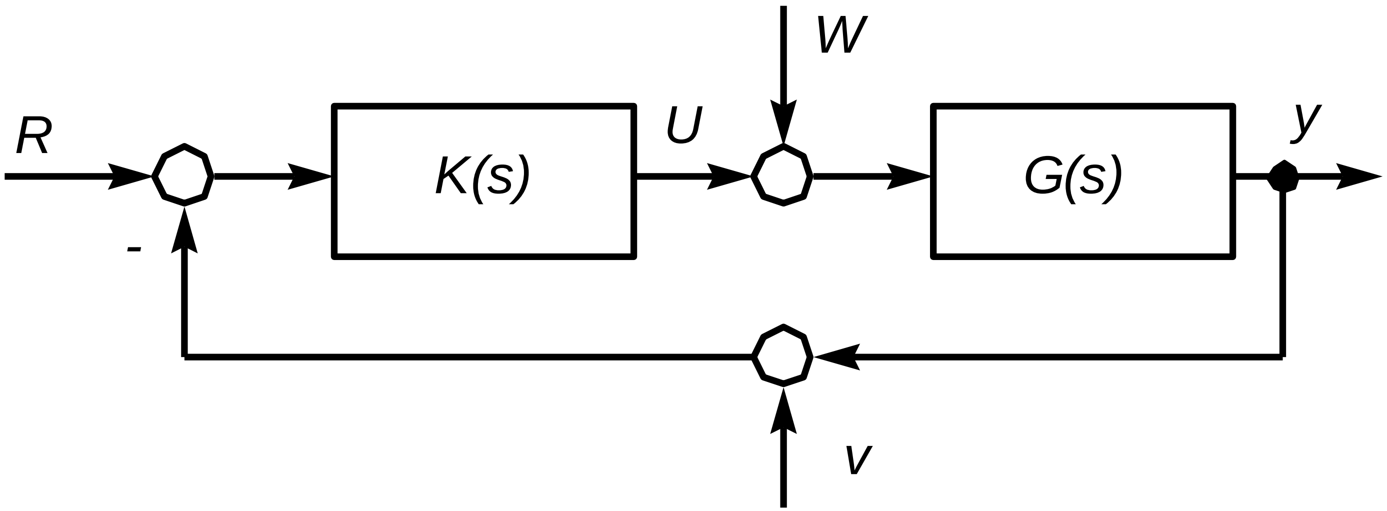
4. – Computer control

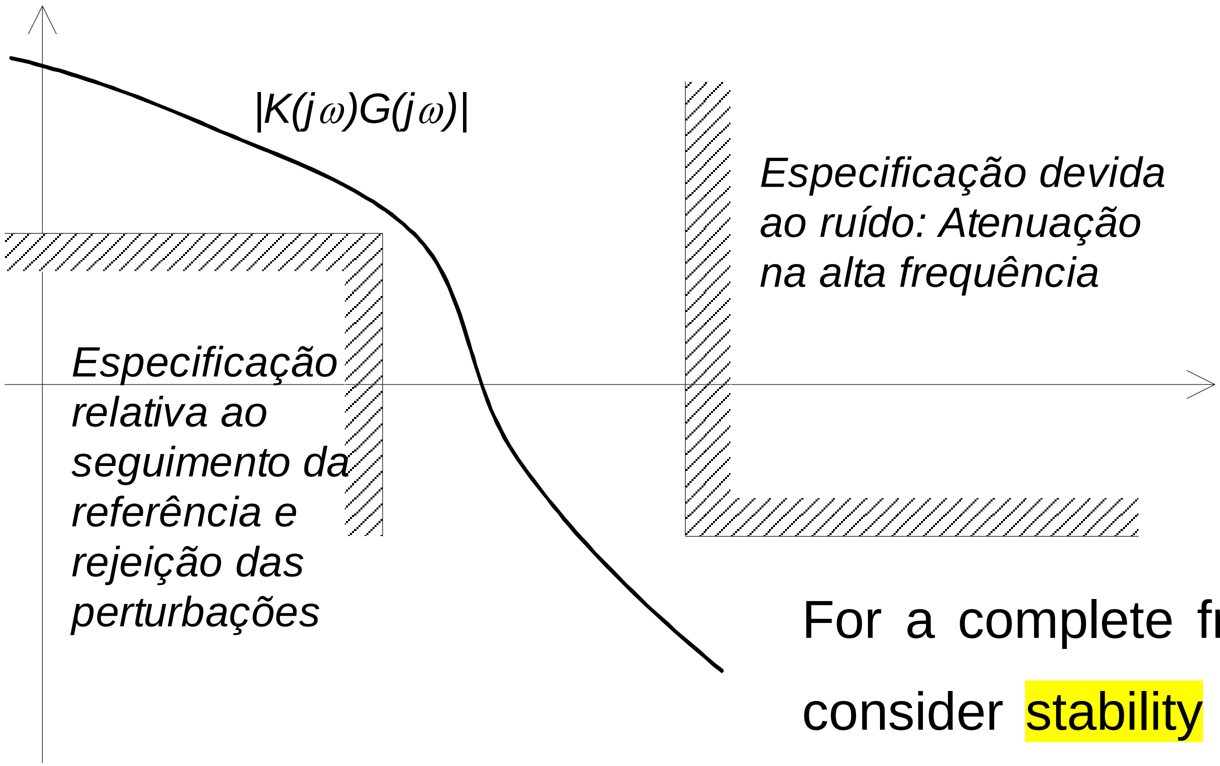
A brief review: Feedback control



$$Y_{cL} = \frac{KG}{1+KG}R + \frac{G}{1+KG}W - \frac{KG}{1+KG}V$$

KG is the loop gain

The controller $K(s)$ is designed such as to “shape” the loop gain:



$$Y_{cL} = \frac{KG}{1+KG}R + \frac{G}{1+KG}W - \frac{KG}{1+KG}V$$

For a complete framework we must also consider stability and uncertainty in plant model.

Main points

- The loop-gain must be large in the low frequency range to track the reference within the specified precision;
- The loop gain must be small in the high frequency range to
  - Reject sensor noise
  - Avoid unmodelled dynamics that can cause instability
- Noise and unmodelled high frequency dynamics impose a limit to the frequency range in which the controlled system may track a time-varying reference
- At 0dB gain the slope of the loop-gain can be at most 20 db/dec. or an unstable closed-loop will result.

4.A – State linear feedback control

Objectivo: *No final desta unidade, o aluno deverá ser capaz de, dado um modelo de estado de um sistema discreto, projectar controladores por retroacção linear de variáveis de estado (RLVE), admitindo que o estado está ou não acessível.*

Franklin, Powell, Workman (1998). *Digital Control of Dynamic Systems*. 3rd ed., Addison Wesley.

Linear state feedback – Regulation

Admissible control laws:

$$u(k) = -Lx(k)$$

State feedback vector of gains

Objective: Find L in order that the dynamic matrix of the closed-loop system has specified values.

Example: Regulation of the sampled double integrator

Model of the sampled double integrator:

$$x(k+1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} u(k)$$

Feedback law:

$$u(k) = -L_1 x_1(k) - L_2 x_2(k)$$

The gains L_1 and L_2 are chosen for the characteristic polynomial of the closed loop to be

$$z^2 + p_1 z + p_2$$

$$z^2 + p_1 z + p_2$$

If we want that this characteristic polynomial corresponds to the sampling with a ZOH of a continuous 2nd order system with given ξ and ω_n, the coefficients p_1 and p_2 are given by:

$$p_1 = -2e^{-\xi\omega_n h} \cos\left(\omega_n h \sqrt{1-\xi^2}\right)$$

$$p_2 = e^{-2\xi\omega_n h}$$

Closed-loop system:

$$x(k+1) = \left(\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \begin{bmatrix} L_1 & L_2 \end{bmatrix} \right) x(k)$$

$$x(k+1) = \begin{bmatrix} 1 - L_1 \dfrac{h^2}{2} & h - \dfrac{h^2}{2} L_2 \\ -hL_1 & 1 - hL_2 \end{bmatrix} x(k)$$

Characteristic polynomial of the closed-loop:

$$\det\left\{\begin{bmatrix} z-1+L_1\dfrac{h^2}{2} & -h+\dfrac{h^2}{2}L_2 \\ hL_1 & z-1+hL_2 \end{bmatrix}\right\} = z^2+\left(L_1\frac{h^2}{2}+hL_2-2\right)z+L_1\frac{h^2}{2}-hL_2+1$$

Design equations, obtained by equating the coefficients of desired and closed-loop characteristic polynomials:

$$\begin{cases} \dfrac{h^2}{2}L_1+hL_2=p_1+2 \\ \dfrac{h^2}{2}L_1-hL_2=p_2-1 \end{cases}$$

In this case there is always a solution for the gains L_1 and L_2.

Exercise (Pole placement with state feedback)

Plant state model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0{,}5 \\ 1 \end{bmatrix} u(k)$$

State feedback controller

$$u(k) = -\begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Find the controller gains L_1 and L_2 such that the poles of the closed-loop system are $0{,}9 \pm j0.1$.

Hints:

1. Obtain the state model of the closed-loop system as a function of L_1 and L_2.
2. Write the characteristic polynomial of the closed-loop system. The coefficients are a function of L_1 and L_2.
3. Write the desired characteristic polynomial. This polynomial has roots equal to the specified poles.
4. Equate the coefficients of the monomials of the same order in both polynomials to obtain a set of linear equations in L_1 and L_2. If the equations are not linear, check your calculations: You've done a mistake.

Solution

Closed-loop dynamics

$$\Phi - \Gamma L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0{,}5 \\ 1 \end{bmatrix} \begin{bmatrix} L_1 & L_2 \end{bmatrix} = \begin{bmatrix} 1 - 0{,}5L_1 & 1 - 0{,}5L_2 \\ -L_1 & 1 - L_2 \end{bmatrix}$$

Closed-loop characteristic polynomial

$$\det(zI - \Phi + \Gamma L) = \begin{vmatrix} z - 1 + 0{,}5L_1 & -1 + 0{,}5L_2 \\ L_1 & z - 1 + L_2 \end{vmatrix} =$$

$$= z^2 + (-2 + 1{,}5L_1)z + 1 + 0{,}5L_1 - L_2$$

Desired characteristic polynomial

$$(z - 0{,}9)^2 + 0.01 = z^2 - 1.8z + 0{,}82$$

Closed-loop characteristic polynomial

$$z^2 + (-2 + 1{,}5L_1)z + 1 + 0{,}5L_1 - L_2$$

Desired characteristic polynomial

$$(z - 0{,}9)^2 + 0.01 = z^2 - 1.8z + 0{,}82$$

Equating the coefficients in both polynomials:

$$-2 + 1{,}5L_1 = -1{,}8$$

$$1 + 0{,}5L_1 - L_2 = 0{,}82$$

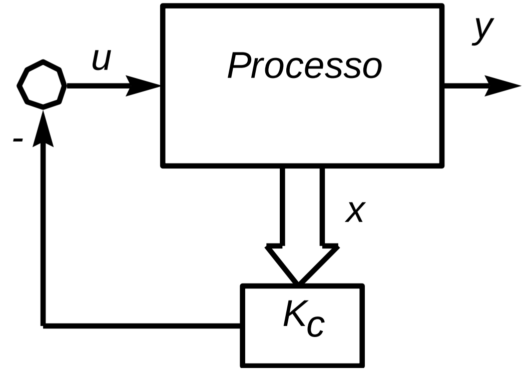
$$L_1 = 0{,}13 \qquad L_2 = 0{,}25$$

----End of Exercise----

Pole placement – General case

Given the state-space model

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k)$$



with the state feedback control law

$$u(k) = -Lx(k)$$

The closed loop system becomes

$$x(k+1) = (\Phi - \Gamma L)x(k)$$

The gain L is to designed such that the dynamic matrix of the closed loop system, $\Phi - \Gamma L$, has eigenvalues at the specified values.

Pole placement (cont.)

Given desired pole locations

$$\beta_1, \beta_2, \dots, \beta_n$$

The desired characteristic polynomial is thus

$$\alpha_c(z) = (z - \beta_1)(z - \beta_2) \dots (z - \beta_n)$$

The pole placement problem consists thus of the solution, with respect to the gains L of

$$\det(zI - \Phi + \Gamma L) = \alpha_c(z)$$

In which case the characteristic polynomial of the closed loop becomes equal to the desired one.

Pole placement problem (condition for solution)

Under what conditions can we place arbitrarily the eigenvalues of the closed loop matrix $\Phi - \Gamma L$ by selecting the vector of gains L?

Answer

The problem is solvable for any α_c if

$$C = [\Gamma \quad \Phi\Gamma \quad \Phi^2\Gamma \; \ldots \; \Phi^{n-1}\Gamma]$$

has rank n (the dimension of the state vector).

The matrix C is called the controllability matrix.

Solution to the pole placement problem

Ackerman's formula

$$L = [0 \quad \dots \quad 1] C^{-1} \alpha_c(\Phi)$$

where

$$\alpha_c(\Phi) = \Phi^n + \alpha_1^c \Phi^{n-1} + \alpha_2^c \Phi^{n-2} + \cdots + \alpha_n^c I$$

Is the desired characteristic polynomial.

Implemented in MATLAB through the function *acker*.
Satisfactory for SISO systems of order less than 10.
Can handle systems with repeated roots.

Better alternative for higher order systems and multivariable systems: *place*.

Optimal solution to the pole placement problem

Select the gains that minimize

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[x^T(k)Qx(k) + Ru^2(k)\right]$$

Select the weight as

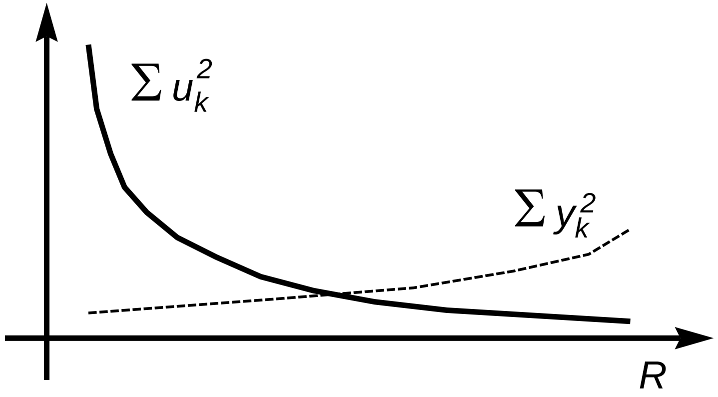
$$Q = H^T H \qquad x^T Qx = x^T H^T Hx = (Hx)^T Hx = y^T y = y^2$$

The cost becomes

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[y^2(k) + Ru^2(k)\right]$$

Use the *dlqr* function of MATLAB Control Systems Toolbox.

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[y^2(k) + Ru^2(k)\right]$$



Increasing the weight R:

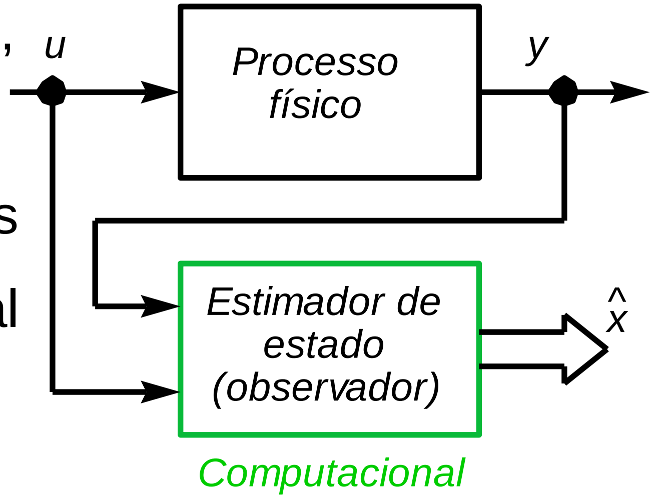
- Decreases the amplitude of u (Good);
- Increases the response of y to disturbances (Bad);
- Reduces the closed-loop bandwidth
    - Makes the system slower (Bad)
    - Reduces sensitivity to high-frequency noise and model errors (Good)

Observers (prediction observer)

Asymptotic estimate of the state based on observations of the input and output:

$$\hat{x}(k+1\,|\,k) = \Phi\hat{x}(k\,|\,k-1) + \Gamma u(k) + K_o\big(y(k) - C\hat{x}(k\,|\,k-1)\big)$$

This observer is a replica of the plant dynamics, driven by the same input, and corrected by a term that is the difference between what is expected to be the output at time k and the actual observation of the output.



Estimation error

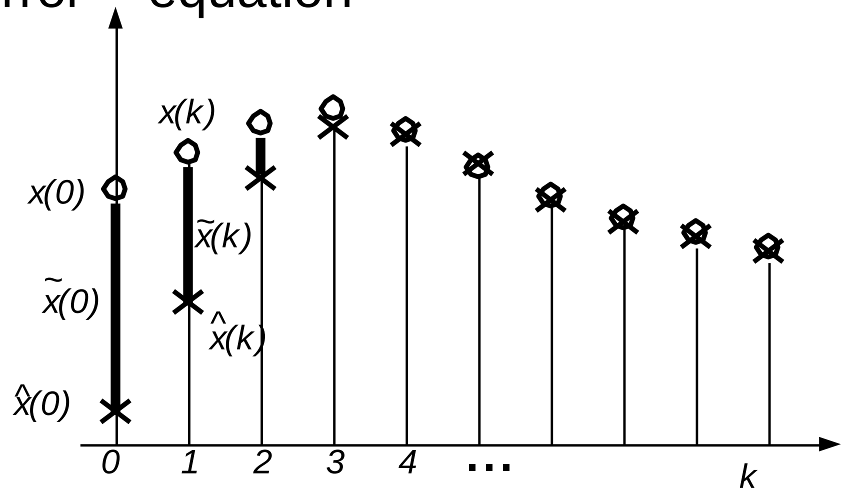
$$\tilde{x} = x - \hat{x}$$

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$\hat{x}(k+1 \mid k) = \Phi \hat{x}(k \mid k-1) + \Gamma u(k) + K_o \big(y(k) - C\hat{x}(k \mid k-1) \big)$$

Subtracting these equations yields the error equation

$$\tilde{x}(k+1) = (\Phi - K_o C)\tilde{x}(k)$$



Solution of the error equation

$$\tilde{x}(k) = (\Phi - K_o C)^k \tilde{x}(0)$$

The error will tend to zero iff

$$|\lambda(\Phi - K_o C)| < 1$$

Polynomial observer: $A_o(\mathrm{z}) = \det(zI - \Phi + K_o C)$

Example

Consider the double integrator

$$x(k+1)=\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}x(k)+\begin{bmatrix}h^2/2\\ h\end{bmatrix}u(k)$$

$$y(k)=[1 \quad 0]x(k)$$

Design a predictive observer such that the roots of the polynomial observer are placed at $0.4 \pm j0.4$.

