambda} \\ &= 30 \quad \text{(minutes)}. \end{aligned}$$

- **Requested probability**
  The probability that nobody is admitted to see the doctor in the first hour is the same as the probability that at most two patients arrive in the first 60 minutes:

$$\begin{aligned} P[N(60) \leq 2] &\overset{N(t)\sim Poisson(\lambda t)}{=} F_{Poisson(0.1\times 60)}(2) \\ &\overset{tables}{=} 0.0620. \end{aligned}$$

(b) Suppose that 20 patients arrived to the clinic during the first four hours. (0.5)
Obtain the probability that at most 10 patients arrived during the first two hours?

- **Relevant r.v.**
  $(N(s) \mid N(t) = n) \overset{form.}{\sim} \text{binomial}(n, s/t), 0 < s < t$
- **Requested probability**

$$\begin{aligned} P[N(120) \leq 10 \mid N(240) = 20] &= F_{binomial(n=20,\, s/t=120/240=0.5)}(10) \\ &\overset{tables}{=} 0.5881. \end{aligned}$$

(c) Now, admit the doctor sees the first patient as soon as he/she arrives. Moreover, suppose that: any patient spends a random time (X, in minutes) being seen by the doctor; X has a Weibull distribution with scale parameter $\alpha = 5\sqrt{2}$ and shape parameter $\beta = 2$. (2.0)

Find the probability that there are at most 2 patients still in the clinic[3] 10 minutes after it opened.

- **R.v.**
  S = time spent by a patient being seen by the doctor
  $S \sim \text{Weibull}(\alpha = 5\sqrt{2}, \beta = 2)$
- **Non-homogenous Bernoulli splitting**
  A patient, who arrived at time s $(0 < s < t)$, will be still in the clinic (being seen by the doctor or in the waiting room) at time t with probability

$$\begin{aligned} p(s) &= P(S > t - s) \\ &= \int_{t-s}^{+\infty} f_S(u)\,du \\ &= \int_{t-s}^{+\infty} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\,dx \\ &= -e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\Big|_{t-s}^{+\infty} \\ &= e^{-\left(\frac{t-s}{\alpha}\right)^{\beta}}. \end{aligned}$$

  Furthermore, the number of patients still in the clinic at time t, $N_D(t)$, results from a non-homogenous Bernoulli splitting of $\{N(t) : t \geq 0\}$. Consequently,

$$N_D(t) \overset{form.}{\sim} \text{Poisson}\left(\lambda \int_0^t p(s)\,ds\right),$$

  where

$$\begin{aligned} \int_0^t p(s)\,ds &\overset{\beta=2}{=} \int_0^t e^{-\left(\frac{t-s}{\alpha}\right)^2}\,ds \\ &= \sqrt{2\pi} \times \alpha/\sqrt{2} \times \int_0^t \frac{1}{\sqrt{2\pi} \times \alpha/\sqrt{2}}\, e^{-\frac{(s-t)^2}{2\times(\alpha/\sqrt{2})^2}}\,ds \\ &= \sqrt{2\pi} \times \alpha/\sqrt{2} \times \left[F_{N(0,(\alpha/\sqrt{2})^2)}(t) - F_{N(0,(\alpha/\sqrt{2})^2)}(0)\right] \\ &= \sqrt{2\pi} \times \alpha/\sqrt{2} \times \left[\Phi\left(\frac{t-t}{\alpha/\sqrt{2}}\right) - \Phi\left(\frac{0-t}{\alpha/\sqrt{2}}\right)\right] \\ &= \sqrt{2\pi} \times \alpha/\sqrt{2} \times \left[\Phi(0) - \Phi\left(-\frac{t}{\alpha/\sqrt{2}}\right)\right] \\ &\overset{\alpha=5\sqrt{2},\, t=10}{=} \sqrt{2\pi} \times 5 \times [0.5 - \Phi(-2)] \\ &= \sqrt{2\pi} \times 5 \times [0.5 - 1 + \Phi(2)] \\ &\overset{tables}{=} \sqrt{2\pi} \times 5 \times (0.5 - 1 + 0.9772) \\ &\simeq 5.98082. \end{aligned}$$

- **Requested probability**

$$\begin{aligned} P[N_D(t) \leq 2] &\simeq F_{Poisson(0.1\times 5.98082)}(2) \\ &\simeq F_{Poisson(0.6)}(2) \\ &\overset{tables}{=} 0.9769. \end{aligned}$$

  [According to *Mathematica*, $F_{Poisson(0.1\times 5.98082)}(2) \simeq 0.977074$.]

2. Suppose that $\{N_0(t) : t \geq 0\}$ is a Poisson process with rate equal to 1. Let: $\lambda(t)$ denote a non-negative function of t; $m(t) = \int_0^t \lambda(s)\,ds$; $N(t) = N_0(m(t))$.

(a) Prove that $\{N(t) : t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t)$. Interpret this result. [4] (2.0)

[3] Either being seen by the doctor or in the waiting room.

[4] **Hint**: $\{N(t) : t \geq 0\} \sim NHPP(\lambda(t))$ iff: $N(0) = 0$; $\{N(t) : t \geq 0\}$ has independent increments; $P[N(t+h) - N(t) = 1] = \lambda(t) \times h + o(h),\ t \geq 0$; $P[N(t+h) - N(t) \geq 2] = o(h),\ t \geq 0$. Moreover, a function f is said to be $o(h)$ iff $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

