DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

MMA, LMAC, MECD

2nd. Semester – 2019/2020
2020/06/09 – 8AM

Duration: 90 minutes **Test 2**

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes 2.0 points

Consider a delayed renewal process, $\{N_D(t): t \geq 0\}$, whose: i) renewal function is equal to $m_D(t) = \frac{\lambda t}{2} + \frac{1-e^{-2\lambda t}}{4}$, $t \geq 0$; ii) first inter-renewal time is exponentially distributed with parameter λ. (2.0)

Derive the distribution of the inter-renewal times X_i ($i \in \mathbb{N}\backslash\{1\}$) of the stochastic process $\{N_D(t): t \geq 0\}$.

- **Delayed renewal process**
  $\{N_D(t): t \geq 0\}$
- **Inter-renewal times**
  $X_1 \sim G \sim \text{exponential}(\lambda)$
  $X_i \overset{i.i.d.}{\sim} X \sim F, \quad i \in \mathbb{N}\backslash\{1\}$
- **LST of** G

$$\begin{aligned}\tilde{G}(s) &= \int_{0^-}^{+\infty} e^{-st}\, dG(t) \quad [\equiv M_{\text{exponential}(\lambda)}(-s) \overset{\text{form.}}{=} \frac{\lambda}{\lambda+s}] \\ &= \int_{0^-}^{+\infty} e^{-st} \times f_{\text{exponential}(\lambda)}(t)\, dt \\ &= \lambda \times LT[e^{-\lambda t}, s] \\ &\overset{\text{form.}}{=} \frac{\lambda}{\lambda+s}\end{aligned}$$

- **LST of the renewal function**

$$\begin{aligned}\tilde{m}_D(s) &= \int_{0^-}^{+\infty} e^{-st}\, dm(t) \quad [\equiv LST[m_D(t), s] = s \times LT[m_D(t), s] - m_D(0) = s \times LT[m_D(t), s]] \\ &= \int_{0^-}^{+\infty} e^{-st} \times \frac{\lambda}{2}\left(1 + e^{-2\lambda t}\right) dt \\ &= \frac{\lambda}{2} \times \left(LT[1, s] + LT[e^{-2\lambda t}, s]\right) \\ &\overset{\text{form.}}{=} \frac{\lambda}{2s} + \frac{\lambda}{2(2\lambda+s)} \\ &= \frac{2\lambda^2 + \lambda s + \lambda s}{2s(2\lambda+s)} \\ &= \frac{\lambda(\lambda+s)}{s(2\lambda+s)}\end{aligned}$$

- **Deriving the inter-renewal distribution** F
  Since $\tilde{m}_D(s) \overset{\text{form.}}{=} \frac{\tilde{G}(s)}{1-\tilde{F}(s)}$ the LST of the inter-renewal distribution can be obtained as follows:

$$\begin{aligned}\tilde{F}(s) &= 1 - \frac{\tilde{G}(s)}{\tilde{m}_D(s)} \\ &= 1 - \frac{\frac{\lambda}{\lambda+s}}{\frac{\lambda(\lambda+s)}{s(2\lambda+s)}} \\ &= 1 - \frac{s(2\lambda+s)}{(\lambda+s)^2} \\ &= \frac{\lambda^2 + 2\lambda s + s^2 - 2\lambda s - s^2}{(\lambda+s)^2}\end{aligned}$$

$$\tilde{F}(s) = \left(\frac{\lambda}{\lambda+s}\right)^2.$$

Taking advantage of the m.g.f. in the formulae, we get:

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dF(x) \\ &= E\left(e^{-sX}\right) \\ &= M_X(-s) \\ \left(\frac{\lambda}{\lambda+s}\right)^2 &\equiv M_{\text{gamma}(2,\lambda)}(-s),\end{aligned}$$

that is, the inter-renewal times are $X_i \overset{i.i.d.}{\sim} X$, $i \in \mathbb{N}\backslash\{1\}$, where $X \sim \text{gamma}(2, \lambda)$.

- **[Comment**
  $\{N_D(t): t \geq 0\}$ consists of all odd arrivals of a $PP(\lambda)$.]