Desired characteristic polynomial

$$\alpha_o(z) = z^2 - 0.8z + 0.32$$

$$\det(zI - \Phi + K_o C) = \det\left(z\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix} + \begin{bmatrix}K_{o1}\\ K_{o2}\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)$$

$$= z^2 + (K_{o1} - 2)z + hK_{o2} + 1 - K_{o1}$$

Comparing the coefficients

$$K_{o1} - 2 = -0.8$$

$$hK_{o2} + 1 - K_{o1} = 0.32$$

Solution:

$$K_{o1} = 1.2 \qquad K_{o2} = \frac{0.52}{h}$$

Separation Principle

$$\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Gamma u(k) + K_o\left(y(k) - C\hat{x}(k|k-1)\right)$$

Control law

$$u(k) = -L\hat{x}(k|k-1)$$

Separation Principle (similar to continuous time): The closed-loop poles are the eigenvalues that are roots of $\det(z\mathrm{I} - \Phi + \Gamma L)$ and $(\Phi - K_o C)$.

The controller gains and the observer gains can de designed separately.

Observers based on the most recent observation

(current observer)

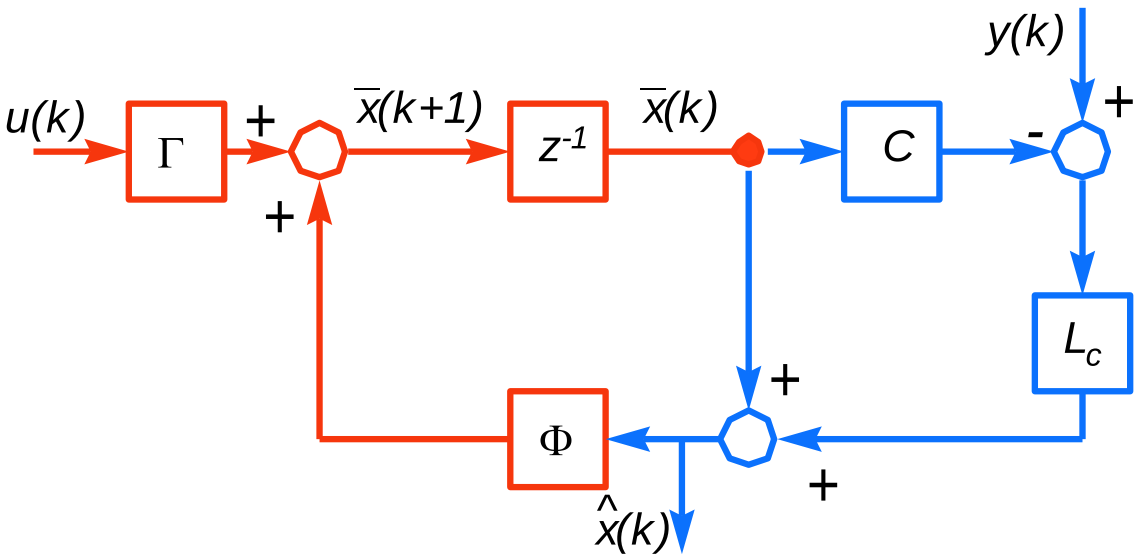
$$\hat{x}(k|k) = \Phi\hat{x}(k-1|k-1) + \Gamma u(k-1)$$
$$+ K_o[y(k) - C(\Phi\hat{x}(k-1|k-1) + \Gamma u(k-1))]$$

The estimate of the state at time k depends on the observations of the output up to time k (current time).

Alternative form of the equations of the current observer

Prediction: $\bar{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k)$

Correction: $\hat{x}(k) = \bar{x}(k) + L_c(y(k) - C\bar{x}(k))$



Optimal design of observer gains: The Kalman filter

Process model with sensor noise and stochastic disturbances

$$x(k+1) = \Phi x(k) + \Gamma u(k) + v_1(k),$$

$$y(k) = Hx(k) + v_2(k).$$

Noise characterization

$$E\left[v_1(k)v_1^T(k)\right] = Q_E,$$

$$E\left[v_2^2(k)\right] = R_E.$$

Use an unbiased estimator ⇒ Use the observer structure

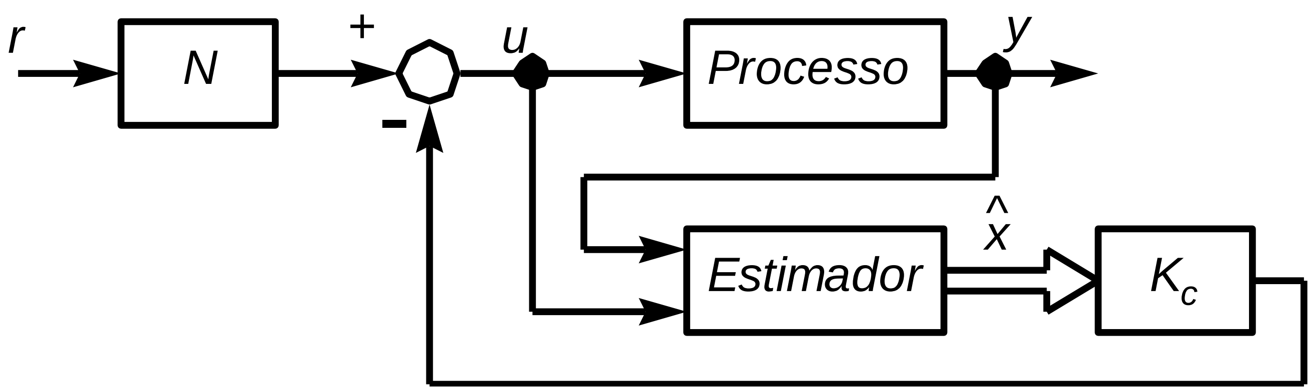
Select the gains by minimizing the steady state power (variance) of the estimation error

$$J_o = E\left[\sum_{k=0}^{\infty} \|x(k) - \hat{x}(k)\|^2\right]$$

Use the MATLAB function *dlqe* to compute the optimal observer gain (Kalman gain).

Select Q_E and R_E as tuning knobs (and not as true noise variances).

Introduction of a reference



The gain N is the inverse of the static gain of the ensemble process/estimator/controller. With the current observer:

$$\hat{x}(k) = \Phi\hat{x}(k-1) + \Gamma u(k-1) + K_o[y(k) - C(\Phi\hat{x}(k-1) + \Gamma u(k-1))]$$

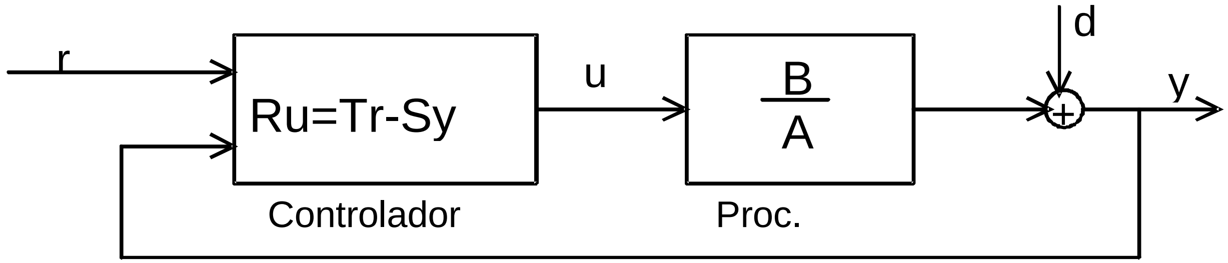
$$u(k) = Nr - K_c\hat{x}(k)$$

4.B. Controller design using polynomial methods

Objective: *Design linear pole-placement controllers in discrete time using input/output process models*

Reference: AW, Computer Controlled Systems

Pole placement based on input/output models



The process is modeled by the transfer function

Why do we have to impose this?

$$H(z) = \frac{B(z)}{A(z)}$$

A and B are coprime polinomials (*i.e.* without common roots).

Objective: Design a causal controller such that the controlled system is

$$H(z) = \frac{B_m(z)}{A_m(z)} \qquad A_m \text{ and } B_m \text{ coprime}$$

Admissible control laws

The admissible control laws have the form

$$R(q)u(k) = T(q)r(k) - S(q)y(k)$$

where R is monic (*i. e.* the coefficient of the largest power in R is 1)

$$R(q) = q^{\partial R} + r_1 q^{\partial R - 1} + \ldots + r_{\partial R}$$

q denotes the forward shift operator:

$$qx(k) := x(k+1)$$

∂R represents the degree of polynomial R.

Furthermore, the control law must be causal.

Example – Digital controller

$$(q + r_1)u(k) = t_0 qr(k) - (s_0 q + s_1)y(k)$$

Corresponds to the control law

$$u(k+1) + r_1 u(k) = t_0 r(k+1) - s_0 y(k+1) - s_1 y(k)$$

Delay by one unit of time

$$u(k) + r_1 u(k-1) = t_0 r(k) - s_0 y(k) - s_1 y(k-1)$$

Solve with respect to $u(k)$:

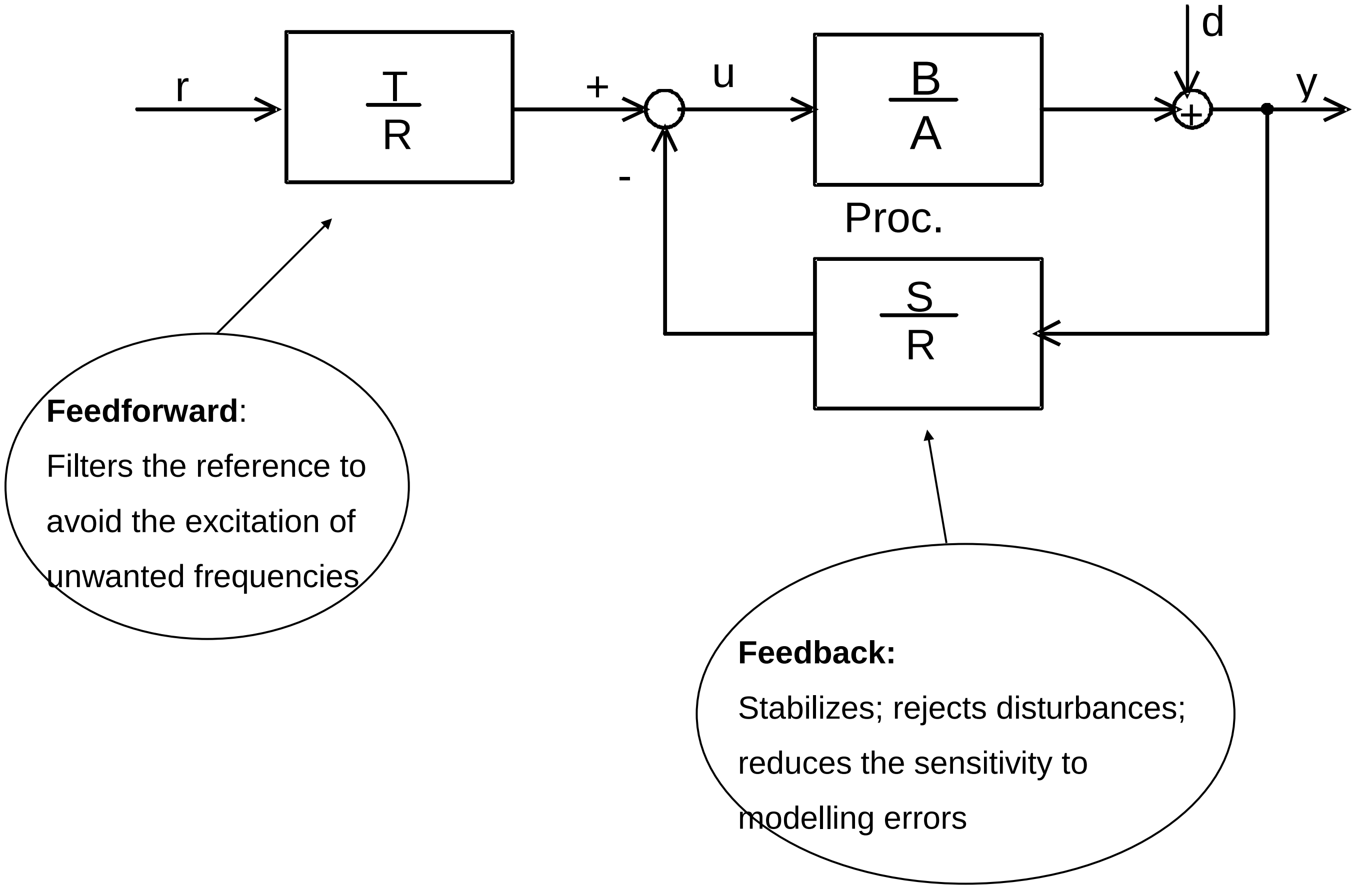
$$u(k) = -r_1 u(k-1) + t_0 r(k) - s_0 y(k) - s_1 y(k-1)$$

$$u(k) = -r_1 u(k-1) + t_0 r(k) - s_0 y(k) - s_1 y(k-1)$$

The realization of this control law in a microcontroller is very simple: The current value of the control variable is a linear combination of

- Previous values of the control
- The reference (and in other cases previous samples of the reference)
- The current value of the process output (read from a A/D) and its past value(s) (kept in memory).

Two degrees of freedom controller



Causality conditions

The causality of the controller means that the manipulated variable u at time k does not depend on future samples of the variables from which it is computed (process output, reference and, eventually, accessible disturbances).

The controller is causal in this sense iff

$$\partial R \geq \partial S$$

$$\partial R \geq \partial T$$

Example - Causality

$$R(q) = q + r_1 \qquad T(q) = q + t_1 \qquad S(q) = q + s_1$$

$$(q + r_1)u(k) = (t_0 q + t_1)r(k) - (s_0 q + s_1)y(k)$$

$$u(k+1) + r_1 u(k) = t_0 r(k+1) + t_1 u_c(k) - \left(s_0 y(k+1) + s_1 y(k)\right)$$

If the second condition is violated, *e. g.* if $S(q) = s_0 q^2 + s_1 q + s_2$

a non causal controller is defined:

$$u(k+1) + r_1 u(k) = t_0 r(k+1) + t_1 r(k) - \left(s_0 y(k+2) + s_1 y(k+1) + s_2 y(k)\right)$$

$$u(k) = -r_1 u(k-1) + t_0 r(k) + t_1 r(k-1) - \left(s_0 y(k+1) + s_1 y(k) + s_2 y(k-1)\right)$$

If this was the case, the manipulated variable at time k would depend on the output at time $k+1$.

Integral action

In oder to reject low frequency disturbances, the loop gain, given by

$$H_{\mathrm{lg}}(z) = \frac{B(z)S(z)}{A(z)R(z)}$$

Must be high in this frequency range.

One way to ensure this is by using integral action, making

$$R(z) = (z-1)^{\lambda} R_1'(z)$$

λ number of integrators. Usually λ=1 (um integrator) or λ=0 (*e.g.* if the process has already an integrator).

Closed-loop transfer function

Process model (no disturbance):

Multiply by $R(q)$

$$A(q)y(k) = B(q)u(k)$$

Controller:

$$R(q)u(k) = T(q)r(k) - S(q)y(k)$$

Multipliby by $B(q)$

Eliminate $u(k)$ between both equations

$$A(q)R(q)y(k) = B(q)\big(T(q)r(k) - S(q)y(k)\big)$$

Or
$$\big(A(q)R(q) + B(q)S(q)\big)y(k) = B(q)T(q)r(k)$$

The design problem consists in finding R, S, and T, such that:

$$\frac{BT}{AR + BS} = \frac{B_m}{A_m}$$

Since there is a polynomial A_o common to R, S, and T, that corresponds to the dynamic of the state estimator,

$$\frac{BT}{AR+BS}=\frac{B_m A_o}{A_m A_o}$$

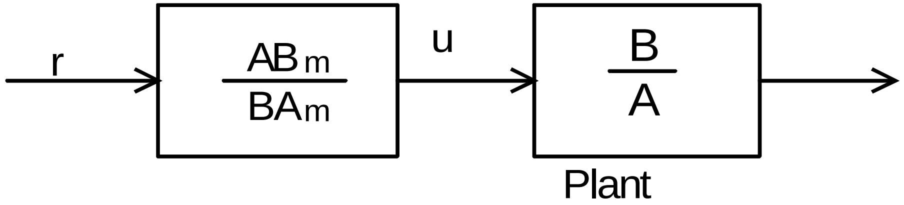
The polynomial A_o must be Hurwitz (i. e., must have all its roots inside the unit circle) and defines the dynamics of the observer.

There are many solutions (actually there are infinitely many) to the equation

$$\frac{BT}{AR+BS}=\frac{B_m}{A_m}$$

Example: Pure feedforward solution

$$R = BA_m \qquad S = 0 \qquad T = AB_m$$

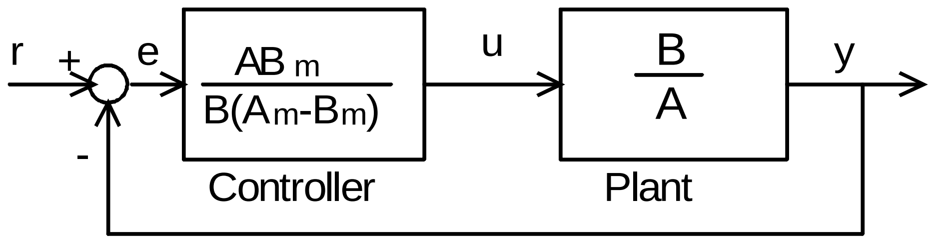


This is not a good solution: Since $S=0$, the loop gain is zero (there is no feedback). Disturbances are not rejected. Model uncertainty is not attenuated.