- **Auxiliary stochastic process**
  $\{N_0(t) : t \geq 0\} \sim PP(\lambda = 1)$
  According to the second definition of PP, we have:
  i) $N_0(0) = 0$;
  ii) $\{N_0(t) : t \geq 0\}$ has independent and stationary increments;
  iii) $P[N_0(t+h) - N_0(t) = 1] = \lambda(t) \times h + o(h), \quad t \geq 0$;
  iv) $P[N_0(t+h) - N_0(t) \geq 2] = o(h), \quad t \geq 0$.
  Moreover, $N_0(t) \sim \text{Poisson}(\lambda t = t)$.
- **Requested proof**
  Let:
  i) $\lambda(t)$ denote a non-negative function of t;
  ii) $m(t) = \int_0^t \lambda(s)\,ds$;
  iii) $N(t) = N_0(m(t))$.
  Now, let us take advantage of the hint and capitalize on the properties of $\{N_0(t) : t \geq 0\}$.
  i) $N(0) = 0$?
  $N(0) = N_0[m(0)] = N_0(\int_0^0 \lambda(s)\,ds) = 0.$ ✓
  ii) Has $\{N(t) : t \geq 0\}$ independent increments?
  Let $0 < t_1 < t_2 < \cdots < t_n$ $(n = 2,3,\ldots)$ and $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$. Since $\{N_0(t) : t \geq 0\}$ has independent increments, the joint probability $P[N(t_1) = i_1, N(t_2) - N(t_1) = i_2 - i_1, \ldots, N(t_n) - N(t_{n-1}) = i_n - i_{n-1}] = \star$ is equal to

$$\begin{aligned} \star &= P[N_0(m(t_1)) = i_1, N_0((t_2)) - N_0(m(t_1)) = i_2 - i_1, \ldots, \\ &\qquad N_0(m(t_n)) - N_0(m(t_{n-1})) = i_n - i_{n-1}] \\ &= P[N_0(m(t_1)) = i_1] \times P[N_0((t_2)) - N_0(m(t_1)) = i_2 - i_1] \\ &\qquad \times \ldots \quad \times P[N_0(m(t_n)) - N_0(m(t_{n-1})) = i_n - i_{n-1}], \end{aligned}$$

  that is, the r.v. $N(t_1), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ are independent, thus $\{N(t) : t \geq 0\}$ also has independent increments. ✓
  iii) $P[N(t+h) - N(t) = 1] = \lambda(t) \times h + o(h),\ t \geq 0$?
  In other words, we have to verify that $\lim_{h\to 0} \frac{P[N(t+h)-N(t)=1]}{h} = \lambda(t)$.
  Capitalizing now on the stationary increments of $\{N_0(t) : t \geq 0\}$ and on the fact that $N_0(t) \sim \text{Poisson}(\lambda t = t)$, we get

$$\begin{aligned} \lim_{h\to 0} \frac{P[N(t+h) - N(t) = 1]}{h} &= \lim_{h\to 0} \frac{P[N_0(m(t+h)) - N_0(m(t)) = 1]}{h} \\ &= \lim_{h\to 0} \frac{P[N_0(m(t+h) - m(t)) = 1]}{h} \\ &= \lim_{h\to 0} \frac{e^{-[m(t+h)-m(t)]} \times [m(t+h) - m(t)]}{h} \\ &= \lim_{h\to 0} \frac{m(t+h) - m(t)}{h} \\ &= \lambda(t). \quad ✓ \end{aligned}$$

  iv) $P[N(t+h) - N(t) \geq 2] = o(h),\ t \geq 0$?
  Equivalently, we have to verify that $\lim_{h\to 0} \frac{P[N(t+h)-N(t)\geq 2]}{h} = 0$. Similarly and invoking L'Hôpital rule,

$$\begin{aligned} \lim_{h\to 0} \frac{P[N(t+h) - N(t) \geq 2]}{h} &= \lim_{h\to 0} \frac{P[N_0(m(t+h)) - N_0(m(t)) \geq 2]}{h} \\ &= \lim_{h\to 0} \frac{P[N_0(m(t+h) - m(t)) \geq 2]}{h} \\ &= \lim_{h\to 0} \frac{1 - e^{-[m(t+h)-m(t)]} \times (1 + [m(t+h) - m(t)])}{h} \\ &= \lim_{h\to 0} \frac{1 - e^{-[m(t+h)-m(t)]}}{h} - \lim_{h\to 0} \frac{m(t+h) - m(t)}{h} \end{aligned}$$

$$\lim_{h\to 0}\frac{P[N(t+h)-N(t)\geq 2]}{h} \overset{Hrule}{=} \lim_{h\to 0}\frac{e^{-[m(t+h)-m(t)]}\times\frac{d[m(t+h)-m(t)]}{dh}}{1}-\lambda(t)$$
$$\overset{Hrule}{=} \lim_{h\to 0}[1\times\lambda(t+h)]-\lambda(t)$$
$$= 0. \quad \checkmark$$

Thus, $\{N(t): t\geq 0\}$ is indeed a NHPP with intensity function $\lambda(t)$. ✓

- **Requested interpretation**
  If we consider $\{N_0(t): t\geq 0\}\sim PP(1)$ and rescale time via $m(t)=\int_0^t \lambda(s)\,ds$, then we end up dealing with a NHPP with intensity function $\lambda(t)$.

(b) Consider (1.0)

$$\lambda(t)=\begin{cases}1, & t\in[2i,2i+1]\\ 0, & t\in(2i+1,2i+2)\end{cases} \quad\text{and}\quad m(t)=\begin{cases}t-i, & t\in[2i,2i+1]\\ i+1, & t\in(2i+1,2i+2),\end{cases}$$

where $i\in\mathbb{N}_0$.

Obtain the probability that the second event of the nonhomogeneous Poisson process $\{N(t): t\geq 0\}$ occurs in the first time unit.

- **Stochastic process**
  $\{N(t): t>0\}\sim NHPP$
- **Intensity and mean value function**
  $$\lambda(t) = \begin{cases}1, & t\in[2i,2i+1]\\ 0, & t\in(2i+1,2i+2)\end{cases}$$
  $$m(t) = \begin{cases}t-i, & t\in[2i,2i+1]\\ i+1, & t\in(2i+1,2i+2),\end{cases}$$
  where $i\in\mathbb{N}_0$.
- **Relevant r.v.**
  S_2 = time of the second event of the NHPP
- **Requested probability**
  $$P(S_2\leq 1) = F_{S_2}(1)$$
  $$\overset{form.}{=} 1-F_{Poisson(m(t))}(n-1)]$$
  $$\overset{n=2,t=1}{=} 1-F_{Poisson(m(1))}(1)$$
  $$= 1-F_{Poisson(1)}(1)$$
  $$\overset{tables}{=} 1-0.7358$$
  $$\simeq 0.2642.$$

3. Customers arrive at the automatic teller machine in accordance with a Poisson process with rate 12 customers per hour. The amount of money withdrawn on each transaction is a r.v. with mean 30 and standard deviation 50.[5] The machine is in use for 15 hours daily. Use the central limit theorem to approximate the probability that the total daily withdrawal does not exceed 6 000. (1.5)

- **R.v. et al.**
  W_i = amount of withdrawal i
  $W_i \overset{i.i.d.}{\sim} W, \quad i\in\mathbb{N}$
  $E(W)=\mu=30,\ V(W)=\sigma^2=50^2$
- **Stochastic processes**
  $\{N(t): t\geq 0\}\sim PP(\lambda=12)$ (independent of W_i, $i\in\mathbb{N}$)

[5] A negative withdrawal means that money was deposited.