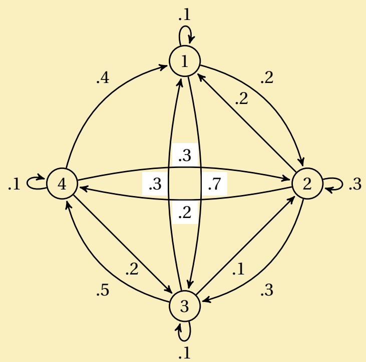
Group 2 — Discrete time Markov chains 8.5 points

1. Suppose a certain customer daily visits one of four restaurants $(1, 2, 3, 4)$ according to the TPM

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0 \\ 0.2 & 0.3 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{bmatrix}.$$

(a) Draw the associated transition diagram and classify the states of this DTMC. Are the states periodic? (1.5)

- **DTMC**
  $\{X_n: n \in \mathbb{N}\}$
  X_n = restaurant visited on day n
- **State space**
  $\mathcal{S} = \{1, 2, 3, 4\}$
- **TPM**
  See text above.
- **Transition diagram**



- **Classification of the states of the DTMC**
  - Judging by the transition diagram, all states communicate with one another, thus $\mathscr{S} = \{1,2,3,4\}$ is a single closed communicating class. Hence the DTMC has a finite state space and is irreducible. With that being said, all states are positive recurrent.
  - The transition diagram leads to the conclusion that we can only return to state 1 after $1,2,3,\ldots$ transitions, thus $d(1) = gcd\{n \in \mathbb{N} : P_{11}^{n} > 0\} = 1$ and this state is aperiodic. The same holds for the remaining states of this irreducible DTMC. [After all, periodicity is a class property.]

(b) Admit that the initial restaurant this customer visits is represented by the r.v. X_1 with p.f. $\underline{\alpha} = [0.1 \quad 0.2 \quad 0.3 \quad 0.4]$. (1.5)

Calculate $P(X_3 = 1, X_5 = 2)$, i.e., the probability that this customer visits restaurant 1 on the 3^{rd} day and restaurant 2 on the 5^{th} day.

- **Initial state**

  X_1

- **Requested probability**

  Since the initial state of this DTMC is X_1 (instead of X_0) we have to adapt the results in the list of formulae:

$$\begin{aligned}\underline{\alpha} &= [P(X_1 = i)]_{i\in\mathscr{S}}\\ &= [0.1 \quad 0.2 \quad 0.3 \quad 0.4]\\ \underline{\alpha}^n &= [P(X_{n+1} = i)]_{i\in\mathscr{S}}\\ &\overset{\text{form.}}{=} \underline{\alpha} \times \mathbf{P}^n.\end{aligned}$$

Thus,

$$P(X_3 = 1, X_5 = 2) = P(X_3 = 1) \times P(X_5 = 2 \mid X_3 = 1)$$

where:

$$\begin{aligned}P(X_{2+1} = 1) &= \underline{\alpha} \times \mathbf{P} \times (1^{st} \text{ column of } \mathbf{P})\\ &= \underline{\alpha} \times \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0\\ 0.2 & 0.3 & 0.3 & 0.2\\ 0.3 & 0.1 & 0.1 & 0.5\\ 0.4 & 0.3 & 0.2 & 0.1\end{bmatrix} \times \begin{bmatrix}0.1\\0.2\\0.3\\0.4\end{bmatrix}\\ &= [0.1 \quad 0.2 \quad 0.3 \quad 0.4] \times \begin{bmatrix}0.26\\0.25\\0.28\\0.20\end{bmatrix}\\ &= 0.24;\\ P(X_5 = 2 \mid X_3 = 2) &= P(X_{2+1} = 2 \mid X_1 = 1)\\ &= P_{12}^2\\ &= (1^{st} \text{ row of } \mathbf{P}) \times (2^{nd} \text{ column of } \mathbf{P})\\ &= [0.1 \quad 0.2 \quad 0.7 \quad 0] \times \begin{bmatrix}0.2\\0.3\\0.1\\0.3\end{bmatrix}\\ &= 0.15.\end{aligned}$$

Consequently,

$$\begin{aligned}P(X_3 = 1, X_5 = 2) &= 0.24 \times 0.15\\ &= 0.036. \qquad [\overset{\text{form.}}{=} (\sum_{i\in\mathscr{S}} \alpha_i \times P_{i1}^2) \times P_{12}^2]\end{aligned}$$