Example: Pure feedback solution

$$S = T = AB_m$$
$$R = B(A_m - B_m)$$
$$Ru = S(r - y)$$

$$\frac{BT}{AR+BS} = \frac{B_m}{A_m}$$



This solution has also problems:

- There is no warranty of high gain at low frequencies;
- The feedforward action is not explored to improve performance.

There are many solutions to the pole placement problem, but they are equally good or even acceptable. We want to find the good ones

To find R, S and T that satisfy

$$\frac{BT}{AR+BS}=\frac{B_m A_o}{A_m A_o}$$

There must be pole-zero cancelations. This must be performed carefully to prevent internal instability. Cancelling a zero outside the unit circle yields an unstable internal mode.

Establishing a design procedure

Consider the equation (the unknowns are R, S, and T):

$$\frac{BT}{AR+BS} = \frac{B_m A_o}{A_m A_o}$$

If a factor of B is not a factor of B_m, it must be a factor of $AR+BS$, in order to be cancelled.

Only "stable" zeros (inside the unit circle) can be cancelled. Therefore, factorize B as

$$B = B^+ B^-$$

B^+ is monic and contains all the zeros to cancel.

$$B = B^{+} B^{-}$$

Mónic, with all the zeros to cancel

Contains the zeros that are NOT cancelled

The specifications must be such that they include B^{-} in the zeros. Thus:

$$B_m = B^{-} B'_m$$

where B'_m is a polynomial to specify.

Since B^+ is not cancelled, it must be a factor of $AR+BS$.

Furthermore, if B^+ is a factor of $AR+BS$, it must be a factor of R (*why?)* and, therefore

$$R = B^+ R'$$

Putting everything together

$$\frac{BT}{AR+BS}=\frac{B_m A_o}{A_m A_o}$$

$$B=B^+B^- \qquad B_m=B^-B'_m \qquad R=B^+R'$$

Hence:

$$\frac{B^+B^-T}{B^+\left(AR'+B^-S\right)}=\frac{B^-B'_m A_o}{A_m A_o}$$

or, simplifying,

$$\frac{T}{AR'+B^-S}B^-=\frac{B'_m A_o}{A_m A_o}B^-$$

$$\frac{T}{AR'+B^{-}S}=\frac{B'_m A_o}{A_m A_o}$$

The following conditions must be satisfied:

$$T=B'_m A_o$$ This equation allows to compute T.

and

$$AR'+B^{-}S=A_m A_o$$

or, since $R'=(z-1)^{\lambda}R'_1$, $A(z-1)^{\lambda}R'_1+B^{-}S=A_m A_o$

This equation is called a *Diophantine equation*, where the unknowns are R_1' and S.

Diophantine equations

Consider the equation

$$AX + BY = C$$

A, B, C known polynomials X, Y unknowns (polynomials to be found)

- Are there solutions?
- Are there several solutions or is the solution unique?

In general, the equation is defined in an algebraic structure called ring (ring of integer numbers, ring of polynomials).

We satart with an example in the ring of integers.

A diophantine equation in the ring of integers

What are the values of x and y (integers) that satisfy

$$3x + 2y = 5$$

$x = 1 \qquad y = 1$ is one solution.

Furthermore, if x_0, y_0 is a solution, we can generate other solutions by

$$x = x_0 + 2n \qquad y = y_0 - 3n$$

with n any integer

Actually, it is possible to ptove that all the solutions are generated in this way..

Example: A diophantine equation without solution

$$\underbrace{4x + 6y}_{\text{Always even}} = \underbrace{1}_{\text{Odd number}}$$

The non-existence of solution is due to the fact that the maximum common divisor of 4 and 6 (which is 2) does not divide the second member.

This fact is general.

Division of polynomials

We say that polynomial A divides polynomial B, if there is a polynomial Q, called quotient, such that

$$B = AQ$$

It is not always possible to find a polynomial Q that verifies this. In general

$$B = AQ + R$$

where the polynomal R, called remainder, is such that

$$\partial R < \partial A$$

The symbol ∂ the degree of the polynomial.

Polynomial division: Examples

$$A(z) = (5z+5)(z+2) \qquad B_1(z) = z+1 \qquad B_2(z) = z+3$$

Polynomial A is divisible by B_1 sin, for $Q = 5(z+2)$, we have

$$A = B_1 Q$$

Polynomial A is not divisible by B_2. In this case

$$A = B_2 \times \underbrace{(5z)}_{Q(z)} + \underbrace{10}_{R(z)}$$

Maximum common divisor

The maximum common divisor of polynomials A and B is a polynomial that divides both A and B, which degree is maximum.

It is represented by (A, B).

For instance, for the polynomials

$$A(z) = (z+1)(z+3)(z+2)^2 \text{ and } B(z) = (z+3)(z+2)(z-5)$$

$$\left(A, B\right) = (z+3)(z+2)$$

The polynomial $z+1$ also divides both A and B but its degree is lower than the one of $\left(A, B\right) = (z+3)(z+2)$.

Solution of Diophantine equations with polynomials

The equation, in which A, B, C, X, Y are polynomials,

$$AX + BY = C$$

Has a solution iff the maximum common divider of A, B divides C

If A and B are coprime (i. e. if they have no common roots), all the solutions are given by

$$X = X_0 + TB \qquad Y = Y_0 - TA$$

where

X_0 and Y_0 verify the equation (they are a particular solution) and

T is any polynomial.

There are infinite many solutions to $AX + BY = C$ if A and B have no common roots. The solution is unique if we impose a suitable degree condition

Consider tghe equation in which A and B have no common roots

$$AX + BY = C$$

There is only one solution that verifies

$$\partial X < \partial B$$

Or, alternatively (another solution, that corresponds to this other condition)

$$\partial Y < \partial A$$

Back to the pole placement problem

Consider now the Diophantine equation that arises in pole placement

$$A(z-1)^{\lambda} R'_1 + B^{-} S = A_m A_o$$

It has a solution if A and B^{-} don't have common roots and B^{-} has no roots on *1* (the process is not a differentiator). Indeed, in this case, the maximum common divisor is 1, that divides $A_m A_o$.

Furthermore (previous slide), there is an unique solution such that

$$\partial S < \lambda + \partial A$$

In this case, it can be shown that

$$\partial R'_1 = \partial A_o + \partial A_m - \partial A - \lambda$$

Causality

If

$$\partial A_m - \partial B_m \geq \partial A - \partial B$$

and, also

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^+ + \lambda - 1$$

then, there is a causal controller that solves the pole placement problem.

Pole placement design algorithm

Data:

Process model

$$\frac{B(q)}{A(q)}$$

Specifications:

Desired model for the closed loop (reference model): $\dfrac{B_m(q)}{A_m(q)}$

It must satisfy $\partial A_m - \partial B_m \geq \partial A - \partial B$

Number on integrators in the controller (usually 0 or 1): λ

Pole placement design algorithm (cont.)

1) Factorize

$$B = B^{+}B^{-}$$

Where all the zeros to cancel, given by the roots of B^{+}, must be inside the unit cycle. Select B^{+} to be monic.

2) Find a polynomial B'_m that satisfies

$$B_m = B^{-}B'_m$$

3) Select A_o that satisfies the causality condition

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^{+} + \lambda - 1$$

4) Compute

$$T = B'_m A_o$$

5) Solve the following diophantine equation to obtain polynomials R' and S:

$$(q-1)^{\lambda} AR' + B^{-} S = A_o A_m$$

where

$$\partial S < \lambda + \partial A$$

$$\partial R' = \partial A_o + \partial A_m - \partial A - \lambda$$

6) Compute

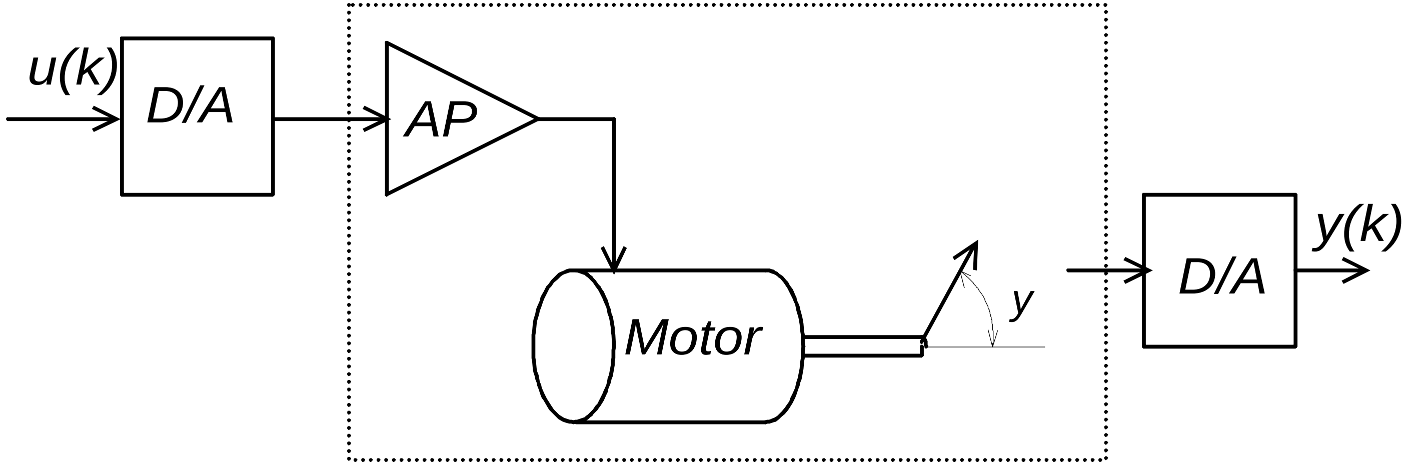
$$R = B^{+} R' (q-1)^{\lambda}$$

Solution of the Diophantine equation

- Constructive prove of the existence of solution (based on Elclides algorithm):
- Redução a uma equação matricial algébrica linear;
- Method of unknown multipliers (given the order of the unknown polynomials, assume general polynomials and equate the coefficients of the corresponding powers).

Digital control of a DC motor

We want to design a position controller to a DC servo-motor.



The transfer function of the DC motor, between applied tension and position

$$\frac{1}{s(s+1)}$$

Process model

Digital transfer function:

$$H(z) = \frac{B(z)}{A(z)} = \frac{K(z-b)}{(z-1)(z-a)}$$

$$K = e^{-h} - 1 + h$$

$$a = e^{-h}$$

$$b = 1 - \frac{h\left(1 - e^{-h}\right)}{e^{-h} - 1 + h}$$

It is remarked that $b < 0$, *i. e.*, the zero is in the negative real axis.

If this zero is cancelled, there will be an internal oscillation that may be undesirable.

Specifications (with zero cancellation)

Transfer function specified to the closed-loop

$$H_m(z) = \frac{B_m(z)}{A_m(z)} = \frac{z(1 + p_1 + p_2)}{z^2 + p_1 z + p_2}$$

Parameters p_1 and p_2 are selected such that they correspond to the sampling of a 2nd order continuous system with given ζ and $\zeta\omega_n$. They are computed from

$$p_1 = -2e^{-\zeta\omega_n h}\cos\left(\sqrt{1-\zeta^2}\,\omega_n h\right)$$

$$p_2 = e^{-2\zeta\omega_n h}$$

$$H_m(z) = \frac{B_m(z)}{A_m(z)} = \frac{z(1 + p_1 + p_2)}{z^2 + p_1 z + p_2}$$

The coice of $B_m(z)$ is made such that:

- The static gain of the closed-loop is one, $H_m(1) = 1$
- The delay of the closed-loop is minimum.

In order for the controller to be causal, the delay of the closed-loop is, at least, the one of the open-loop.

In open-loop the delay is 1. Hence, we add 1 zero at the origin in order for the closed-loop delay to be also 1 (the best possible).

Design of the pole placement controller for the motor

Factorize B as $B = B^+ B^-$ where

- B^+ is monic and contains all the zeros to cancel
- B^- has the remaining zeros of B

Since $B = K(z-b)$

$$B^+ = z - b \qquad B^- = K$$

and

$$B_m^{'} = \frac{B_m}{K} = \frac{z\left(1 + p_1 + p_2\right)}{K}$$

Causality condition for the observer polynomial:

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^+ - 1 + \lambda = 2 \times 2 - 2 - 1 - 1 + 0 = 0$$

We select

$$A_o(z) = 1$$

Degree of the unknown polynomials in the Diophantine equation

$$\partial S < \lambda + \partial A = 0 + 2$$

Select $S(z)$ as generic polynomial of degree 1

$$S(z) = s_0 z + s_1$$

Remark that we may conclude that $s_0 = 0$.

With this choice

$$\partial R' = \partial A_o + \partial A_m - \partial A - \lambda = 0 + 2 - 2 - 0 = 0$$

The Diophantine equation is $(z-1)^{\lambda} AR' + B^{-}S = A_o A_m$

Introduce $R'(z) = r_0$ and $S(z) = s_0 z + s_1$ to get

$$(z-1)(z-a)r_0 + K(s_0 z + s_1) = z^2 + p_1 z + p_2$$

Equate the coefficients of the corresponding powers of z to get the system of algebraic equations

$$r_0 = 1$$
$$-(1+a)r_0 + Ks_0 = p_1$$
$$ar_0 + Ks_1 = p_2$$

with solution

$$r_0 = 1$$
$$s_0 = \frac{1+a+p_1}{K}$$
$$s_1 = \frac{p_2 - a}{K}$$

Furthermore

$$R = B^{+}R'(z-1)^{\lambda} = z - b$$

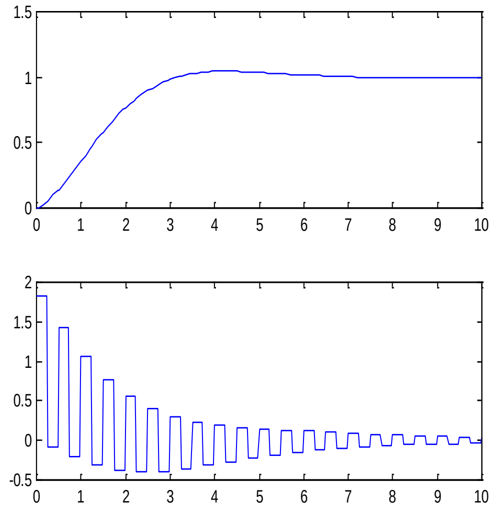
$$T(z) = A_o(z)B_m^{'}(z) = \frac{z(1+p_1+p_2)}{K} = t_0 z$$

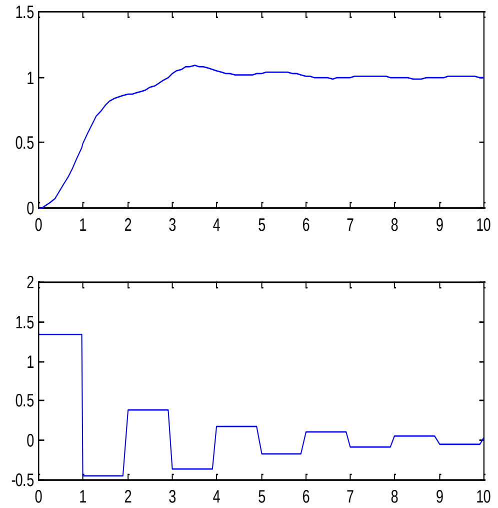
The control law is thus

$$u(k) = t_0 r(k) - s_0 y(k) - s_1 y(k-1) + bu(k-1)$$

Results with $\zeta = 0.7, \omega_n = 1$:

h=0.25





h=1

Due to the cancelation of a negative zero there is a high frequency oscillation in the control. Observe that when the sampling interval is reduced the amplitude of the control increases.

Specifications (without zero cancelation)

The desired transfer function is now:

$$H_m(z) = \frac{B_m(z)}{A_m(z)} = \frac{z-b}{z^2 + p_1 z + p_2} \cdot \frac{1+p_1+p_2}{1-b}$$

Since the process zero is a zero of the desired transfer function, it is not canceled.

$$A_m(z) = z^2 + p_1 z + p_2$$

$$B_m(z) = \frac{1+p_1+p_2}{1-b}\left(z-b\right)$$

Pole placement design without zero cancelation

Factorize $B(z)$ as

$$B^{+}=1 \qquad\qquad B^{-}=K(z-b)$$

Therefore

$$B_m'=\frac{B_m}{B^{-}}=\frac{1+p_1+p_2}{1-b}.(z-b).\frac{1}{K(z-b)}=\frac{1+p_1+p_2}{K(1-b)}$$

Observer polynomial A_o:

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^+ - 1 + \lambda = 2 \times 2 - 2 - 0 - 1 + 0 = 1$$

We choose

$$A_o(z) = z$$

It is convenient to select a monic polynomial.

Convém escolher o polinómio do observador mónico. If A, A_m, B^+ and A_o are monic, then R' will always be monic (i. e. $r_0 = 1$), a fact that reduces the number of unknowns.