$N(t)\sim \text{Poisson}(\lambda t)$
$\{X(t)=\sum_{i=1}^{N(t)} W_i : t\geq 0\}\sim CompoundPP(\lambda, W)$
$X(t)$ = total of withdrawals over $[0,t]$
$X(15)$ = total daily withdrawals **Mean and variance of** $X(15)$

$$E[X(t)] \overset{form.}{=} \lambda\, t\times E(W)$$
$$\overset{\lambda=12,t=15,etc.}{=} 12\times 15\times 30$$
$$= 5400$$
$$V[X(t)] \overset{form.}{=} \lambda\, t\times E(W^2)$$
$$= \lambda\, t\times[V(W)+E^2(W)]$$
$$\overset{\lambda=12,t=15,etc.}{=} 12\times 15\times(50^2+5400^2)$$
$$= 612\,000$$

- **Requested probability**
  We can apply the CLT to provide the following approximate value:
  $$P[X(15)\leq 6000] \simeq \Phi\left(\frac{6000-5400}{\sqrt{612000}}\right)$$
  $$\simeq \Phi(0.77)$$
  $$\overset{tables}{=} 0.7794.$$

Group 3 — Renewal Processes 9.0 points

1. Let $N(t)$ represent the number of emitted signals recorded by a receptor. Admit that the inter-renewal times X_n $(n\in\mathbb{N})$ of a renewal process $\{N(t): t\geq 0\}$ have common p.f. $P(X=x)=(1-p)^{x-1}p$, for $x\in\mathbb{N}$.

(a) Identify the distributions of the event times S_n and use them to derive $P[N(t)=n]$, for $n\in\mathbb{N}_0$ and $t\geq 1$. (1.5)

- **Renewal process**
  $\{N(t): t\geq 0\}$
  $N(t)$ = number of renewals up to time t
- **Inter-renewal times**
  X_i = time between the $(i-1)^{th}$ and i^{th} renewals
  $X_i \overset{i.i.d.}{\sim} X\sim \text{geometric}(p),\ i\in\mathbb{N}$
  $\mu=E(X)=\frac{1}{p}$
- **Event/renewal times**
  $S_n = n^{th}$ renewal time, $n\in\mathbb{N}$
  $S_n=\sum_{i=1}^n X_i$
  $S_n\sim \text{negativeBin}(n,p)$
- **P.f. of** $N(t)$
  We are essentially dealing with a binomial counting process $\{N(t): t\geq 0\}$. For $t\geq 1$, $N(t)\sim$ binomial$(\lfloor t\rfloor, p)$, where $\lfloor t\rfloor$ represents the integer part of t. In fact, for $t\geq 1$:
  $$P[N(t)=0] = P(S_1=X_1>t)$$
  $$= 1-F_{geometric(p)}(\lfloor t\rfloor)$$
  $$= (1-p)^{\lfloor t\rfloor};$$
  $$P[N(t)=n] = P(S_n\leq t)-P(S_{n+1}\leq t)$$
  $$= F_{negativeBin(n,p)}(\lfloor t\rfloor)-F_{negativeBin(n+1,p)}(\lfloor t\rfloor)$$

$$P[N(t)=n] = [1-F_{binomial(\lfloor t\rfloor,p)}(n-1)] - [1-F_{binomial(\lfloor t\rfloor,p)}(n)]$$
$$= \binom{\lfloor t\rfloor}{n} p^n (1-p)^{\lfloor t\rfloor - n}, \quad n \in \{1,\ldots,\lfloor t\rfloor\}.$$

(b) Admit that the time unit is an hour and $p=0.8$. Calculate the exact value and the approximate value to the probability that less than 15 arrivals occur in the first 20 hours. Comment. (1.5)

- **Requested exact probability**
  Since $N(t) \sim \text{binomial}(\lfloor t\rfloor, p)$, we get

$$\begin{aligned}
P[N(t)<x] &= P[\lfloor t\rfloor - N(t) > \lfloor t\rfloor - x] \\
&= 1 - P[\lfloor t\rfloor - N(t) \le \lfloor t\rfloor - x] \\
&= 1 - F_{binomial(\lfloor t\rfloor, 1-p)}(\lfloor t\rfloor - x) \\
&\overset{t=20,\,p=0.8,\,x=15}{=} 1 - F_{binomial(20,1-0.8)}(20-15) \\
&\simeq 1 - F_{binomial(20,0.2)}(5) \\
&\overset{tables}{=} 1 - 0.8042 \\
&= 0.1958.
\end{aligned}$$

- **IRT**
  $X_i \overset{i.i.d.}{\sim} X,\ i \in \mathbb{N}$

  $X \sim \text{geometric}(p=0.8)$

  $\mu = E(X) \overset{form.}{=} \frac{1}{p} = 1.25$

  $\sigma^2 = V(X) \overset{form.}{=} \frac{1-p}{p^2} = 0.3125$

- **Requested approximate probability**

$$\begin{aligned}
P[N(t)<n] &\overset{form.}{\simeq} \Phi\left(\frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right) \\
&\overset{t=20,\,n=5}{=} \Phi\left(\frac{15 - 20/1.25}{\sqrt{20 \times 1.25/0.3125^3}}\right) \\
&\simeq \Phi(-0.56) \\
&= 1 - \Phi(0.56) \\
&\overset{tables}{=} 1 - 0.7123 \\
&= 0.2877.
\end{aligned}$$

- **Comment**
  Unsurprisingly, the approximate value is substantially different to the exact value (after all, t is not sufficiently large).

