Functional Analysis

Test 2 - 4th June of 2021

1. (1.5 val) Let X and Y be two normed spaces and $T \in L(X, Y)$. Show that $T^{'} \in L(Y^{'} X^{'})$ and
$$\|T^{'}\| = \|T\|.$$

2. (2.0 val) Let A be a unital complex Banach algebra and x an invertible element of $\mathcal{A}$.

   (a) Prove that if $s \in \mathcal{A}$ is such that $\|x - s\| < \|x^{-1}\|^{-1}$ then s is invertible and find an expression for s^{-1}.

   (b) Is it true that $\varphi(x) = 0$ for some non-zero linear multiplicative functional φ on $\mathcal{A}$?

3. (2.5 val) Let $C([0, 1])$ denote the Banach space of continuous complex-valued functions on $[0, 1]$ with the maximum norm. Let $T : C([0, 1]) \to C([0, 1])$ be the operator defined by
$$(Tf)(x) = \int_0^x (x - t) f(t) dt.$$

   (a) Show that T is continuous and compute $\|T\|$.

   (b) Is T a compact operator?

   (c) Assume that, $(T^n f)(x) = \frac{1}{n!} \int_0^x (x - t)^n f(t) dt$ for every $n \in \mathbb{N}$. Compute $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

4. (2.0.val) Let X be a Banach Space and f_n a sequence of linear functionals in X such that the operator $T : X \to l^1(\mathbb{N})$, $T(x) = (f_n(x))_{n=1}^{\infty}$ is well defined. Show that
$$T \text{ is continuous if and only if } f_n \text{ is continuous } \forall n \in \mathbb{N}.$$

5. (2.0 val) Let X be a Banach Space. Suppose $T \in L(X) \setminus \{0\}$ is such that $T^2 = 0$ and let M be the algebraic complement of $\ker T$ in X.

   (a) Show that the map $\| \cdot \|_1 : X = \ker T \oplus M \to \mathbb{R}_0^+$ defined by
$$\|y + m\|_1 = \|y\| + \|m + \ker T\|_{X/\ker T} \quad y \in \ker T,\ m \in M$$
   is another norm in X and X is complete for this norm.

   (b) Show that the two norms are equivalent if and only if M is closed.

   (c) Prove that T is continuous for the norm $\| \cdot \|_1$.