Degree of the unknowns in the Diophantine equationa

$$\partial S < \lambda + \partial A = 0 + 2$$

Assume $S(z)$ to be a generic polynomial of degree 1

$$S(z) = s_0 z + s_1$$

With this choice

$$\partial R' = \partial A_o + \partial A_m - \partial A - \lambda = 1 + 2 - 2 - 0 = 1$$

Hence:

$$R = B^+ R' \left(z-1\right)^{\lambda} = R' = z + r_1$$

The Diophantine equation is $(z-1)^{\lambda} AR' + B^{-}S = A_o A_m$ or

$$(z-1)(z-a)(z+r_1)+K(z-b)(s_0 z+s_1)=z^3+p_1 z^2+p_2 z \qquad (*)$$

Make $z=b$ in (*), to get

$$r_1=-b+\frac{b\left(b^2+p_1 b+p_2\right)}{(b-1)(b-a)}$$

$$(z-1)(z-a)(z+r_1)+K(z-b)(s_0 z+s_1)=z^3+p_1 z^2+p_2 z$$

Making $z=1$ and $z=a$ we get the following system of algebraic equations verified by s_0 and s_1:

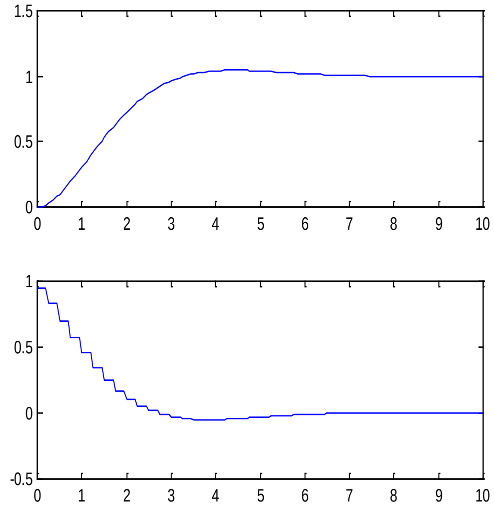
$$K(1-b)(s_0+s_1)=1+p_1+p_2$$

$$K(a-b)(s_0 a+s_1)=a^3+p_1 a^2+p_2 a$$

Finally:

$$T(z)=A_o B_m' = z\frac{1+p_1+p_2}{K(1-b)}=t_0 z$$

Results with $\zeta = 0.7, \omega_n = 1$:



The fast oscillation of the manipulated variable has been eliminated.

The structure of the controller is the same (with respect to the design in which the zero is canmcelled) , but the gains are different.

Exercice on pole placement design with polynomial methods

Given the open-loop digital transfer function

$$\frac{B(z)}{A(z)} = \frac{0.4z + 0.3}{z^2 - 1.6z + 0.65}$$

Find a causal controller that satisfies:

- Static gain of the controlled system = 1
- Minimum degree of the observer polynomial with all the roots at the origin
- Cancelation of the process zero
- No integral action
- Desired closed-loop characteristic polynomial: $A_m(z) = z^2 - 0.7z + 0.25$

$$B = B^+ B^- \qquad B^+ \text{ monic}$$

$$B^+ = z + \frac{0.3}{0.4} = z + 0.75 \qquad\qquad B^- = 0.4$$

$$B_m = B^- B'_m = 0.4 \times \frac{A_m(1)}{0.4} z = 0.4 \times \frac{1 - 0.7 + 0.25}{0.4} z = \underbrace{0.4}_{B^-} \times \underbrace{1.375z}_{B'_m}$$

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^+ - 1 + \lambda = 2 \times 2 - 2 - 1 - 1 + 0 = 0$$

$$A_o(z) = 1$$

$$T = B'_m A_o = 1.375z$$

Since integral action is not forced , $\lambda = 0$ and

$$\partial S < \partial A \qquad \partial R' = \partial A_o + \partial A_m - \partial A$$

$$\partial S < 2 \qquad \partial R' = 0 + 2 - 2 = 0$$

$$S(z) = s_0 z + s_1 \qquad R'(z) = 1$$

$$AR' + B^- S = A_o A_m$$

$$\left[z^2 - 1.6z + 0.65\right] \times 1 + 0.4 \times \left[s_0 z + s_1\right] = z^2 - 0.7z + 0.25$$

$$-1.6z + 0.4 s_0 z = -0.7z \qquad s_0 = 2.25$$

$$0.65 + 0.4 s_1 = 0.25 \qquad s_1 = -1$$

$$R = B^{+}R' \qquad\qquad R = z + 0.75$$

Control law:

$$Ru = Tr - Sy$$

$$(q + 0.75)u(t) = 1.375qr(t) - (2.25q - 1)y(t)$$

Or:

$$u(t) = -2.25y(t) + y(t-1) - 0.75u(t-1) + 1.375r(t)$$

Another design: Inclusion of an integrator $\lambda = 1$

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^+ - 1 + \lambda = 2 \times 2 - 2 - 1 - 1 + 1 = 1$$

$$A_o(z) = z$$

$$\partial S < 1 + \partial A$$

$$\partial R' = \partial A_o + \partial A_m - \partial A - \lambda$$

$$\partial S < 3$$

$$\partial R' = 1 + 2 - 2 - 1 = 0$$

$$S(z) = s_0 z^2 + s_1 z + s_2$$

$$R'(z) = 1$$

$$(z-1)AR' + B^- S = A_o A_m$$

$$(z-1)\left(z^2 - 1.6z + 0.65\right) \times 1 + 0.4 \times \left(s_0 z^2 + s_1 z + s_2\right) = z^3 - 0.7z^2 + 0.25z$$

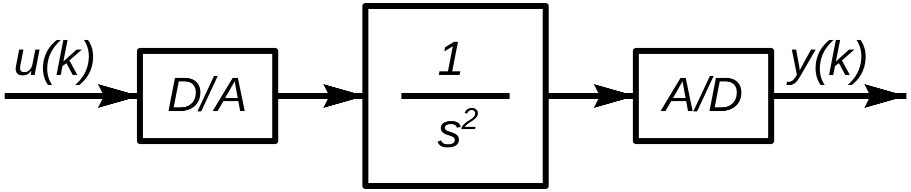
Solution by the method of unknown multipliers

$$s_0 = 4.75$$

$$s_1 = -5$$

$$s_2 = 1.625$$

Example: Control of the double integrator



Discrete equivalent: $$H(z) = \frac{h^2}{2} \cdot \frac{z+1}{(z-1)^2}$$

Design a controller that satisfies the following specifications:

- No zero cancelation; No extra integrator; minimum delay
- Desired characteristic polynomial: $A_m(z) = z^2 + a_1 z + a_2$
- Observer polynomial with poles at a_o

Desired characteristic polynomial:

$$A_m(z) = z^2 + a_1 z + a_2$$

$$a_1 = -2e^{-\varsigma\omega_n h} \cos\left(\omega_n h\sqrt{1-\varsigma^2}\right)$$

$$a_2 = e^{-2\varsigma\omega_n h}$$

This choice emulates the continuous time characteristic polynomial:

$$s^2 + 2\varsigma\omega_n z + \omega_n^2$$

$$H(z)=\frac{h^2}{2}\cdot\frac{z+1}{(z-1)^2} \qquad B(z)=\frac{h^2}{2}(z+1) \qquad A(z)=(z-1)^2=z^2-2z+1$$

No zero cancellation:

$$B^+(z)=1 \qquad B^-(z)=\frac{h^2}{2}(z+1)$$

Minimum delay and unit static gain:

$$B_m(z)=\frac{1+a_1+a_2}{2}(z+1)$$

$$B'_m(z)=\frac{B_m(z)}{B^-(z)}=\frac{1+a_1+a_2}{h^2}$$

Observer polynomial:

$$\partial A_o \geq 2\partial A - \partial A_m - \partial B^+ + \lambda - 1 = 2\times 2 - 2 - 0 + 0 - 1 = 1$$

$$A_o(z) = z - a_o$$

Controller $T(z)$ polynomial:

$$T(z) = B'_m(z)A_o(z) = \frac{1+a_1+a_2}{h^2}(z-a_o)$$

or:

$$T(z) = t_0 z + t_1 \quad \text{with} \quad t_0 = \frac{1+a_1+a_2}{h^2}z \quad \text{and} \quad t_1 = \frac{1+a_1+a_2}{h^2}a_o$$

Since $\lambda = 0$, the Diophantine equation is

$$AR' + B^- S = A_o A_m$$

Degree conditions on polynomials S and R':

$$\partial S < \lambda + \partial A = 0 + 2 \quad \text{from which:} \quad S(z) = s_0 z + s_1$$

$$\partial R' = \partial A_o + \partial A_m - \partial A - \lambda = 1 + 2 - 2 - 0 = 1 \quad \text{and hence} \quad R'(z) = z + r_1$$

Recall that, when A, A_m, A_o and B^+ are monic, then, R and R' are also monic and hence $r_0 = 1$.

$$R'(z) = z + r_1 \qquad R(z) = (z-1)^{\lambda} B^{+}(z) R'(z) = z + r_1$$

$$S(z) = s_0 z + s_1 \qquad T(z) = t_0 z + t_1$$

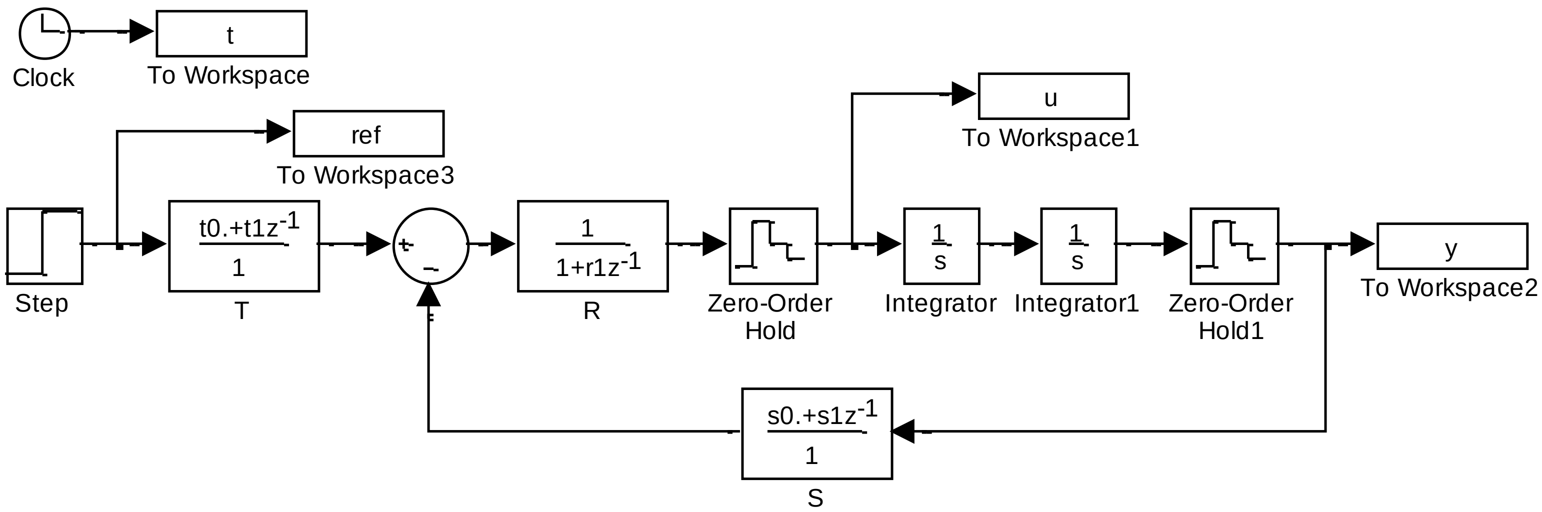
Controller structure:

$$(q + r_1)u(k) = (t_0 q + t_1)r(k) - (s_0 q + s_1)y(k)$$

In terms of the delay operator q^{-1} and solving with respect to $u(k)$:

$$u(k) = \frac{1}{1 + r_1 q^{-1}} \cdot \left[(t_0 + t_1 q^{-1}) r(k) - (s_0 + s_1 q^{-1}) y(k) \right]$$

$$u(k) = \frac{1}{1 + r_1 q^{-1}} \cdot \left[(t_0 + t_1 q^{-1}) r(k) - (s_0 + s_1 q^{-1}) y(k) \right]$$



Solving the Diophantine equation: $AR' + B^{-}S = A_o A_m$

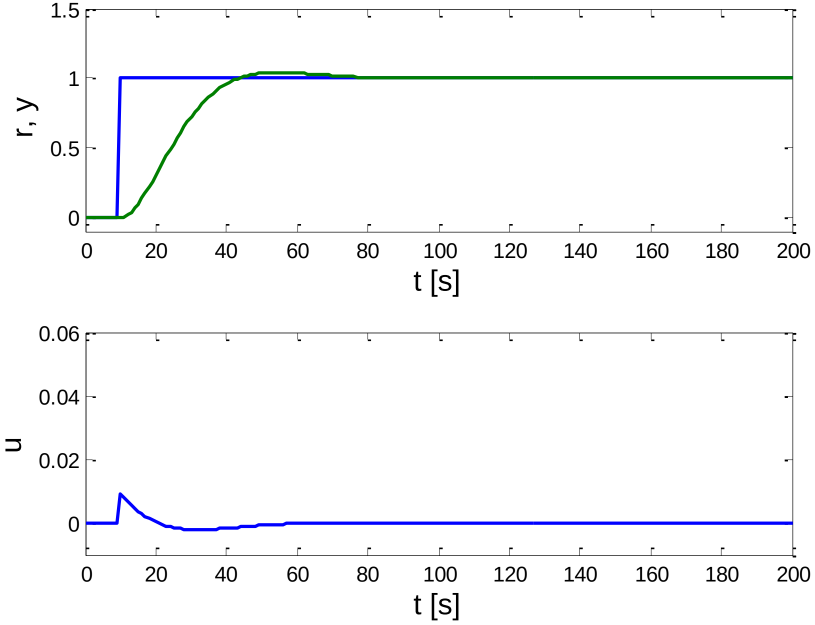
$$(z-1)^2(z+r_1) + \frac{h^2}{2}(z+1)(s_0 z + s_1) = (s-a_o)(z^2 + a_1 z + a_2)$$

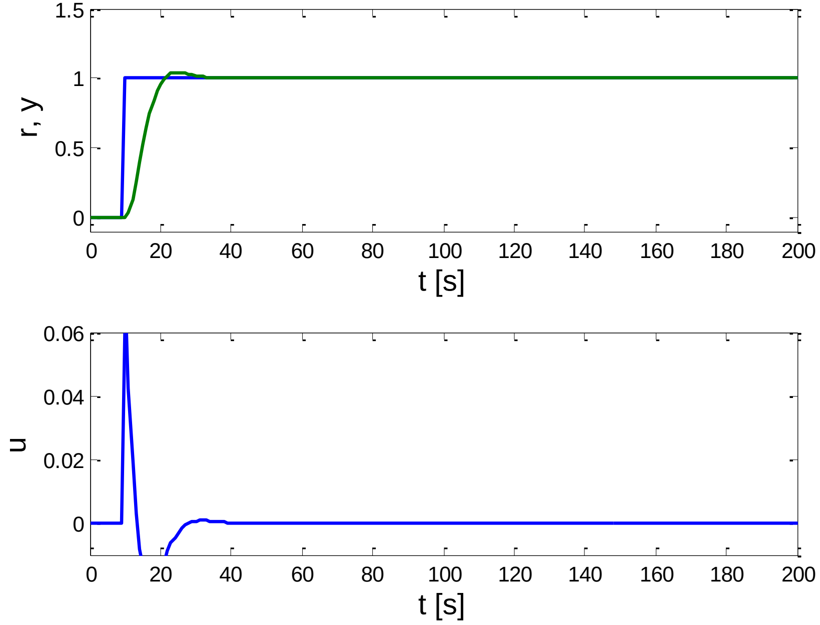
$$z^3 + \left(-2 + r_1 + \frac{h^2}{2}s_0\right)z^2 + \left(1 - 2r_1 + \frac{h^2}{2}\left(s_0 + s_1\right)\right)z + \left(r_1 + \frac{h^2}{2}s_1\right) =$$

$$= z^3 + \left(a_1 - a_o\right)z^2 + \left(a_2 - a_o a_1\right)z - a_o a_2$$

$$\begin{bmatrix} \frac{h^2}{2} & 0 & 1 \\ \frac{h^2}{2} & \frac{h^2}{2} & -2 \\ 0 & \frac{h^2}{2} & 1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} a_1 - a_o + 2 \\ a_2 - a_o a_1 - 1 \\ -a_o a_2 \end{bmatrix}$$

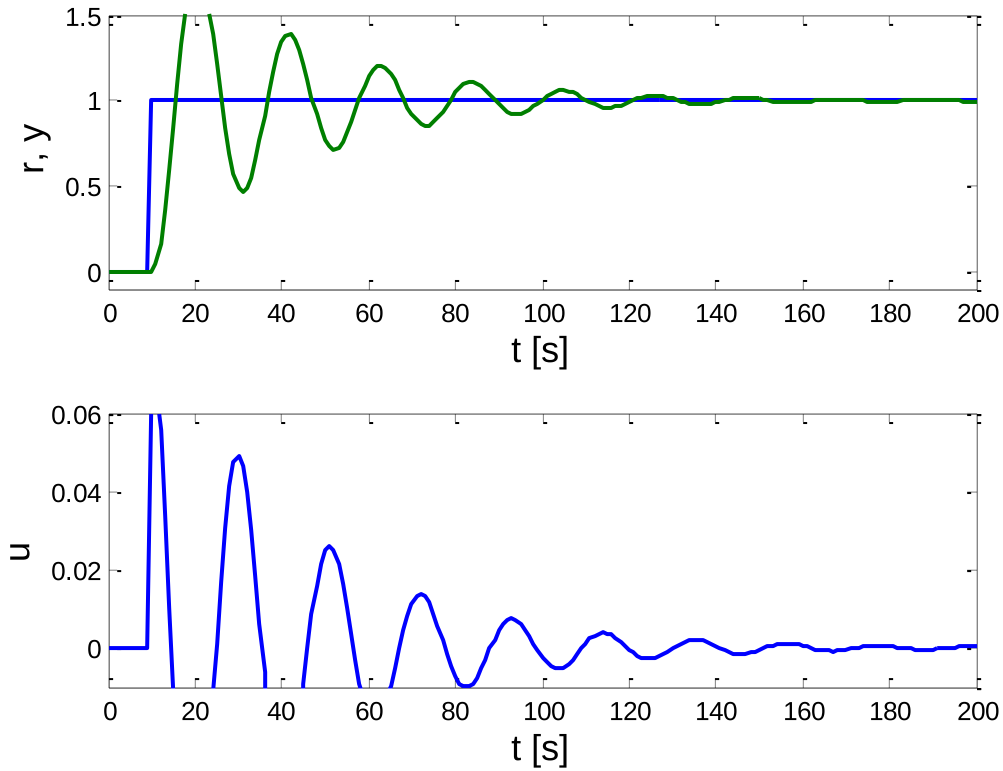
$$\varsigma = 0.707, \omega_n = 0.1 \qquad \varsigma = 0.707, \omega_n = 0.3$$