(c) Obtain the renewal function $m(t)$, for $t \ge 1$, and show that it satisfies the elementary renewal theorem. (1.0)

- **Renewal function**
  Since $N(t) \sim \text{binomial}(\lfloor t\rfloor, p)$, for $t \ge 1$, we get

  $m(t) = \lfloor t\rfloor \times p, \quad t \ge 1.$

- **Verification of the ERT**

$$\begin{aligned}
\lim_{t\to+\infty} \frac{m(t)}{t} &= \lim_{t\to+\infty} \frac{\lfloor t\rfloor \times p}{t} \\
&= \lim_{t\to+\infty} \frac{(t-\epsilon)\times p}{t} \qquad (0 \le \epsilon < 1) \\
&= \lim_{t\to+\infty} \left(p - \frac{\epsilon p}{t}\right) \\
&= p \equiv \frac{1}{\mu}. \qquad \checkmark
\end{aligned}$$

(d) Determine $\lim_{n\to+\infty} E[\text{number of renewals at } n]$ and $\lim_{t\to+\infty}[m(t+1)-m(t)]$. Comment. (1.5)

- **Requested limits**
  Since the inter-renewal distribution is lattice with period $d=1$, we can apply Blackwell's theorem and get

$$\begin{aligned}
\lim_{n\to+\infty} E[\text{number of renewals } \textbf{at } n] \quad [&= \lim_{n\to+\infty}[m(n)-m(n-1)] \\
&= \lim_{n\to+\infty} \frac{\lfloor n\rfloor \times p}{n} \\
&= \lim_{n\to+\infty} \frac{n \times p}{n} \qquad (n \in \mathbb{N}) \\
&= p] \\
&\overset{form.}{=} \frac{1}{\mu} \\
&= p
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to+\infty}[m(t+1)-m(t)] &= \lim_{t\to+\infty}[\lfloor t+1\rfloor \times p - \lfloor t\rfloor \times p] \\
&= \lim_{t\to+\infty}[(t+1-\epsilon)-(t-\epsilon)] \times p \qquad (0 \le \epsilon < 1) \\
&= p] \\
&\overset{form.}{=} \frac{1}{\mu} \\
&= p.
\end{aligned}$$

- **Comment**
  The probability that an emitted signal is recorded at time n is p, thus the expected number of emitted signals recorded by the receptor:
  – at an integer time point n is equal to p, for any n;
  – in the interval $(t, t+1]$ containing exactly one integer time point, equals p, for any $t \ge 0$.

  Moreover, these two limits coincide because they follow from two equivalent statements of Blackwell's theorem.
  [The inter-renewal distribution distribution is lattice with period $d=1$, thus the relevant limit is that of the expected number of renewals at $n\,d$, which is proportional to the period (d) and to the long-run rate at which renewals occur ($1/\mu$).
  Furthermore, $\lim_{t\to+\infty}[m(t+d)-m(t)]$ does not exist because renewals can only occur at integral multiples of d and therefore the expected number of renewals in an interval $(t,t+1]$ far from the origin would clearly depend on how many points integer multiples of the period d it contains and not on the interval length.

2. A seller is recruiting subscribers by phone. The duration of the phone calls are independent r.v. with common c.d.f. given by

$$F(x) = \begin{cases} 0, & x \le 0 \\ x(2-x), & 0 < x < \tau \\ 1, & x \ge \tau, \end{cases}$$

where τ is a constant in $(0,1]$. If the seller manages to persuade a customer to subscribe before time τ, then a subscriber has been recruited.

(a) Derive the expected number of recruited subscribers per time unit in the long-run.[6] (2.5)

- **Renewal process**
  $\{N(t) : t \ge 0\}$

  $N(t) =$ number of phone calls by time t

- **IRT**
  $X_n \overset{i.i.d.}{\sim} X, \quad n \in \mathbb{N}$

[6]**Hint**: Recall that for any non-negative r.v. X, $E(X) = \int_0^{+\infty}[1-F_X(x)]\,dx$.

$$F(x) = F_X(x) = \begin{cases} 0, & x \le 0 \\ x(12-x), & 0 < x < \tau \\ 1, & x \ge \tau \end{cases}$$

- **Reward renewal process**

  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \ge 0\}$

  $R(t)$ = number of recruited subscriber until time t

  $$R_n = \begin{cases} 1, & \text{if } X_n < \tau \text{ (i.e., if the } n^{th} \text{ phone call led to a recruited subscriber)} \\ 0, & \text{otherwise} \end{cases}$$

  $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R),\ n \in \mathbb{N}$

- **Expected IRT**

$$\begin{aligned} E(X) &\overset{X \ge 0}{=} \int_0^{+\infty} [1 - F_X(x)]\, dx \\ &= \int_0^{\tau} [1 - x(2-x)]\, dx \\ &= \int_0^{\tau} (1 - 2x + x^2)\, dx \\ &= \left(x - x^2 + \frac{x^3}{3} \right)\Big|_0^{\tau} \\ &= \tau - \tau^2 + \frac{\tau^3}{3} \end{aligned}$$

- **Expected number of recruited subscribers per phone call**

$$\begin{aligned} E(R) &= 1 \times P(X < \tau) + 0 \times P(X \ge \tau) \\ &= \tau(2-\tau) \end{aligned}$$

- **Number of recruited subscribers per time unit in the long-run**

  Since $E(X), E(R) < +\infty$, we can add that the expected number of recruited subscribers per time unit in the long-run is given by

$$\begin{aligned} \lim_{t \to +\infty} \frac{E[R(t)]}{t} &= \frac{E(R)}{E(X)} \\ &= \frac{\tau(2-\tau)}{\tau - \tau^2 + \frac{\tau^3}{3}} \\ &= \frac{6 - 3\tau}{3 - 3\tau + \tau^2} \equiv h(\tau). \end{aligned}$$

(b) Find the value of τ that maximizes the expected number of recruited subscribers per time unit in the long-run. (1.0)