(c) What is the expected time between consecutive visits of this customer to restaurant 1 ?[1] (2.5)

- **Stationary distribution**

  Since the DTMC is irreducible, positive recurrent and aperiodic we can add that the unique stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathscr{S}}$, is given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

where: $\underline{1} = [1 \quad \cdots \quad 1]$ is a row vector with $\#\mathscr{S}$ ones; $\mathbf{I}$ = identity matrix with rank $\#\mathscr{S}$; $\mathbf{ONE}$ is the $\#\mathscr{S} \times \#\mathscr{S}$ matrix with all entries equal to one. Hence

$$\begin{aligned}\underline{\pi} &= \underline{1} \times \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \begin{bmatrix}0.1&0.2&0.7&0\\0.2&0.3&0.3&0.2\\0.3&0.1&0.1&0.5\\0.4&0.3&0.2&0.1\end{bmatrix} + \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\right)^{-1}\\ &= \underline{1} \times \begin{bmatrix}1.9&0.8&0.3&1\\0.8&1.7&0.7&0.8\\0.7&0.9&1.9&0.5\\0.6&0.7&0.8&1.9\end{bmatrix}^{-1}\\ &\simeq [1 \quad 1 \quad 1 \quad 1] \times \begin{bmatrix}0.686432&-0.264867&0.106118&-0.277683\\-0.236329&0.890294&-0.208305&-0.195660\\-0.120130&-0.292208&0.646104&0.016234\\-0.079118&-0.121326&-0.228811&0.679255\end{bmatrix}\\ &= [0.250855 \quad 0.211893 \quad 0.315106 \quad 0.222146].\end{aligned}$$

- **Requested expected value**

  The expected time between consecutive visits to restaurant 1 is equal to

$$\begin{aligned}\mu_{11} &\overset{\text{form.}}{=} \frac{1}{\pi_1}\\ &\simeq \frac{1}{0.250855}\\ &\simeq 3.986367.\end{aligned}$$

(d) Admit the daily amounts spent in restaurants 1, 2, 3, 4 are: $c(1) = 10$, $c(2) = 15$, $c(3) = 20$, $c(4) = 25$. (0.5)

What is the long-run amount spent per day by this customer in visits to these four restaurants?

- **Vector of costs**

$$\underline{c} = [c(j)]_{j\in\mathscr{S}} = \begin{bmatrix}10\\15\\20\\25\end{bmatrix}$$

- **Long-run expected profit per time unit**

$$\begin{aligned}\underline{\pi} \times \underline{c} &= \sum_{j\in\mathscr{S}} \pi_j \times c(j)\\ &= \underline{\pi} \times \underline{c}\\ &\simeq 0.250855 \times 10 + 0.211893 \times 15 + 0.315106 \times 20 + 0.222146 \times 25\\ &= 17.542715.\end{aligned}$$

2. Let $\{X_n : n \in \mathbb{N}_0\}$ be a branching process describing the size of generation n of a colony of cells. Admit that $X_0 = 1$ and the number of offspring per individual has p.g.f. given by

$$P(s) = 0.2 + 0.2\,s + 0.6\,s^2, \quad s \in [0,1].$$

(a) Calculate the expected number of offspring per individual and the extinction probability. (1.5)

[1] The following result may come handy: $\begin{bmatrix}1.9&0.8&0.3&1\\0.8&1.7&0.7&0.8\\0.7&0.9&1.9&0.5\\0.6&0.7&0.8&1.9\end{bmatrix}^{-1} = \begin{bmatrix}0.686432&-0.264867&0.106118&-0.277683\\-0.236329&0.890294&-0.208305&-0.195660\\-0.120130&-0.292208&0.646104&0.016234\\-0.079118&-0.121326&-0.228811&0.679255\end{bmatrix}.$

- **Branching process**
  $\{X_n : n \in \mathbb{N}_0\}$
  X_n = size of generation n a colony of cells

  $X_0 = 1$
  $X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n \in \mathbb{N}$

- **Number of offspring per individual and its p.g.f.**
  $Z_l \equiv Z_{l,n}$ = number of offspring of the l^{th} individual of generation n