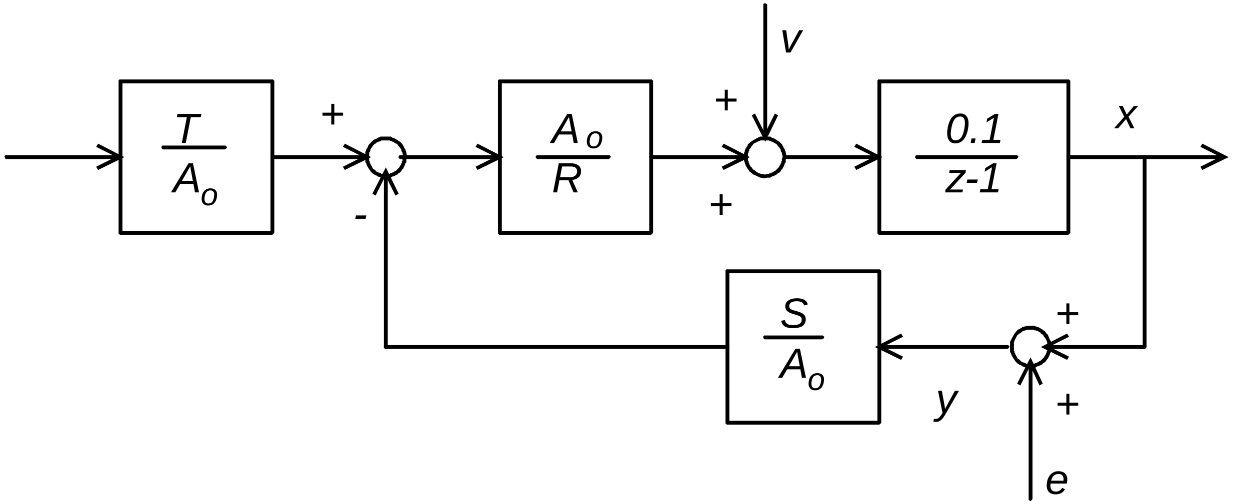


Increasing the specified bandwidth the response becomes faster, but the amplitude of the control signal increases.

$$\varsigma = 0.1, \ \omega_n = 0.3$$



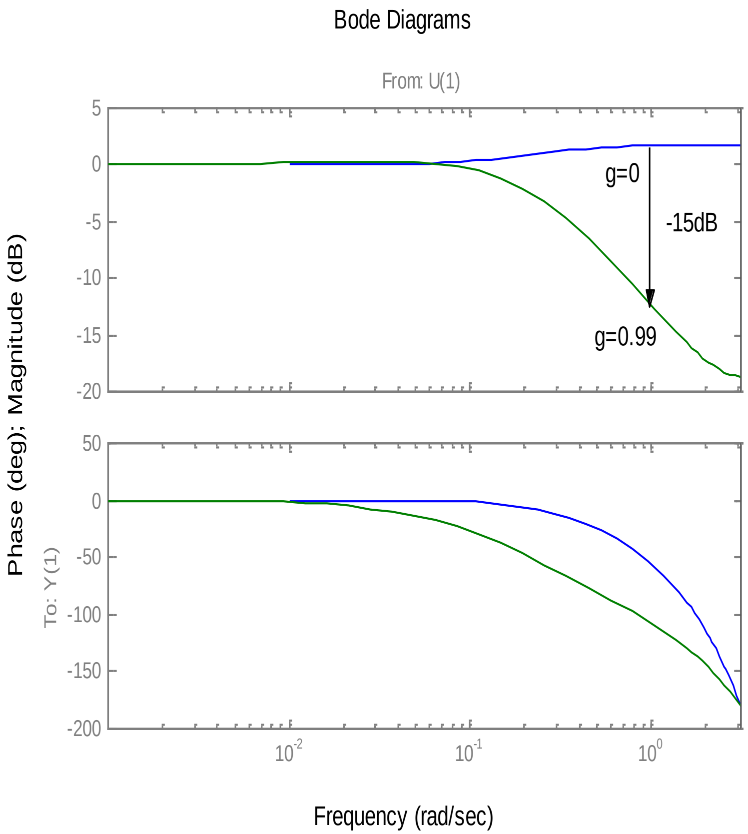
Choice of tghe observer polynomial

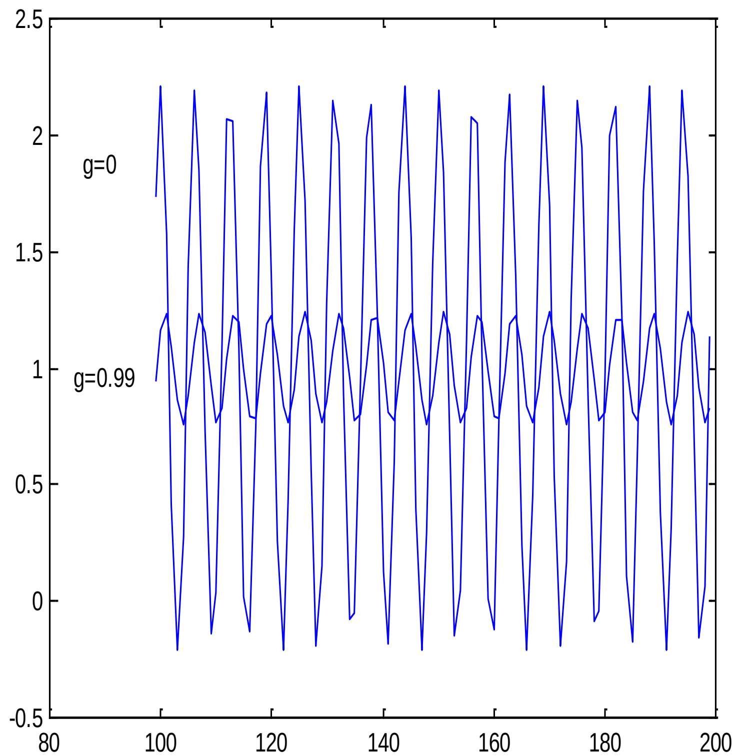


Process: $H(z) = \dfrac{0.1}{z-1}$ Specification: $H_m(z) = \dfrac{0.2}{z-0.8}$ $\lambda = 1, \quad A_o(z) = z - g$

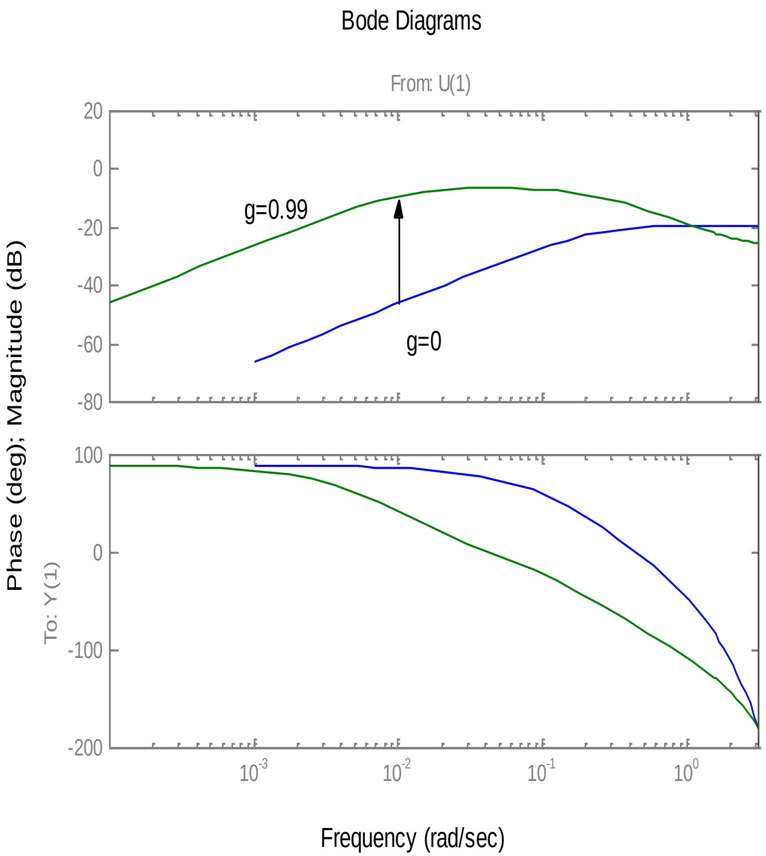
$$x = \frac{0.2}{z-0.8} r + \frac{0.1(z-1)}{(z-g)(z-0.8)} v - \frac{(1.2-g)z - 1 + 0.8g}{(z-g)(z-0.8)} e$$

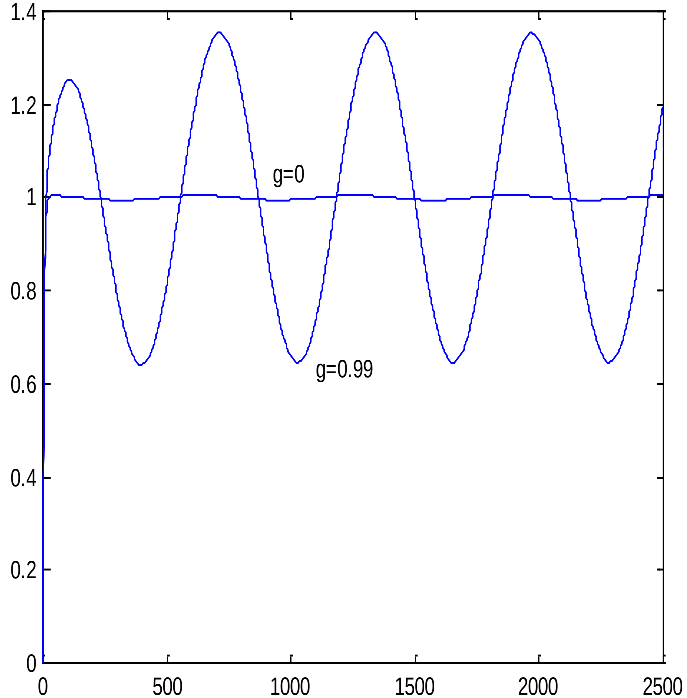
Efect of the observation noise, *e*





Efect of the input disturbances, *v*



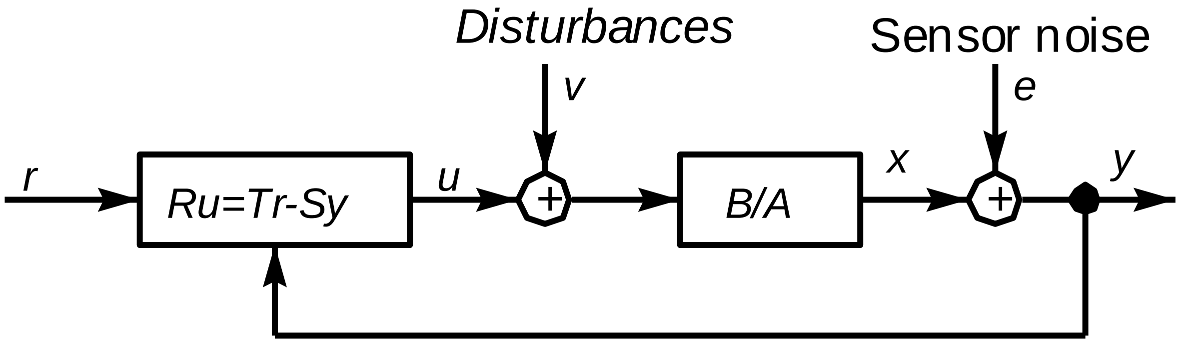


The choice of the observer polynomial implies a trade-off:

Making the observer slower (moving the observer pole away from the origin, towards 1):

- Reduces the sensitivity to sensor noise;
- Increases the sensitivity to the load (disturbance) in the low frequency.

High frequency noise and low frequency disturbances



$$x = \frac{BT}{AR + BS} r + \frac{BR}{AR + BS} v - \frac{BS}{AR + BS} e$$

How to select R and S so as to decrease the sensitivity to high frequency noise and low frequency disturbances?

Rejecting constant disturbances

$$x = \frac{BT}{AR+BS} r + \frac{BR}{AR+BS} v - \frac{BS}{AR+BS} e$$

The rejection of constant disturbances v can be achieved by the inclusion of an integrator in the controller.

For this sake, polynomial R must be of the form $R = R'(z-1)$.

This choice creates a zero for $z = 1$ (corresponding to the analog frequency zero) that blocks the disturbances in the transfer function

$$\frac{BR}{AR+BS}$$

Rejecting high frequency noise

$$x = \frac{BT}{AR+BS} r + \frac{BR}{AR+BS} v - \frac{BS}{AR+BS} e$$

In a similar way, to block noise at frequency ω, we must create a zero in polynomial S at the corresponding digital frequency $z = e^{j\omega h}$.

The highest frequency "seen" by a sampled system is the Nyquist frequency, that corresponds to half the sampling frequency. This corresponds to a zero in

$$z = e^{j\omega_N h} = e^{j\frac{2\pi}{2h}h} e^{j\pi} = -1$$

Conclusion: Forcing the polynomial S to be of the form

$$S = S'(z+1)$$

Has the effect of rejecting high frequency noise.

For this sake, the design equations are the same, with the degree of the observer polynomial increased by 1 unit.

4.C. Linear prediction and minimum variance control

Objective: *Design linear controllers in discrete time for systems with stochastic disturbances. Preparing the set for predictive and adaptive control.*

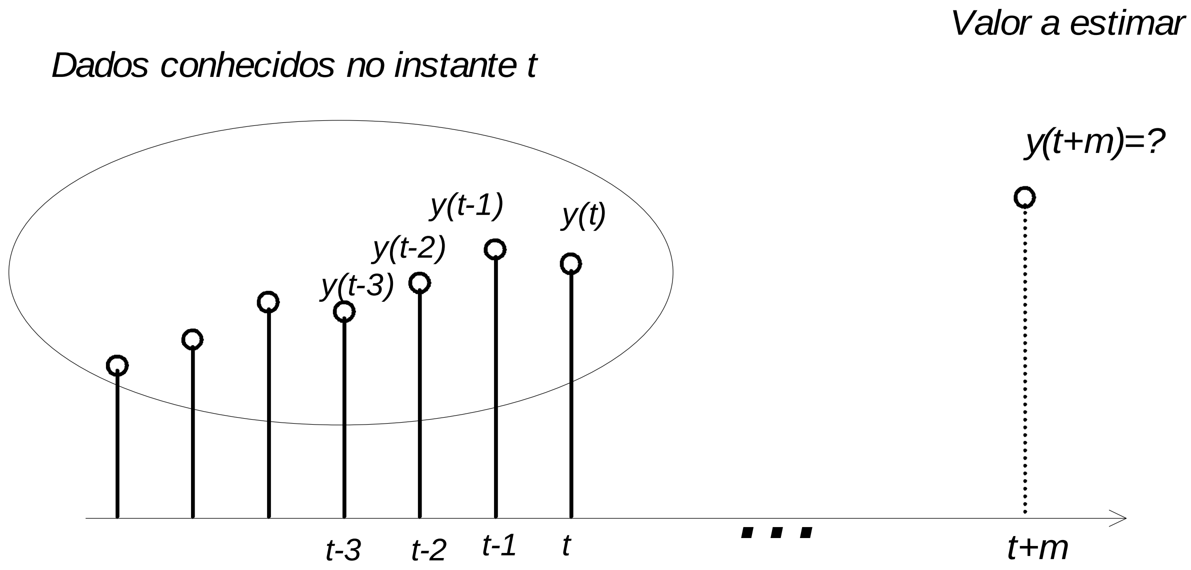
Referência: AW

The prediction problem

We observe a time series of numbers:

$$y(0),\, y(2),\, y(3),\, ...,\, y(t)$$

The objective is to estimate the value m steps ahead.



O problema da predição tem várias motivações. A mais óbvia (e porventura mais atraente…) é constituída pelas aplicações em Economia e Gestão. Um outro exemplo diz respeito às aplicações em Controlo de sistemas sujeitos a perturbações aleatórias. Este problema foi tratado pela primeira vez por Kolmogorov e Wiener.

A. Kolmogorov (1903-1987)

Personalidades muito diferentes, ambos foram não só notáveis matemáticos que dedicaram a sua atenção aos sistemas estocásticos, mas também tiveram um amplo leque de interesses em áreas diversificadas.

N. Wiener (1894-1964)

The solution implies:

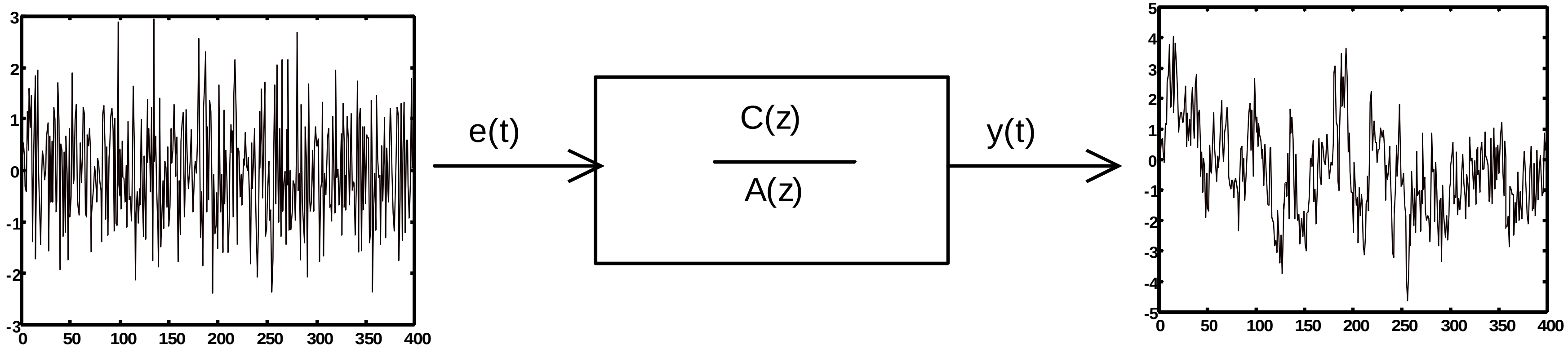
- A data generation model;
- A criteria to compare different possible predictions.