- **Maximizing** $h(\tau)$

$$\begin{aligned} \tau \in (0,1] \quad : \quad & \frac{dh(\tau)}{d\tau} = 0 \quad \left(\text{and } \frac{d^2 h(\tau)}{d\tau^2} < 0 \right) \\ & -3(3 - 3\tau + \tau^2) - (6 - 3\tau)(-3 + 2\tau) = 0 \\ & -9 + 9\tau - 3\tau^2 + 18 - 9\tau - 12\tau + 6\tau^2 = 0 \\ & 3\tau^2 - 12\tau + 9 = 0 \\ & \tau^2 - 4\tau + 3 = 0 \\ & \tau = \frac{4 \pm \sqrt{16 - 12}}{2} \\ & \tau = 1. \end{aligned}$$

Group 4 — Renewal Processes (cont'd) 2.0 points

Orders arrive to a warehouse according to a delayed renewal process, with inter-renewal times (2.0) $X_1 \sim \text{gamma}(2, \lambda)$ and $X_i \sim \text{gamma}(4, \lambda)$, $i \in \mathbb{N} \backslash \{1\}$. Let $N_D(t)$ represent the number of orders that arrived to the warehouse by time t.

Derive an expression for $P[N_D(t) = n]$, $n \in \mathbb{N}_0$, and calculate its value when $\lambda = 1$, $t = 5$ and $n = 2$.

- **Delayed renewal process**

  $\{N_D(t) : t \le 0\}$

  $N_D(t) =$ number of orders that arrived to the warehouse by time t

- **Inter-renewal times**

  $X_1 \sim \text{gamma}(2, \lambda)$

  $X_i \overset{i.i.d.}{\sim} \text{gamma}(4, \lambda),\ i \in \mathbb{N} \backslash \{1\}$

- **Auxiliary r.v.**

  S_n = arrival time of order n, $\quad n \in \mathbb{N}$

  $S_n = X_1 + \sum_{i=2}^{n} X_i, \quad n \in \mathbb{N} \backslash \{1\}$

  $S_1 = X_1 \sim \text{gamma}(2, \lambda)$. Furthermore, since the sum of independent gamma distributions with a common scale parameter,

  $$S_n \sim \text{gamma}(2 + 4(n-1), \lambda), \quad n \in \mathbb{N}.$$

- **Requested expression**

$$\begin{aligned} P[N_D(t) = 0] &= P(X_1 > t) \\ &= 1 - F_{gamma(2,\lambda)}(t) \\ &= 1 - \left[1 - F_{Poisson(\lambda t)}(2-1) \right] \\ &= F_{Poisson(\lambda t)}(1) \\ &= e^{-\lambda t}(1 + \lambda t) \end{aligned}$$

$$\begin{aligned} P[N_D(t) = n] &\overset{form.}{=} P(S_n \le t) - P(S_{n+1} \le t) \\ &= F_{gamma(2+4(n-1),\lambda)}(t) - F_{gamma(2+4n,\lambda)}(t) \\ &\overset{form.}{=} \left[1 - F_{Poisson(\lambda t)}(2 + 4(n-1) - 1) \right] - \left[1 - F_{Poisson(\lambda t)}(2 + 4n - 1) \right] \\ &= F_{Poisson(\lambda t)}(4n+1) - F_{Poisson(\lambda t)}(4n-3) \\ [&= \sum_{i=0}^{4n+1} e^{-\lambda t} \frac{(\lambda t)^i}{i!} - \sum_{i=0}^{4n-3} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \\ &= \sum_{i=4n-2}^{4n+1} e^{-\lambda t} \frac{(\lambda t)^i}{i!}], \quad n \in \mathbb{N}. \end{aligned}$$

- **Requested probability**

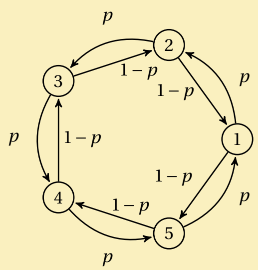
$$\begin{aligned} P[N_D(t) = n] &\overset{\lambda=1, t=5, n=2}{=} F_{Poisson(1\times5)}(4 \times 2 + 1) - F_{Poisson(1\times5)}(4 \times 2 - 3) \\ &= F_{Poisson(5)}(9) - F_{Poisson(5)}(5) \\ &\overset{tables}{=} 0.9682 - 0.6160 \\ &= 0.3522. \end{aligned}$$

Group 5 — Discrete time Markov chains 8.5 points

1. A particle moves among 5 vertices that are situated on a circumference in the following manner. At each step it moves one step either in the clockwise direction with probability p $(0 < p < 1)$ or the counterclockwise direction with probability $1 - p$. Let $\{X_n : n \in \mathbb{N}_0\}$ be a DTMC, where X_0 denotes the initial state and X_n represents the position of the particle at step n.

(a) Draw the associated transition diagram and determine the TPM. (1.0)

- **DTMC**
  $\{X_n : n \in \mathbb{N}_0\}$
  X_n = position of the particle at step n
- **State space**
  $\mathscr{S} = \{1,2,3,4,5\}$
- **Transition diagram**
  According to the description above, we are dealing with the following transition diagram:



- **TPM**
  From the transition diagram above, we get

$$
\begin{aligned}
\mathbf{P} &= [P_{ij}]_{i,j\in\mathscr{S}} \\
&= [P(X_{n+1} = j \mid X_n = i)]_{i,j\in\mathscr{S}}, \quad n \in \mathbb{N}_0 \\
&= [P(X_1 = j \mid X_0 = i)]_{i,j\in\mathscr{S}} \\
&= \begin{bmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{bmatrix}.
\end{aligned}
$$

(b) Classify the states of this DTMC. Is it periodic? (1.5)