  $Z_l \overset{i.i.d.}{\sim} Z,\ l \in \mathbb{N}$

  $P(s) = E(s^Z) = \sum_j s^j \times P(Z = j) = 0.2 + 0.2\,s + 0.6\,s^2, \quad s \in [0,1]$

- **Probability of extinction**

$$\begin{aligned} E(Z) &\overset{\text{form.}}{=} \left.\frac{dP(s)}{ds}\right|_{s=1} \\ &= \left.\frac{d\left(0.2 + 0.2\,s + 0.6\,s^2\right)}{ds}\right|_{s=1} \\ &= (0.2 + 1.2\,s)|_{s=1} \\ &= 0.2 + 1.2 \\ &= 1.4. \end{aligned}$$

- **Probability of extinction**
  Since $E(Z) > 1$, the probability of extinction, $\pi \overset{\text{form.}}{=} \lim_{n\to+\infty} P(X_n = 0 \mid X_0 = 1)$, is the smallest positive number satisfying

$$\begin{aligned} s &\overset{\text{form.}}{=} \sum_{j=0}^{+\infty} s^j \times P_j \\ &= P(s) \\ &= 0.2 + 0.2\,s + 0.6\,s^2 \\ 0.6\,s^2 - 0.8\,s + 0.2 &= 0 \\ s &= \frac{0.8 \pm \sqrt{0.8^2 - 4 \times 0.6 \times 0.2}}{2 \times 0.6} \\ &= \frac{0.8 \pm 0.4}{1.2} \\ &= \frac{2}{3} \pm \frac{1}{3}, \end{aligned}$$

  thus $\pi = \frac{1}{3}$.

(b) Obtain the probability that this colony of cells is extinct in the third generation. (1.0)

- **Requested probability**

$$\begin{aligned} \pi_3 &= P(X_3 = 0 \mid X_0 = 1) \\ &= \left.\sum_{j=0}^{+\infty} s^j \times P(X_3 = j \mid X_0 = 1)\right|_{s=0} \\ &= P_3(0) \\ &\overset{\text{form.}}{=} P(P[P(0)]) \\ &= P[P(0.2 + 0.2 \times 0 + 0.6 \times 0^2)] \\ &= P[P(0.2)] \\ &= P(0.2 + 0.2 \times 0.2 + 0.6 \times 0.2^2) \\ &= P(0.264) \\ &= P(0.2 + 0.2 \times 0.264 + 0.6 \times 0.264^2) \\ &\simeq 0.294618. \end{aligned}$$

Group 3 — Continuous time Markov chains 9.5 points

1. A rent-a-car maintenance facility has 4 servers. Assume that: cars arrive at this facility according to a Poisson process with rate equal to 6 cars per day; service times are independent and exponentially distributed with mean $\frac{1}{3}$. Moreover, admit that we are dealing with impatient customers, who, upon arrival: i) wait for service if they find at most 4 cars already at this facility; ii) wait for service with probability $\frac{1}{2}$ if they see exactly 5 cars at the facility; iii) leave if they spot precisely 6 cars at this maintenance facility.

Let $X(t)$ be the number of cars at this facility at time t.

(a) Draw the rate diagram and identify the infinitesimal generator $\mathbf{R}$ of the CTMC $\{X(t) : t \ge 0\}$. (2.0)

- **CTMC**
  $\{X(t) : t \ge 0\}$
  $X(t)$ = number of cars at the maintenance facility at time t

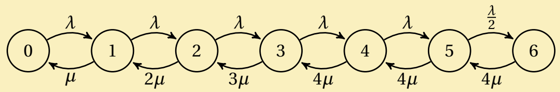
- **State space**
  $\mathscr{S} = \{0,1,2,3,4,5,6\}$

- **Birth/death rates**

$$\lambda_i = \begin{cases} \lambda = 6, & i = 0,1,2,3,4 \\ \frac{\lambda}{2} = 3, & i = 5 \\ 0, & i = 6,7,\ldots \end{cases}$$

$$\mu_i = \begin{cases} i \times \mu = 3i, & i = 1,2,3,4 \\ 4\mu = 12, & i = 5,6 \\ 0, & i = 7,8,\ldots \end{cases}$$

- **Rate diagram**
  [Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij} > 0$) with q_{ij} written on it. These rates coincide with the birth and death rates...]