We are going to use:

- Data generation model: White noise filtered by a linear system
- Criteria: Quadratic cost

The ARMA model

The stationary signal $y(t)$ is modelled as generated by filtering white noise by a linear system $C(z)/A(z)$



Thuis model is called ARMA (Auto-Regresssive, Moving Average)

The ARMA model in the delay operator

Reciprocal polynomial

$$A^*(q^{-1}) = q^{-n} A(q) \qquad\qquad A^*(q^{-1}) = 1 + \sum_{i=1}^{n} a_i q^{-i}$$

Multiply the ARMA model by q^{-n}:

$$\underbrace{q^{-n} A(q)}_{A^*(q^{-1})} y(t) = \underbrace{q^{-n} C(q)}_{C^*(q^{-1})} e(t)$$

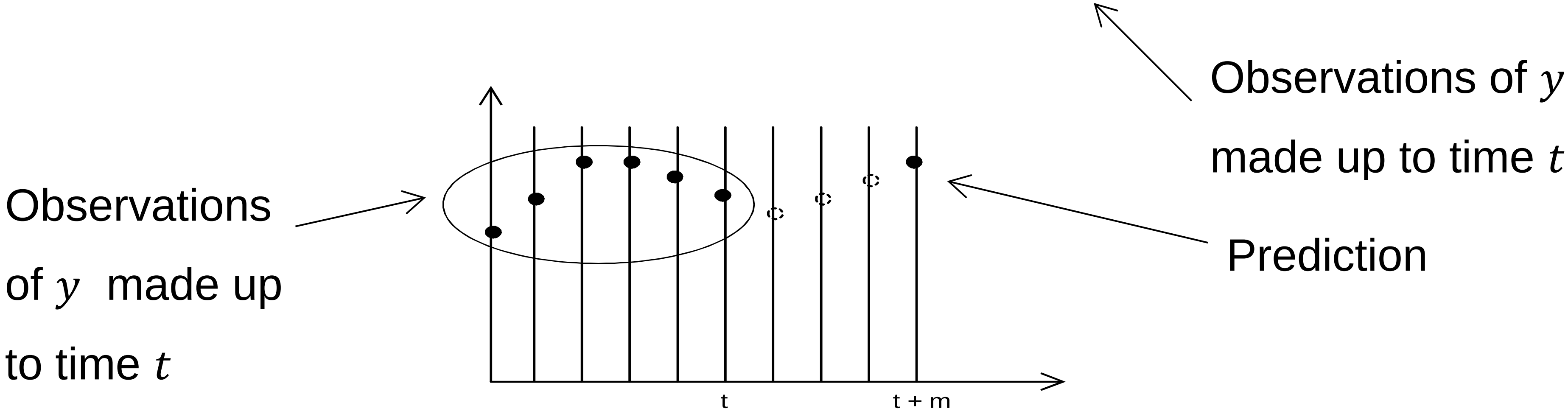
The ARMA model in the unit delay operator is

$$A^*(q^{-1})\, y(t) = C^*(q^{-1})\, e(t)$$

The prediction problem with mean square error

Findr $\hat{y}(t+m|t), m \geq 1$ that depends only on the observations of y made up to time t, that minimizes the stationary variance of the prediction error:

$$E\left[(y(t+m) - \hat{y}(t+m \mid t))^2 \mid O^t\right]$$



Solution of the prediction problem

$$y(t+m) = \frac{C^*(q^{-1})}{A^*(q^{-1})} e(t+m)$$

Expand C* / A* by long division

$$\frac{C^*(q^{-1})}{A^*(q^{-1})} = \underbrace{1 + f_1 q^{-1} + \ldots + f_{m-1} q^{-m+1}}_{F_m^*(q^{-1})} + \underbrace{f_m q^{-m} + f_{m-1} q^{-m-1} + \ldots}_{q^{-m} \frac{G_m^*(q^{-1})}{A^*(q^{-1})}}$$

$$y(t+m) = \underbrace{e(t+m) + f_1 e(t+m-1) + \ldots + f_{m-1} e(t+1)}_{\varepsilon_m(t) \text{ Depends only on future noise values}} + \underbrace{f_m e(t) + f_{m+1} e(t-1) + \ldots}_{\text{Depends on noise up to time } t}$$

The equation

$$\frac{C^*(q^{-1})}{A^*(q^{-1})} = F_m^*(q^{-1}) + q^{-m} \frac{G_m^*(q^{-1})}{A^*(q^{-1})}$$

Provides the separation of the terms that make $y(t+m)$ in two parts:

- The terms that occur up to time t
- The parcel $\varepsilon_m(t)$ that depends on the terms $t+1$ and on to the future

$$y(t+m) = F_m^*(q^{-1}) + \frac{G_m^*(q^{-1})}{A^*(q^{-1})} e(t) = \varepsilon_m(t) + \frac{G_m^*(q^{-1})}{A^*(q^{-1})} e(t)$$

The expectation of the product of this two parcels is zero (why?).

To obtain the optimal control consider the cost:

$$J = E\left[(y(t+m) - \hat{y}(t+m|t))^2 | O^t \right]$$

Use the model, in the form of the expression presented before:

$$J = E\left[(\underbrace{\frac{G_m^*(q^{-1})}{A^*(q^{-1})} e(t) - \hat{y}(t+m|t)}_{A} + \underbrace{\varepsilon_m(t)}_{B})^2 | O^t \right]$$

Parcels *A* and *B* are statistically independent and their expectation is zero, and hence, the expectation of their product is zero. Hence

$$J = E\left[\left(\frac{G_m^*(q^{-1})}{A^*(q^{-1})} e(t) - \hat{y}(t+m|t) \right)^2 |O^t \right] + E\left[\varepsilon_m^2(t) | O^t \right]$$

$$J = E\left[\left(\frac{G_m^*(q^{-1})}{A^*(q^{-1})}e(t) - \hat{y}(t+m|t)\right)^2 \middle| O^t\right] + E\left[\varepsilon_m^2(t)|O^t\right]$$

Since the expectation is conditioned, we get (why?):

$$J = \left(\frac{G_m^*(q^{-1})}{A^*(q^{-1})}e(t) - \hat{y}(t+m|t)\right)^2 + E\left[\varepsilon_m^2(t)\right]$$

This term is minimum when it vanishes

This term does not depend on $\hat{y}(t+m|t)$

The optimal predictor is thus

$$\hat{y}(t+m|t) = \frac{G_m^*(q^{-1})}{A^*(q^{-1})}e(t)$$

The optimal predictor is therefore given by

$$\hat{y}(t+m|t)=\frac{G_m^*(q^{-1})}{A^*(q^{-1})}e(t)$$

This expression is not useful because it expresses the prediction on e.

However, since A and C have the same degree

$$e(t)=\frac{A^*(t)}{C^*(t)}y(t)$$

And the optimal predictor in stochastic steady state is

$$\hat{y}(t+m|t)=\frac{G_m^*(q^{-1})}{C^*(q^{-1})}y(t)$$

Mean square error predictor for ARMA signals

Given the ARMA model

$$y(t) = \frac{C^*(q^{-1})}{A^*(q^{-1})} e(t) \quad \text{com} \quad E\left[e^2(t)\right] = \sigma_e^2$$

the optimal m-steps ahead predictor is given by

$$\hat{y}(t+m|t) = \frac{G_m^*(q^{-1})}{C^*(q^{-1})} y(t)$$

where

$$\frac{C^*(q^{-1})}{A^*(q^{-1})} = F_m^*(q^{-1}) + q^{-m} \frac{G_m^*(q^{-1})}{A^*(q^{-1})}$$

with $F_m(q^{-1})$ a polynomial of degree $m-1$.

Variance of the prediction error

The variance of the prediction error is

$$E\left[\varepsilon_m^2(t)\right] = E\left[\left(e(t+m) + f_1 e(t+m-1) + \ldots + f_{m-1} e(t+1)\right)^2\right]$$

Since $\{e\}$ is a sequence of zero mean, independent random, the cross terms vanish and

$$E\left[\varepsilon_m^2(t)\right] = (1 + f_1^{\,2} + f_2^{\,2} + \ldots + f_{m-1}^{\,2})\sigma_e^2$$

Example: Linear prediction

Compute the 1, 2 and 3 steps ahead predictions for the signal modelled by

$$y(k) = \frac{q^2 - 1.4q + 0.5}{q^2 - 1.2q + 0.4} e(k)$$

where $\{e\}$ is a sequence of identically distributed, independent, random variables, with zero mean and unit variance. Compute the standard deviation of the estimates.

$$\frac{C^*(q^{-1})}{A^*(q^{-1})} = \frac{1 - 1.4q^{-1} + 0.5q^{-2}}{1 - 1.2q^{-1} + 0.4q^{-2}}$$

$$\begin{array}{llllll}
1 & -1.4q^{-1} & +0.5q^{-2} & & & \\
1 & -1.2q^{-1} & +0.4q^{-2} & & & \\ \hline
 & -0.2q^{-1} & +0.1q^{-2} & & & \\
 & -0.2q^{-1} & +0.24q^{-2} & -0.08q^{-3} & & \\ \hline
 & & -0.14q^{-2} & +0.08q^{-3} & & \\
 & & -0.14q^{-2} & +0.168q^{-3} & -0.056q^{-4} & \\ \hline
 & & & -0.088q^{-3} & +0.056q^{-4} &
\end{array}
\qquad
\begin{array}{|l}
1 - 1.2q^{-1} + 0.4q^{-2} \\ \hline
1 - 0.2q^{-1} - 0.14q^{-2}
\end{array}$$

$$
\begin{array}{l|l}
-0.16q^{-3} \quad +0.056q^{-4} & 1-1.2q^{-1}+0.4q^{-2} \\
\hline
1 \quad -1.2q^{-1} \quad +0.4q^{-2} & 1-0.2q^{-1}-0.14q^{-2}
\end{array}
$$

$$\boxed{-0.2q^{-1} \quad +0.1q^{-2}} \qquad q^{-1}G_1^*(q^{-1})$$

$$-0.2q^{-1} \quad +0.24q^{-2} \quad -0.08q^{-3}$$

$$\boxed{-0.14q^{-2} \quad +0.08q^{-3}} \qquad q^{-2}G_2^*(q^{-1})$$

$$-0.14q^{-2} \quad +0.168q^{-3} \quad -0.056q^{-4}$$

$$\boxed{-0.088q^{-3}+0.056q^{-4}} \qquad q^{-3}G_3^*(q^{-1})$$

1 step ahead predictor:

$$\hat{y}(t+1|t) = \frac{-0.2 + 0.1q^{-1}}{1 - 1.4q^{-1} + 0.5q^{-2}}\, y(t)$$

$$\hat{y}(t+1|t) = 1.4\hat{y}(t|t-1) - 0.5\hat{y}(t-1|t-2) - 0.2y(t) + 0.1y(t-1)$$

Variance of the 1-step ahead prediction error

$$\sigma_1 = 1$$

2-steps ahead predictor:

$$\hat{y}(t+2|t) = \frac{-0.14 + 0.08q^{-1}}{1 - 1.4q^{-1} + 0.5q^{-2}} y(t)$$

$$\hat{y}(t+2|t) = 1.4\hat{y}(t+1|t-1) - 0.5\hat{y}(t|t-2) - 0.14y(t) + 0.08y(t-1)$$

Variance of the 2-steps ahead prediction error

$$\sigma_2 = 1 + (-0.2)^2 = 1.04$$

3-steps ahead predictor:

$$\hat{y}(t+3|t) = \frac{-0.088 + 0.056q^{-1}}{1 - 1.4q^{-1} + 0.5q^{-2}}\, y(t)$$

$$\hat{y}(t+3|t) = 1.4\hat{y}(t+2|t-1) - 0.5\hat{y}(t+1|t-2) - 0.088y(t) + 0.056y(t-1)$$

Variance of the 3-steps ahead prediction error

$$\sigma_3 = 1 + (-0.2)^2 + (-0.14)^2 = 1.06$$

Predictors for ARX models

$$A(q)\, y(t) = B(q)\, u(t) + q^{n}\, e(t)$$

$\partial A - \partial B = 1$ "process delay"

$$A^*(q^{-1})\, y(t) = B^*(q^{-1})\, u(t-1) + e(t)$$

$$A^*(q^{-1}) = 1 + \sum_{i=1}^{n} a_i\, q^{-i}$$

$$B^*(q^{-1}) = \sum_{i=0}^{m} b_i\, q^{-i}$$

Let

$$F_j^*(q^{-1}) = 1 + f_1\, q^{-1} + ... + f_j\, q^{-j}$$

$$G_j^*(q^{-1}) = g_0^j + g_1^j\, q^{-1} + ... + g_{n-1}^j\, q^{-n+1}$$

that verify

$$1 = F_j^*(q^{-1})\, A^*(q^{-1}) + q^{-j-1}\, G_j^*(q^{-1})$$

or

$$F_j^*(q^{-1})\, A^*(q^{-1}) = 1 - q^{-j-1}\, G_j^*(q^{-1})$$

$$A^*(q^{-1})\, y(t + j + 1) = B^*(q^{-1})\, u(t + j) + e(t + j + 1)$$

Multiply by $F_j^*(q^{-1})$

$$\underbrace{F_j^*(q^{-1})\, A^*(q^{-1})}_{1 - q^{-j-1}\, G_j^*(q^{-1})}\, y(t + j + 1) = F_j^*(q^{-1})\, B^*(q^{-1})\, u(t + j) + F_j^*(q^{-1})\, e(t + j)$$

$$y(t + j + 1) = \underbrace{G_j^*(q^{-1})\, y(t) + F_j^*(q^{-1})\, B^*(q^{-1})}_{\hat{y}(t + j + 1 \mid t)}\, u(t + j) + F_j^*(q^{-1})\, e(t + j)$$

$$\hat{y}(t + j + 1 \mid t) = G_j^*(q^{-1})\, y(t) + F_j^*(q^{-1})\, B^*(q^{-1})\, u(t + j)$$

Break F_j^* B^* such as to separate the "free" control samples (that occur between t and $t + j - 1$) from the ones that were already defined in the past

$$F_j^*(q^{-1})\, B^*(q^{-1}) = W_j^*(q^{-1}) + q^{-j-1}\, H_j^*(q^{-1})$$

$$W_j^*(q^{-1}) = w_1 + w_2\, q^{-1} + \ldots + w_{j+1}\, q^{-j}$$

$$H_j^*(q^{-1}) = h_0^j + h_1^j\, q^{-1} + \ldots + h_{m-1}^j\, q^{-m+1}$$

$$\hat{y}(t + j + 1 \mid t) = G_j^*(q^{-1})\, y(t) + W_j^*(q^{-1})\, u(t + j) + H_j^*(q^{-1})\, u(t - 1)$$

$$\hat{y}(t + j + 1 \mid t) = W_j^*(q^{-1})\, u(t + j) + G_j^*(q^{-1})\, y(t) + H_j^*(q^{-1})\, u(t - 1)$$

Linear combination of u(t)...u(t+j)

Linear combination of y(t) y(t-1)...y(t-n+1)

Linear comb. of u(t-1)...u(t-m)

$$\hat{y}(t + j + 1) = \sum_{i=0}^{j} w_{i+1}\, u(t + j - i) + \pi'_{j+1}\, s(t)$$

$$s(t) = [y(t)\ y(t - 1)\ \ldots\ y(t - n + 1)\ u(t - 1)\ \ldots\ u(t - m)]'$$

1-step ahead predictor

$$\hat{y}(t+1) = w_1 u(t) + \pi_1^{'} s(t)$$

$$s(t) = \begin{bmatrix} y(t) & \cdots & y(t-n+1) & u(t-1) & \cdots & u(t-m) \end{bmatrix}$$

The output differs from the prediction by a term that is independent from data

$$y(t+1) = \hat{y}(t+1) + e(t+1)$$

2-steps ahead predictor

$$\hat{y}(t+1) = w_1 u(t+1) + w_2 u(t) + \pi_2^{'} s(t)$$

Minimum variance control of plants with stable inverse

Given the ARMAX plant

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

Assume

$$d = \partial A - \partial B \geq 1 \qquad \partial A = \partial B = n$$

And that the polynomial $B(z)$ has all its roots inside the unit circle (i. e. the plant to control has a stable inverse).

Minimum variance functional

The objective is, at each discrete time t, to find $u(t)$so as to minimize

$$J(u(t)) = E\left[y^2(t+d)|O^t\right]$$

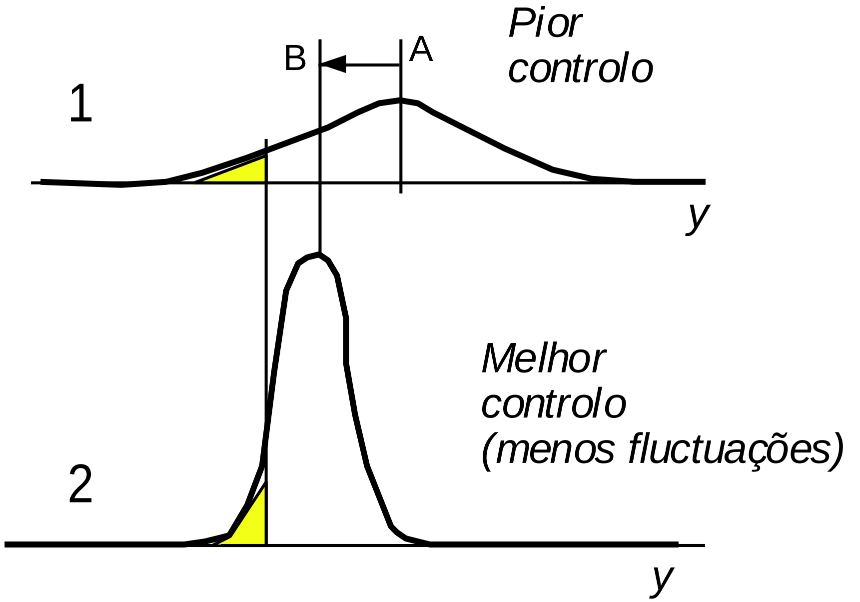
We want to keep the output y close to zero, despite stochastic disturbances.