- **Classification of the states of the DTMC**
  - Judging by the transition diagram, all states communicate with one another, i.e., the state space $\mathscr{S} = \{1,2,3,4,5\}$ is a single closed communicating class. Thus, the DTMC has a finite state space and is irreducible. As a result, all states are positive recurrent.
  - **Periodicity of the states**
    A close inspection of the transition diagram leads to the conclusion that we can from state 1 to state 1 in $2,4,5,6,7,\ldots$ steps. Thus, state 1 had period $d = \gcd\{n : P_{11}^n > 0\} = 1$. Moreover, since periodicity is a class property and the DTMC is irreducible, all its states have the same period $d = 1$ and the DTMC is aperiodic.
    [In fact, if we obtain the entry $(1,1)$ of a few powers of $\mathbf{P}$, for $0 < p < 1$:

$$
\begin{aligned}
(\mathbf{P}^2)_{11} &= \text{1st. row of } \mathbf{P} \times \text{1st. column of } \mathbf{P} \\
&= [0 \quad p \quad 0 \quad 0 \quad 1-p] \times \begin{bmatrix} 0 \\ 1-p \\ 0 \\ 0 \\ p \end{bmatrix} = 2p(1-p) \neq 0; \\
(\mathbf{P}^3)_{11} &= \text{1st. row of } \mathbf{P} \times \text{1st. column of } \mathbf{P}^2 \\
&= [0 \quad p \quad 0 \quad 0 \quad 1-p] \times \begin{bmatrix} 2p(1-p) \\ 0 \\ (1-p)^2 \\ p^2 \\ 0 \end{bmatrix} = 0; \\
(\mathbf{P}^4)_{11} &= \text{1st. row of } \mathbf{P}^2 \times \text{1st. column of } \mathbf{P}^2 \\
&= [2p(1-p) \quad 0 \quad p^2 \quad (1-p)^2 \quad 0] \times \begin{bmatrix} 2p(1-p) \\ 0 \\ (1-p)^2 \\ p^2 \\ 0 \end{bmatrix} \neq 0 \\
&\vdots \quad]
\end{aligned}
$$

(c) Calculate the limiting probabilities. (1.5)

- **Requested limiting probabilities**
  For an irreducible positive recurrent and aperiodic DTMC,
  $$\lim_{n\to+\infty} P_{ij}^n = \pi_j, \quad i,j \in \mathscr{S},$$
  where the limiting probabilities $\{\pi_j : j \in \mathscr{S}\}$ are the unique non-negative solution of
  $$\begin{cases} \pi_j = \sum_{i\in\mathscr{S}} \pi_i P_{ij}, & j \in \mathscr{S} \\ \sum_{j\in\mathscr{S}} \pi_j = 1, \end{cases}$$
  Capitalizing on the fact that $\mathbf{P}$ is a doubly stochastic TPM, that is, $\sum_{k\in\mathscr{S}} P_{ik} = \sum_{k\in\mathscr{S}} P_{kj} = 1$, $i,j \in \mathscr{S}$, we get:
  $$\begin{cases} \pi_j = \sum_{i\in\mathscr{S}, i\neq j} \pi_i P_{ij} + \pi_j P_{ij}, & j \in \mathscr{S} \\ - \end{cases}$$
  $$\begin{cases} \pi_j = \sum_{i\in\mathscr{S}, i\neq j} \pi_i P_{ij} + \pi_j \left(1 - \sum_{k\in\mathscr{S}, k\neq j} \pi_i P_{ij}\right), & j \in \mathscr{S} \\ - \end{cases}$$
  $$\begin{cases} 0 = \sum_{i\in\mathscr{S}, i\neq j} (\pi_i - \pi_j) P_{ij}, & j \in \mathscr{S} \\ - \end{cases}$$
  $$\begin{cases} \pi_i = \pi_j, & i,j \in \mathscr{S} \\ \sum_{j\in\mathscr{S}} \pi_j = 1. \end{cases}$$
  Hence $\pi_j = \frac{1}{\#\mathscr{S}} = \frac{1}{5}$, i.e., the stationary distribution is uniform$(\{1,\ldots,5\})$.

(d) Now, consider $p \neq \frac{1}{2}$ and the DTMC $\{X_m : m \in \mathbb{Z}\}$ governed by the same TPM as $\{X_n : n \in \mathbb{N}_0\}$. Are we dealing with a time reversible DTMC? (0.5)

- **New DTMC and associated TPM**
  $\{X_m : m \in \mathbb{Z}\}$
- **Checking time reversibility**
  $\{X_m : m \in \mathbb{Z}\}$ is time reversible iff the detailed balance equations
  $$\pi_i \times P_{ij} = \pi_j \times P_{ji}, \quad i,j \in \mathscr{S},$$
  are verified.
  The stationary distribution is uniform in $\{1,2,3,4,5\}$ (i.e., $\pi_j = \frac{1}{5}$, $j \in \mathscr{S}$), as seen in (c), and the TPM $\mathbf{P}$ is not symmetric if $p \neq \frac{1}{2}$. Consequently, the detailed balance equations failed to be verified.
  [Using Kolmogorov's criterion for time reversibility leads to the same conclusion.]

(e) Consider $p = \frac{1}{2}$ and determine $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$, for $i = 1,2,3,4,5$, $n = 1,2$ and $j = 1$. (2.0)

- **TPM**

$$\mathbf{P} = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

- **Requested probabilities**

  Let:

  i) $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$ be the probability of reaching state j for the first time starting from state i, for $i, j \in \mathcal{S}$ and $n \in \mathbb{N}$;

  ii) $\underline{f}_j^n = [f_{ij}^n]_{i\in\mathcal{S}}$ be the associated vector, for fixed $j \in \mathcal{S}$ and $n \in \mathbb{N}$.

  According to the formulae,

$$\underline{f}_j^n = \begin{cases} \underline{f}_j^1 = [P_{ij}]_{i\in\mathcal{S}}, & n = 1 \\ {}^{(j)}\mathbf{P} \times \underline{f}_j^{n-1} = \left[{}^{(j)}\mathbf{P}\right]^{n-1} \times \underline{f}_j^1, & n = 2, 3, \ldots, \end{cases}$$

  where ${}^{(j)}\mathbf{P}$ is obtained by setting all the entries of the j^{th} column of $\mathbf{P}$ equal to 0.