- **Infinitesimal generator**
  This matrix has entries

$$r_{ij} = \begin{cases} q_{ij}, & i \ne j \\ -\nu_i = -\sum_{m \in \mathscr{S}} q_{im}, & j = i \end{cases}$$

  and in this case it is equal to

$$\begin{aligned} \mathbf{R} &= [r_{ij}]_{i,j \in \mathscr{S}} \\ &= \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & 0 & 0 \\ 0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda & 0 & 0 \\ 0 & 0 & 0 & 4\mu & -(\lambda+4\mu) & \lambda & 0 \\ 0 & 0 & 0 & 0 & 4\mu & -(\frac{\lambda}{2}+4\mu) & \frac{\lambda}{2} \\ 0 & 0 & 0 & 0 & 0 & 4\mu & -4\mu \end{bmatrix}. \end{aligned}$$

(b) Write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \mathbb{N}_0$. (Do not try to solve the differential equations!) (2.0)

- **Kolmogorov's forward differential equations**
  These equations can be written in matrix form:

$$\frac{d\mathbf{P}(t)}{dt} = \left[\frac{dP_{ij}(t)}{dt}\right]_{i,j\in\mathscr{S}} \overset{\text{form.}}{=} \mathbf{P}(t)\times\mathbf{R}.$$

Since $i=0$, we are only interested in the first row of the previous matrix. Hence the following Kolmogorov's forward differential equations

$$\frac{dP_j(t)}{dt} \overset{\text{form.}}{=} P_{j-1}(t)\times\lambda_{j-1} - P_j(t)\times(\lambda_j+\mu_j) + P_{j+1}(t)\times\mu_{j+1}, \quad j\in\mathscr{S}.$$

They read as follows:

$$\begin{aligned}
\frac{dP_0(t)}{dt} &= -P_0(t)\times\lambda + P_1(t)\times\mu \\
\frac{dP_j(t)}{dt} &= P_{j-1}(t)\times\lambda - P_j(t)\times(\lambda+j\,\mu) + P_{j+1}(t)\times\min\{j+1,4\}\times\mu, \quad j=1,2,3,4 \\
\frac{dP_5(t)}{dt} &= P_4(t)\times\lambda - P_5(t)\times\left(\frac{\lambda}{2}+4\mu\right) + P_6(t)\times 4\mu, \quad j=5 \\
\frac{dP_6(t)}{dt} &= P_5(t)\times\frac{\lambda}{2} - P_6(t)\times 4\mu,
\end{aligned}$$

where $\lambda=6$ and $\mu=3$.

(c) Derive the equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t)$ for this birth and death process. (2.0)

- **[Obs.**
  Since this CTMC has a finite state space, we only need to verify that $\rho = \frac{\lambda}{\mu} < +\infty$ to guarantee the existence of equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t)$. And that is the case.]
- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_j(t)$

$$\begin{aligned}
P_0 &= \left[1+\sum_{n=1}^{+\infty}\left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1} \\
&= \left(1 + \frac{6}{3} + \frac{6}{3}\times\frac{6}{6} + \frac{6}{3}\times\frac{6}{6}\times\frac{6}{9} + \frac{6}{3}\times\frac{6}{6}\times\frac{6}{9}\times\frac{6}{12} + \frac{6}{3}\times\frac{6}{6}\times\frac{6}{9}\times\frac{6}{12}\times\frac{6}{12}\right. \\
&\quad \left. + \frac{6}{3}\times\frac{6}{6}\times\frac{6}{9}\times\frac{6}{12}\times\frac{6}{12}\times\frac{3}{12}\right)^{-1} \\
&= \left(1+2+2+\frac{4}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{12}\right)^{-1} \\
&= \left(\frac{5\times 12+7\times 4+1}{12}\right)^{-1} \\
&= \frac{12}{89} \quad [\simeq 0.134831] \\
P_j &= \left(\prod_{i=0}^{j-1}\frac{\lambda_i}{\mu_{i+1}}\right)\times P_0, \quad j=1,2,\dots,6 \\
&= \begin{cases}
2\,P_0 = \frac{24}{89} & [\simeq 0.269663], \quad j=1,2 \\
\frac{4}{3}\,P_0 = \frac{16}{89} & [\simeq 0.179775], \quad j=3 \\
\frac{2}{3}\,P_0 = \frac{8}{89} & [\simeq 0.089888], \quad j=4 \\
\frac{1}{3}\,P_0 = \frac{4}{89} & [\simeq 0.044944], \quad j=5 \\
\frac{1}{12}\,P_0 = \frac{1}{89} & [\simeq 0.011236], \quad j=6.
\end{cases}
\end{aligned}$$