At time t, in which we select $u(t)$, we can only influence $y(t+d)$ or the output in times subsequent to $t+d$.

Astrom, 1962

Peterka, 1962

Examplo: Paper manufacturing



Reducing the fluctuations of the paper thickness (process output) allows to decrease the set-point , and therefore save raw material.

Solution of the minimum variance control problem

In the delay operator, the plant is represented by the model

$$A^*(q^{-1})y(t+d) = B^*(q^{-1})u(t) + C^*(q^{-1})e(t+d)$$

Let $F_d^*(q^{-1}) = 1 + f_1 q^{-1} + f_2 q^{-2} + \ldots + f_{d-1} q^{-d+1}$

and $G_d^*(q^{-1})$ of degree $n-1$ such that

$$\frac{C^*(q^{-1})}{A^*(q^{-1})} = F_d^*(q^{-1}) + q^{-d}\,\frac{G_d^*(q^{-1})}{A^*(q^{-1})}$$

or $A^*(q^{-1})F_d^*(q^{-1}) = C^*(q^{-1}) - q^{-d} G_d^*(q^{-1})$

Multiply the model by $F_d^*(q^{-1})$:

$$A^*(q^{-1})F_d^*(q^{-1})y(t+d) = B^*(q^{-1})F_d^*(q^{-1})u(t) + C^*(q^{-1})F_d^*(q^{-1})e(t+d)$$

$$\boxed{A^*(q^{-1})F_d^*(q^{-1}) = C^*(q^{-1}) - q^{-d}G_d^*(q^{-1})}$$

We get

$$(C^*(q^{-1}) - q^{-d}G_d^*(q^{-1}))y(t+d) = B^*(q^{-1})F_d^*(q^{-1})u(t) + C^*(q^{-1})F_d^*(q^{-1})e(t+d)$$

or

$$y(t+d) = \frac{1}{C^*(q^{-1})}(G_d^*(q^{-1})y(t) + B^*(q^{-1})F_d^*(q^{-1})u(t)) + F_d^*(q^{-1})e(t+d)$$

We have expressed $y(t+d)$ as the sum of two uncorrelated terms (the mean of their product is zero), one that depends of all the data up to time t and the other that depends on what happens after $t+1$:

$$y(t+d) = \underbrace{\boxed{\frac{1}{C^*(q^{-1})}(G_d^*(q^{-1})y(t) + B^*(q^{-1})F_d^*(q^{-1})u(t))}}_{\alpha(u(t))} + \underbrace{\boxed{F_d^*(q^{-1})e(t+d)}}_{\beta}$$

α(u(t)), this term is

The predictor of *y(t+d)*

β

$$y(t+d) = \alpha(u(t)) + \beta$$

Consider again the minimum variaance cost, and replace this expresion on $y(t+d)$

$$J(u(t)) = E\left[y^2(t+d)|O^t\right]$$

$$y(t+d) = \alpha(u(t)) + \beta$$

We get

$$J(u(t)) = E\left[\left(\alpha(u(t)) + \beta\right)^2 |O^t\right] = E\left[\alpha^2(u(t))|O^t\right] + E\left[\beta^2|O^t\right]$$

$$J(u(t)) = \alpha^2(u(t)) + E\left[\beta^2\right]$$

$$J(u(t)) = \alpha^2(u(t)) + E\left[\beta^2\right]$$

The second term does not depend on $u(t)$. The second is always positive or zero. Itrs minimum value is zero, being attained for a value of $u(t)$ that satisfies

$$\alpha(u(t)) = 0$$

i. e., the optimal value of $u(t)$ verifies

$$G_d^*(q^{-1})y(t) + B^*(q^{-1})F_d^*(q^{-1})u_{VM}(t) = 0$$

The minimum variance control is therefore given by

$$\boxed{u_{VM}(t) = -\frac{G_d^*(q^{-1})}{B^*(q^{-1})F_d^*(q^{-1})}y(t) = -\frac{G_d(q)}{B(q)F_d(q)}y(t)}$$

$$J(u(t)) = \alpha^2(u(t)) + E\left[\beta^2\right]$$

Minimum variance control

makes this term vanish

The smallest value of J is given by

$$J(u_{VM}(t)) = E\left[y^2(t+d)|O^t\right] = E\left[\beta^2\right]$$

or

$$\boxed{E\left[y^2(t+d)|O^t\right] \geq \left[1 + f_1^2 + f_2^2 + \ldots + f_{d-1}^2\right]\sigma_e^2}$$

Minimum variance control: Example 1

Consider the process modelled by

$$A^*(q^{-1})y(t) = B^*(q^{-1})u(t-1) + C^*(q^{-1})e(t)$$

where

$$A^*(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2} \qquad B^*(q^{-1}) = 1 + 0.5q^{-1}$$

$$C^*(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2}$$

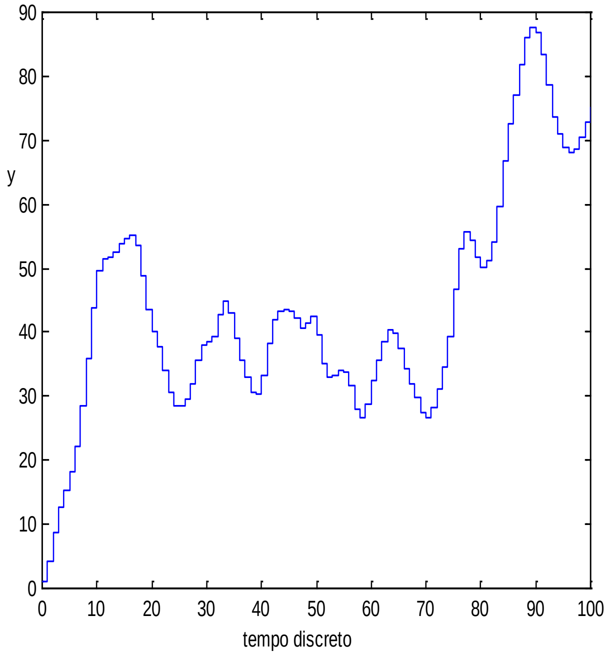
$$\sigma_e^2 = E\left[e_t^2\right] = 1$$

The plant delay is 1.

We want to find the minimum variance control.

When the control is zero, the output changes a lot, due to the disturbance given by passing white noise through the filter $C^*(q^{-1})/A^*(q^{-1})$.

A possible response is given in the figure



Computing polynomials $F_1^*(q^{-1})$ and $G_1^*(q^{-1})$:

$$
\begin{array}{l|l}
1+1.5q^{-1}+0.9q^{-2} & 1-1.7q^{-1}+0.7q^{-2} \\ \hline
\underline{1-1.7q^{-1}+0.7q^{-2}} & 1 \\
\quad 3.2q^{-1}+0.2q^{-2} &
\end{array}
$$

Therefore:

$$G_1^*(q^{-1}) = 3.2 + 0.2q^{-1} \qquad F_1^*(q^{-1}) = 1$$

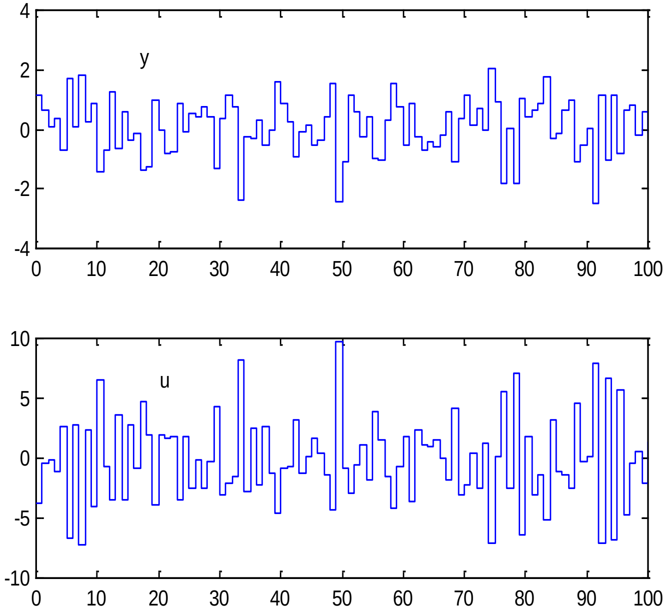
The minimum variance controller is

$$u(t) = -\frac{G_1^*(q^{-1})}{B^*(q^{-1})F_1^*(q^{-1})}y(t) = -\frac{3.2+0.2q^{-1}}{1+0.5q^{-1}}y(t)$$

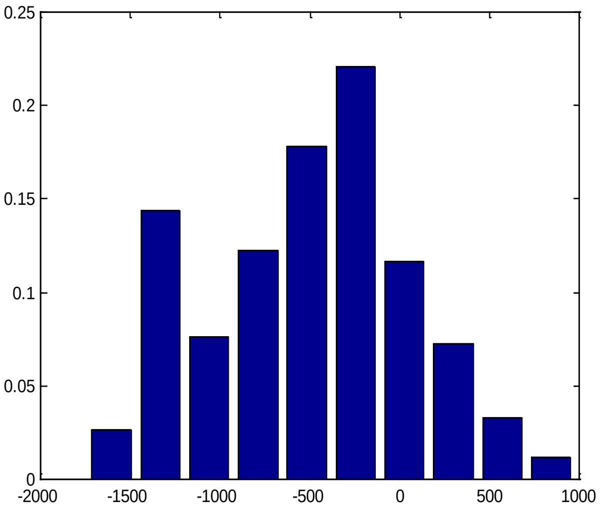
In stochastic steady-state, the variance of the output is

$$E\left[y^2(t)\right] = 1$$

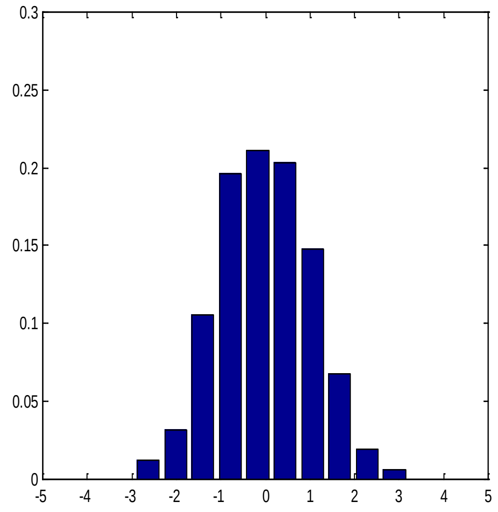
The following figure shows the response with miniomum variance control:



Compare the histograms of the output samples without (left) and with minimum variance control (right)



Without control



With MV control

Minimum variance control: Example 2

Consider the process modelled by

$$A^*(q^{-1})y(t) = B^*(q^{-1})u(t-2) + C^*(q^{-1})e(t)$$

Where (same as before)

$$A^*(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2} \qquad B^*(q^{-1}) = 1 + 0.5q^{-1}$$

$$C^*(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2}$$

$$\sigma_e^2 = E\left[e_t^2\right] = 1$$

The delay of this other system is 2.

Find the minimum variance control.

Computation of polynomials $F_1^*(q^{-1})$ e $G_1^*(q^{-1})$:

$$
\begin{array}{l|l}
1 + 1.5q^{-1} + 0.9q^{-2} & 1 - 1.7q^{-1} + 0.7q^{-2} \\ \hline
\underline{1 - 1.7q^{-1} + 0.7q^{-2}} & 1 + 3.2q^{-1} \\
\quad 3.2q^{-1} + 0.2q^{-2} & \\
\quad \underline{3.2q^{-1} - 5.44q^{-2} + 2.24q^{-3}} & \\
\qquad\qquad 5.64q^{-2} - 2.24q^{-3} &
\end{array}
$$

Therefore:

$$G_2^*(q^{-1}) = 5.64 - 2.24q^{-1} \qquad F_2^*(q^{-1}) = 1 + 3.2q^{-1}$$

The controller is now

$$u(t) = -\frac{G_2^*(q^{-1})}{B^*(q^{-1})F_2^*(q^{-1})}\,y(t) = -\frac{5.64 - 2.24q^{-1}}{1 + 3.7q^{-1} + 1.6q^{-2}}\,y(t)$$

In stochastic steady-state the output variance is now

$$E\left[y^2(t)\right] = 1 + 3.2^2 = 11.24$$

It is remarked that the output variance is now substantially higher than in example 1.

Closed-loop poles with minimum variance control

Start by rewritting:

$$\frac{C^*(q^{-1})}{A^*(q^{-1})} = F_d^*(q^{-1}) + q^{-d}\,\frac{G_d^*(q^{-1})}{A^*(q^{-1})}$$

or

$$A^*(q^{-1})F_d^*(q^{-1}) = C^*(q^{-1}) - q^{-d}G_d^*(q^{-1})$$

Multiplying both members by q^{n+d-1} we get:

$$\boxed{q^{d-1}C(q) = A(q)F_d(q) + G_d(q)}$$

The controlled system is described by the matrix model:

$$\begin{bmatrix} A(q) & -B(q) \\ G_d(q) & F(q)_d\,B(q) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} C(q) \\ 0 \end{bmatrix} e(t)$$

The chatacteristic equation of this system, with one input and 2 outputs is:

$$\det\left(\begin{bmatrix} A(q) & -B(q) \\ G_d(q) & F(q)_d\,B(q) \end{bmatrix} \right) = 0$$

or

$$B(q)\left[A(q)F_d(q) + G_d(q) \right] = 0$$

$$B(q)\left[A(q)F_d(q)+G_d(q)\right]=0$$

$$q^{d-1}C(q)=A(q)F_d(q)+G_d(q)$$

Hence, we conclude that the characteristic polynomial of the plant controlled with minimum variance control is

$$q^{d-1}B(q)C(q)=0$$

It is remarked that, if the open-loop system has zeros outside the unit circle, there will be unstable modes.

Example 3 – Minimum variance control

Consider the system in which:

$$A(q) = q^2 - 1.7q + 0.7 \qquad B(q) = 0.9q + 1 \qquad C(q) = q^2 - 0.7q$$

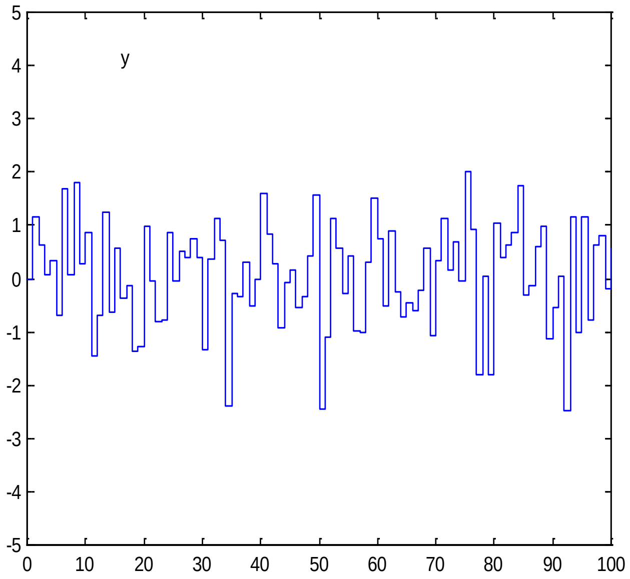
There is a zero at -1.11 that originates an unstable mode.

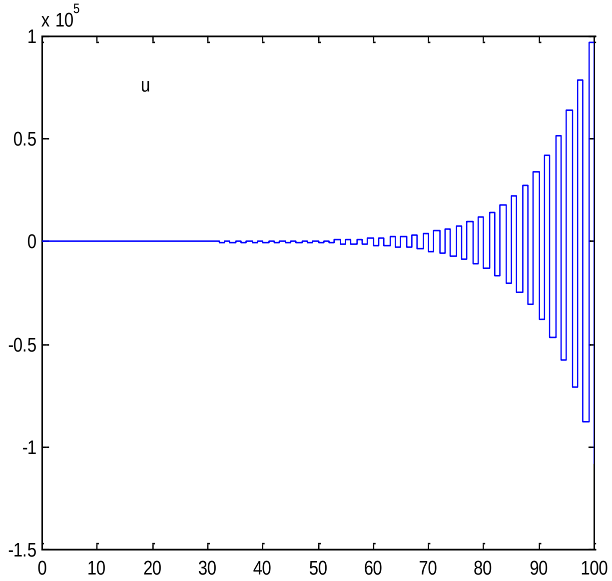
The minimum variance controller is

$$-\frac{G(q)}{B(q)F(q)} = -\frac{q - 0.7}{0.9q + 1}$$

The mode caused by the zero in -1.11 creates an oscillation that can be seen at the input but not at the outpu.

The example 3 simulation results are:





The incapacity of tackling non-minimum phase systems is a serious drawback of minimum variance control.

Detuned minimum variance control

To solve the previous drawback, include a penalty on the manipulated variable. The new cost functional is

$$J(u(t)) = E\left[y^2(t+d) + \rho u^2(t) | O^t\right]$$

Parameter ρ is a positive number that weights the manipulated variable and prevents it to grow.

Clarke e Gawthrop, 1974

The minimization of this cost functional allows to stabilize open-loop stable non-minimum-phase systems.

However, it does not allow to stabilize open-loop unstable nonminimum-phase systems.