  When $j = 1$, we get

$$\begin{aligned} {}^{(1)}\mathbf{P} &= \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \\ \underline{f}_1^1 &= [P_{i1}]_{i\in\mathcal{S}} \\ &= \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} \\ \underline{f}_1^2 &= {}^{(2)}\mathbf{P} \times \underline{f}_2^1 \\ &= \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \\ 0 \\ 0.25 \\ 0.25 \\ 0 \end{bmatrix}. \end{aligned}$$

2. Let $\{X_n : n \in \mathbb{N}_0\}$ be a branching process such that the number of offspring per individual has p.f. given by $(1-p)^x p$, for $x \in \mathbb{N}_0$. Suppose that we are dealing with a random number of initial individuals, X_0, with the same p.f. as the number of offspring per individual. (2.0)

Derive the extinction probability of this branching process and write it in terms of p, when $0 < p < \frac{1}{2}$.

- **Simple branching process**

  $\{X_n : n \in \mathbb{N}_0\}$

  $X_0 \sim \text{geometric}^*(p)$,

  X_n = size of generation n

  $X_n = \sum_{l=1}^{X_{n-1}} Z_l, \quad n \in \mathbb{N}$

- **Number of offspring per individual and its p.g.f.**

  $Z_l \equiv Z_{l,n}$ = number of offspring of the l^{th} individual of generation n

  $Z_l \stackrel{i.i.d.}{\sim} Z \sim \text{geometric}^*(p), \quad l \in \mathbb{N}$

  $E(Z) \stackrel{form.}{=} \frac{1-p}{p} > 1$, when $0 < p < \frac{1}{2}$

  $P_Z(s) = E(s^Z) \stackrel{form.}{=} \frac{p}{1-(1-p)s}, \quad |s| \leq 1$

- **Probability of extinction** (with one single initial individual)

  $E(Z) > 1$, therefore the probability of extinction, $\pi \stackrel{form.}{=} \lim_{n\to+\infty} P(X_n = 0)$, is the smallest positive number satisfying

$$\begin{aligned} s &\stackrel{form.}{=} P_Z(s) \\ &= \frac{p}{1-(1-p)s} \\ (1-p)s^2 - s + p &= 0 \\ s &= \frac{1 \pm \sqrt{1-4(1-p)p}}{2(1-p)} \\ s &= \frac{1 \pm \sqrt{1-4p+4p^2}}{2(1-p)} \\ s &= \frac{1 \pm (1-2p)}{2(1-p)} \\ s &= 1 \quad \text{or} \quad \frac{p}{1-p}. \end{aligned}$$

  Since $0 < p < \frac{1}{2}$, the smallest positive root is $\pi = \frac{p}{1-p} = \frac{1}{E(Z)}$.

- **New initial state**

  $X_0 = \text{geometric}^*(p)$

- **New probability of extinction**

  Using the total probability law, the fact that the offspring are produced independently and $X_0 \sim Z$, we get

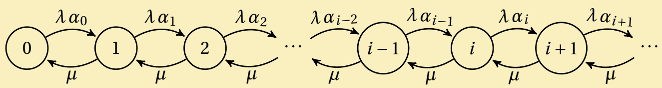
$$\begin{aligned} P(\text{extinction}) &= \sum_{j=0}^{+\infty} P(\text{extinction} \mid X_0 = j) \times P(X_0 = j) \\ &= \sum_{j=0}^{+\infty} \pi^j \times P(X_0 = j) \\ &= P_{X_0}(\pi) \\ &= P_Z(\pi) \\ &= \pi \\ &= \frac{p}{1-p}. \end{aligned}$$

Group 6 — Continuous time Markov chains	9.5 points

1. Potential customers arrive at a single-server station in accordance with a Poisson process with rate λ. However, if the arriving customer finds n customers already in the station, then he/she will enter the system with probability α_n $(n \in \mathbb{N}_0)$. Assuming an exponential service rate μ, set this up as a birth and death process $\{X(t) : t \geq 0\}$, where $X(t)$ represents the number of customers in this staition at time t.

(a) Draw the rate diagram and identify the infinitesimal generator $\mathbf{R}$ of $\{X(t) : t \geq 0\}$. (1.5)

- **CTMC**
  $\{X(t) : t \ge 0\}$
  $X(t)$ = number of customers in the queueing system at time t
- **State space**
  $\mathscr{S} = \mathbb{N}_0$
- **Birth/death rates**
  $\lambda_i = \lambda\,\alpha_n, \quad i \in \mathbb{N}_0$
  $\mu_i = \mu, \quad i \in \mathbb{N}$
- **Rate diagram**
  [Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij} > 0$) with q_{ij} written on it. These rates coincide with the birth and death rates...]



- **Infinitesimal generator**
  This matrix has entries

$$r_{ij} \;=\; \begin{cases} q_{ij}, & i \neq j \\ -\nu_i = -\sum_{m\in\mathscr{S}} q_{im}, & j = i \end{cases}$$

  and in this case $\mathbf{R} = [r_{ij}]_{i,j\in\mathscr{S}}$ is equal to

$$\begin{bmatrix} -\lambda\alpha_0 & \lambda\alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu & -(\lambda\alpha_1+\mu) & \lambda\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu & -(\lambda\alpha_2+\mu) & \lambda\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \mu & -(\lambda\alpha_{i-2}+\mu) & \lambda\alpha_{i-2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \mu & -(\lambda\alpha_{i-1}+\mu) & \lambda\alpha_{i-1} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda\alpha_i+\mu) & \lambda\alpha_i & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\lambda\alpha_{i+1}+\mu) & \lambda\alpha_{i+1} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

(b) Write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \mathbb{N}_0$, when $\alpha_n = \xi(1-\xi)^n$, $n \in \mathbb{N}_0$ $(0 < \xi < 1)$. (Do not solve the differential equations!) (1.5)

- **Kolmogorov's forward differential equations**
  These equations can be written in matrix form:

$$\frac{d\,\mathbf{P}(t)}{dt} = \left[\frac{d\,P_{ij}(t)}{dt}\right]_{i,j\in\mathscr{S}} \overset{form.}{=} \mathbf{P}(t)\times\mathbf{R}.$$

  Since $i = 0$, we are only interested in the first row of the previous matrix. Hence the following Kolmogorov's forward differential equations

$$\frac{d\,P_j(t)}{dt} \overset{form.}{=} P_{j-1}(t)\times\lambda\alpha_{j-1} - P_j(t)\times(\lambda\alpha_j+\mu_j) + P_{j+1}(t)\times\mu_{j+1}, \quad j \in \mathscr{S}.$$

  They read as follows:

$$\frac{d\,P_0(t)}{dt} = -P_0(t)\times\lambda\xi + P_1(t)\times\mu$$
$$\frac{d\,P_j(t)}{dt} = P_{j-1}(t)\times\lambda\xi(1-\xi)^{j-1} - P_j(t)\times\left(\lambda\xi(1-\xi)^j+\mu\right) + P_{j+1}(t)\times\mu, \quad j \in \mathbb{N}.$$

(c) After checking the ergodicity condition, derive the equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t)$ for this birth and death process, when $\lambda = \mu = 1$ and $\xi = 0.5$.[7] (1.5)

[7] **Note**: It might be useful to know that $\sum_{n=1}^{+\infty} 0.5^{\frac{n(n+1)}{2}} \simeq 0.641633$.