2. Satellites are launched from country A (resp. B) according to a Poisson process with rate $\lambda_A = 50$ (resp. $\lambda_B = 15$). Suppose each satellite will operate for a random time with uniform distribution in the interval [5, 15] after it has been launched.

Let $X_A(t)$ (resp. $X_B(t)$) denote the number of satellites launched from country A (resp. B) that are still operating at time t.

(a) Determine $E[X_A(t)+X_B(t) \mid X_A(0)=X_B(0)=0]$, for $t=10$. (2.0)

- **Analogy with the $M/G/\infty$**
  We can view this as two independent $M/G/\infty$ queueing systems where:
  - a satellite launched from country i corresponds to an arrival, and these arrivals occur according to a $PP(\lambda_i)$, $i=A,B$, where $\lambda_A=50$ and $\lambda_B=15$ (satellites per year);
  - uniform(5,15) is the operating or self-service distribution with expected value $\mu^{-1} = \frac{5+15}{2} = 10$, and the associated service rate is $\mu = 0.1$ (satellites per year).
- **State variables**
  $(X_i(t) \mid X_i(0)=0)$ = number of satellites launched from country i still operating at time t, given that there were no satellites from this country operating at time 0, $i=A,B$.
- **Transient distributions**
  According to the formulae, we have for these two $M/G/\infty$ queueing systems

$$(X_i(t)\mid X(0)=0) \sim \text{Poisson}\left(\lambda_i\times\int_0^t [1-G(t-s)]\,ds\right), \quad i=A,B,$$

where

$$\begin{aligned}
\int_0^t [1-G(t-s)]\,ds &= \int_0^t [1-G(s)]\,ds \\
&= \int_0^t [1-F_{\text{uniform}(5,15)}(s)]\,ds \\
&\overset{t=10}{=} \int_0^5 ds + \int_5^{10}\left(1-\frac{s-5}{15-5}\right)ds \\
&= 5+\int_5^{10}\frac{15-s}{10}\,ds \\
&= 5+\frac{1}{10}\int_5^{10}(15-s)\,ds \\
&= 5+\left.\frac{15\times 5}{10}-\frac{s^2}{20}\right|_5^{10} \\
&= 5+7.5-(5-1.25) \\
&= 8.75.
\end{aligned}$$

- **Requested expected value**

$$\begin{aligned}
E[X_A(t)+X_B(t)\mid X_A(0)=X_B(0)=0] &= (\lambda_A+\lambda_B)\times\int_0^t [1-G(t-s)]\,ds \\
&= (50+15)\times 8.75 \\
&= 568.75.
\end{aligned}$$

(b) Admit that countries A and B launch satellites independently and obtain an approximate value to the probability that the total number of satellites still operating exceeds 650 in the long-run. (1.5)

- **Long-run distributions**
  Since countries A and B launch satellites independently, the number of satellites launched from each country and operating in the long-run is

$$X_i = \lim_{t\to+\infty}(X_i(t)\mid X_i(0)=0) \overset{indep}{\sim} \text{Poisson}\left(\frac{\lambda_i}{\mu}\right), \quad i=A,B$$

(see formulae). As a consequence,

$$X_A+X_B \sim \text{Poisson}\left(\frac{\lambda_A+\lambda_B}{\mu} \overset{(a)}{=} (50+15)\times 10 = 650\right).$$

- **Requested probability**

$$\begin{aligned}
P[X_A+X_B>650\mid X_A(0)=X_B(0)=0] &= 1-P[X_A+X_B\le 650\mid X_A(0)=X_B(0)=0] \\
&= 1-F_{\text{Poisson}(650)}(650) \\
&\overset{CLT}{\simeq} 1-\Phi\left(\frac{650-650}{\sqrt{650}}\right) \\
&= 1-\Phi(0) \\
&= 0.5.
\end{aligned}$$