- **Ergodicity condition**
  In this particular case, it reads as follows:

$$\exists k_0 \in \mathbb{N} : \forall k \ge k_0, \quad \frac{\lambda_k}{\mu_k} < 1$$
$$\frac{\lambda_k}{\mu_k} \overset{\lambda=\mu=1,\xi=0.5}{=} 0.5\times 0.5^k = 0.5^{k+1} < 1.$$

  The ergodicity condition is verified for any $k \in \mathbb{N}_0$, thus for any $k \ge k_0 = 1$.
- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_j(t)$

$$P_0 = \left[1 + \sum_{n=1}^{+\infty}\left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1}$$
$$\overset{\lambda=\mu=1,\xi=0.5}{=} \left\{1 + \sum_{n=1}^{+\infty}\left[\prod_{i=0}^{n-1}\left(0.5^{i+1}\right)\right]\right\}^{-1}$$
$$= \left[1 + \sum_{n=1}^{+\infty} 0.5^{\frac{n(n+1)}{2}}\right]^{-1}$$
$$\simeq (1+0.641633)^{-1}$$
$$\simeq 0.609150$$
$$P_j = \left(\prod_{i=0}^{j-1}\frac{\lambda_i}{\mu_{i+1}}\right)\times P_0$$
$$\simeq 0.5^{\frac{j(j+1)}{2}}\times 0.609150, \quad j \in \mathbb{N}_0.$$

2. Calls to a technical support centre arrive according to a Poisson process with rate equal to 30 calls per hour. The time for a support person to serve one customer is exponentially distributed with a mean of 5 minutes. The support centre has 3 technical staff to assist callers. Assume that customers do not abandon their calls.

(a) Obtain and interpret $P(L_s \ge 3)$. (2.0)

- **Birth-death queueing system**
  $M/M/m$
  $m = 3$
- **State space**
  $\mathscr{S} = \mathbb{N}_0$
- **Birth/death rates**
  $\lambda_k = \lambda = 30, \quad k \in \mathbb{N}_0$

$$\mu_k = \begin{cases} k\mu = 12k, & k \in \{1,2,\dots,m-1\} \\ m\mu = 12m, & k \in \{m, m+1,\dots,\} \end{cases}$$

- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{m\mu} = \frac{30}{3\times 12} = 5/6 < +\infty$
- **Performance measure (in the long-run)**
  L_s = number of customers requesting technical support
- **Requested probability and interpretation**

$$P(L_s \ge m) \overset{form.}{=} C(m, m\rho)$$
$$= \frac{\frac{(m\rho)^m}{m!(1-\rho)}}{\sum_{j=0}^{m-1}\frac{(m\rho)^j}{j!} + \frac{(m\rho)^m}{m!(1-\rho)}}$$
$$\overset{m=3,\lambda=30,\mu=12,m\rho=2.5}{=} \frac{\frac{2.5^3}{3!(1-5/6)}}{\sum_{j=0}^{3-1}\frac{2.5^j}{j!} + \frac{2.5^3}{3!(1-5/6)}}$$

$$P(L_s \geq 3) = \frac{\frac{2.5^3}{3!(1-5/6)}}{1+2.5+\frac{2.5^2}{2}+\frac{2.5^3}{3!(1-5/6)}} = 0.702247.$$

$P(L_s \geq m)$ represents the probability of positive delay, i.e., the probability that a customer is unable to immediately access a support staff, that is, the probability of a customer being put on hold.

(b) On average, how many customers are put on hold and how long does a customer spend until he/she ends the call with a member of the technical staff? (1.5)

- **Other performance measures (in the long-run)**

  L_q = number of customers that are put on hold (in the long-run)

  W_s = total time spent by a customer to get technical support

- **Requested expected values**

$$E(L_q) \overset{form.}{=} \frac{\rho}{1-\rho} \times C(m, m\rho)$$
$$\overset{(a),\, m=3, \lambda=30, \mu=12, \rho=5/6}{\simeq} \frac{5/6}{1-5/6} \times 0.702247$$
$$\simeq 3.51123$$
$$E(W_s) = \frac{1}{\mu} + \frac{C(m, m\rho)}{m\mu(1-\rho)}$$
$$\simeq \frac{1}{12} + \frac{0.702247}{3 \times 12 \times (1-5/6)}$$
$$= 0.200374 \quad [\simeq 12\text{minutes}].$$

(c) The technical support centre management is considering hiring another person to join the technical staff if customers spend more than 15 minutes on hold with a probability larger than 0.25. Is this the case? (1.5)

- **Another performance measures (in the long-run)**

  W_q = time spent by a customer on hold

- **Requested probability**

  A customer spends more than a quarter of an hour on hold with a probability

$$P(W_q > t) = 1 - F_{W_q}(t)$$
$$\overset{form.}{=} C(m, m\rho) \times [1 - F_{exponential(m\mu(1-\rho))}(t)$$
$$= C(m, m\rho) \times e^{-m\mu(1-\rho)\,t}$$
$$\overset{(a),\, m=3, \mu=12, \rho=5/6}{\simeq} 0.702247 \times e^{-3\times 12\times(1-5/6)\times 0.25}$$
$$\simeq 0.156692.$$

- **Comment**

  Since $P(W_q > 0.25) \simeq 0.156692 \not> 0.25$ there is no need to hire an additional person to join the technical staff.