To solve this later problem , one has to resort to stochastic optimnal control methods or to Model Predictive Control (MPC)

For an ARX plant, we have shown that the model can be written as (for simplicity, we consider the case $d=1$):

$$y(t+1) = w_1 u(t) + \pi_1^{'} s(t) + e(t+1)$$

Inserting this model in the cost, we get

$$J(u(t)) = E\left[y^2(t+1) + \rho u^2(t) \middle| O^t \right]$$

$$J(u(t)) = E\left[\left(w_1 u(t) + \pi_1^{'} s(t) + e(t+1) \right)^2 + \rho u^2(t) \middle| O^t \right]$$

$$J(u(t)) = E\left[\left(w_1 u(t) + \pi_1^{'} s(t) + e(t+1)\right)^2 + \rho u^2(t) | O^t\right]$$

$$J(u(t)) = \underbrace{\left(w_1 u(t) + \pi_1^{'} s(t)\right)^2 + \rho u^2(t)}_{\boxed{\alpha}} + E\left[e^2(t+1)\right]$$

This term does not depend on *u(t)*

The value of $u(t)$ that minimizes $J(u(t))$ satisfies

$$\frac{d\alpha}{du(t)} = 0$$

$$\frac{d}{du(t)}\left[\left(w_1 u(t) + \pi_1^{'} s(t)\right)^2 + \rho u^2(t)\right] = 0$$

$$w_1\left(w_1 u(t) + \pi_1^{'} s(t)\right) + \rho u(t) = 0$$

The solution of this equation is

$$u(t) = -\frac{w_1}{w_1^2 + \rho}\pi_1^{'} s(t)$$

This copntrol law can be written in the form of feedback of the "state" $s(t)$, where the vector of feedback gains is

$$F = -\frac{w_1}{w_1^2 + \rho}\pi_1$$

5. Adaptive and Predictive Control (MPC)

Objective: *To show how it is possible to integrate the blocks previously studied on identification and control to obtain more powerful control tools, including adaptive control and predictive control (MPC)*

Referência: AW, Cap. 14

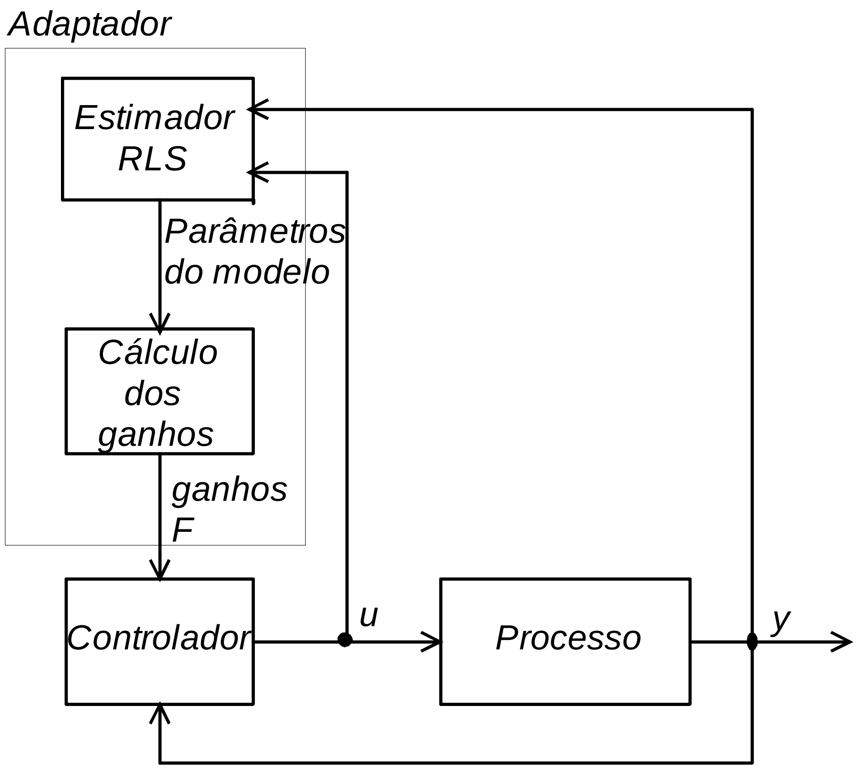
Adaptive Control: Years 1960 (Idea of Kalman in 1957).

MPC linear: Starting years 1980

MPC non linear: Starting yearss 1990

Adaptive control

Couple a parameter estimator with a control law. The control law is redesigned on-line using parameter estimates.



Self-tuning adaptive control *Astrom, Wittenmark, 1965*

Actual process:

$$y(k+1)+ay(k)=u(k)+e(k+1)+ce(k)$$

Model:

$$y(k+1)+\theta y(k)=u(k)+v(k+1)$$

Minimum variance control for the model, in which the noise is assumed white

$$u(k)=\hat{\theta}y(k)$$

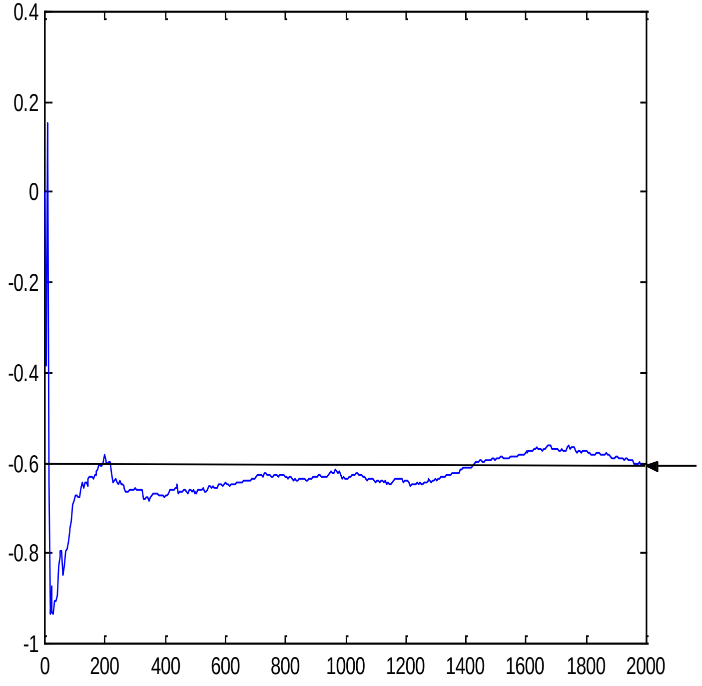
Since the noise is coloured, the estimate of θ **does not converge to the true value,** a .

However, the controller gain (given by the estimate of θ) approaches the minimum variance gain given by

$$F_{VM} = a - c$$

For instance, if $a = -0.9$ and $c = -0.3$, the gain approches -0,6 .

Gain convergence in the presence of colored noise. The error indicates the gain that yields minimum variance control.



Adaptive pole-placement control of a DC-motor

Structure of the transfer function of the plant to control

$$H(z) = \frac{K(z-b)}{(z-1)(z-a)}$$

Corresponding difference equation:

$$y(t) = (1-q^{-1})(1-aq^{-1}) = K(1-bq^{-1})u(t-1)$$

Take advantage of the fact that the plant has an integrator:

Defining $\Delta y(t) \coloneqq (1-q^{-1})y(t)$, we have the simpler model

$$\Delta y(t) = a\Delta y(t-1) + Ku(t-1) - Kbu(t-2)$$

$$\Delta y(t) = a\Delta y(t-1) + Ku(t-1) - Kbu(t-2)$$

Define:

$$\varphi(t-1) := \begin{bmatrix} \Delta y(t-1) & u(t-1) & u(t-2) \end{bmatrix}^T$$

$$\theta_o := \begin{bmatrix} \theta_1 = a & \theta_2 = K & \theta_3 = -Kb \end{bmatrix}^T$$

The model is written in the linear regression form:

$$\Delta y(t) = \varphi^T(t-1)\theta_o$$

Estime the parameters using RLS. Compute the controller gains from these estimates.

Adaptive explicit pole-placement control algorithm

At each time step, recursively execute the following procedure:

1. Compute $\Delta y(t) = y(t) - y(t-1)$

2. Using RLS, recursively estimate the vector of parameters θ_o in the linear regression model $\Delta y(t) = \varphi^T(t-1)\theta_o$

   a. Kalman gain: $$\gamma(t) = \frac{P(t)\varphi(t-1)}{1+\varphi^T(t-1)P(t)\varphi(t-1)}$$

   b. $\hat{\theta}(t) = \hat{\theta}(t-1) + \gamma(t)\left[\Delta y(t) - \varphi^T(t-1)\hat{\theta}(t-1)\right]$

   c. $P(t-1) = \left[I - \gamma(t)\varphi^T(t-1)\right]P(t)$

3. $\hat{a} = \hat{\theta}_1 \qquad \hat{K} = \hat{\theta}_2 \qquad \hat{b} = -\hat{\theta}_3 / \hat{\theta}_2$

4. Compute the controller gains from the parameter estimates (p_1 and p_2 are the coefficients of the desired closed-loop polynoimial $z^2 + p_1 z + p_2$):

   a. $$t_0 = \frac{1 + p_1 + p_2}{\hat{K}} \qquad s_0 = \frac{1 + \hat{a} + p_1}{\hat{K}} \qquad s_1 = \frac{p_2 - \hat{a}}{\hat{K}}$$

   b. $$r_1 = -\hat{b} + \frac{\hat{b}(\hat{b} + p_1 \hat{b} + p_2)}{(\hat{b} - 1)(\hat{b} - a)}$$ (without zero cancellation)

5. Control to apply to the motor

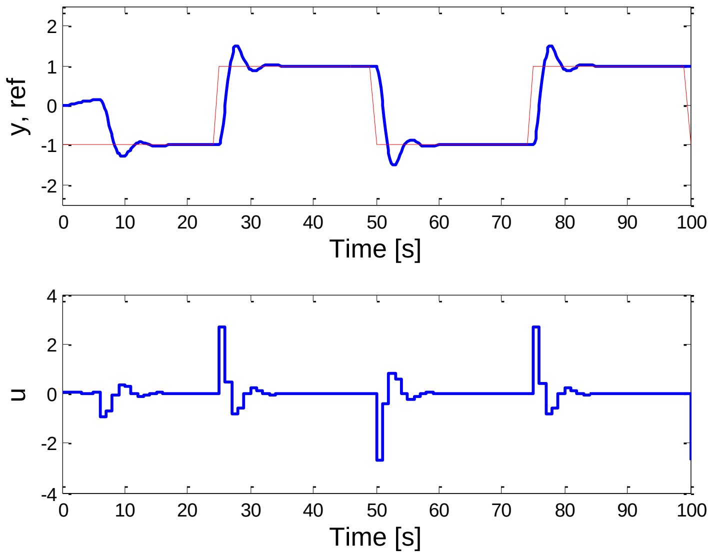
$$u(t) = t_0 u_c(t) - s_0 y(t) - s_1 y(t-1) - r_1 u(t-1)$$

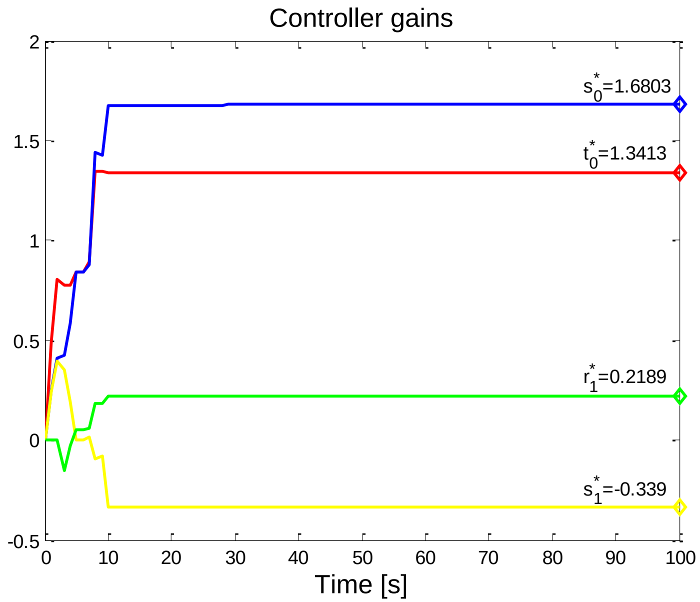
The RLS estimator is initialized as

$$\hat{\theta}(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad P(0) = 1000I$$

The initial value of $\hat{\theta}_2$ is chosen to be different from zero (could be any value) to avoid division by zero in the first step.

Results with no dither noise added to the manipulated variable





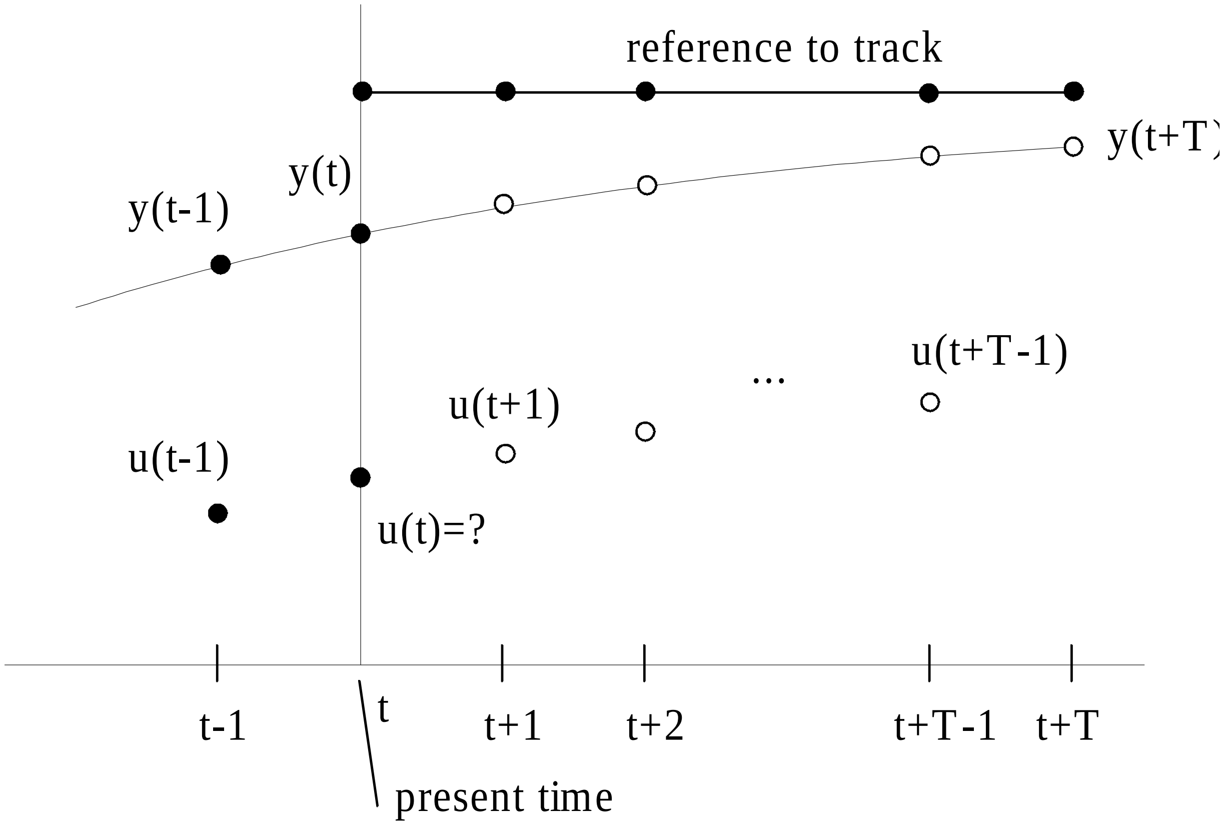
During the first 5 seconds the motor is in open loop.

During the first 10 seconds a low power dither noise is added to the manipulated variable, in order to improve the convergence of RLS.

Adaptive Model Predictive Control (MPC)

Idea: minimize a cost associated to an horizon that slides in front of you.

Apply just the first value of the resulting controller sequence and repeat the optimization at the next step (this is the Receding Horizon strategy).



At time *t*, the value of *u(t)* to apply to the plant is sought.

The value of *u(t)* applied to the plant is obtained as follows:

- Using suitable predictive models, find the sequence

  $$u(t),\ u(t+1),\ \ldots,\ u(t+T-1)$$

  which minimizes the quadratic cost

Information available up to time *t*. Includes past *i/o* observations, but may include other signals, such as state variables and accessible disturbances for feedforward

$$J_T = E\left[\sum_{i=1}^{T} y^2(t+i) + \rho u^2(t+i-1) \middle| O^t \right] \qquad \rho \geq 0$$

- According to a receding horizon strategy, apply to the plant only *u(t)* and repeat the process at time t+1.

The ARX plant *(Auto-Regressive with Exogenous variable)*:

$$A^*(q^{-1})y(t) = B^*(q^{-1})u(t-1) + e(t)$$

admits predictors of the form

$$\hat{y}(t+j+1) = \sum_{i=0}^{j} w_{i+1} u(t+j-i) + \pi'_{j+1} s(t)$$

$$s'(t) = \begin{bmatrix} y(t) & y(t-1) & \dots & y(t-n+1) & u(t-1) & \dots & u(t-m) \end{bmatrix}$$

$$y(t+j+1) = \hat{y}(t+j+1) + \varepsilon_{j+1}(t)$$

Residue orthogonal

to the data

An Example: GPC control law (position form)

Clarke, Mothadi, 1980, 1984

Write the output predictors for

$$y(t+1) = w_1 u(t) + \pi_1' s(t) + \varepsilon_1(t)$$

$$y(t+2) = w_1 u(t+1) + w_2 u(t) + \pi_2' s(t) + \varepsilon_2(t)$$

$$\bullet\bullet\bullet$$

$$y(t+T) = w_1 u(t+T-1) + \ldots + w_T u(t) + \pi_T' s(t) + \varepsilon_T(t)$$

Let

$$Y_{t+1}^{t+T} = \begin{bmatrix} y(t+1) \\ \vdots \\ y(t+T) \end{bmatrix} \qquad U_t^{t+T-1} = \begin{bmatrix} u(t) \\ \vdots \\ u(t+T-1) \end{bmatrix} \qquad \mathrm{E}_{t+1}^{t+T} = \begin{bmatrix} \varepsilon_1(t) \\ \vdots \\ \varepsilon_T(t) \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 & 0 & 0 & \cdots & 0 \\ w_2 & w_1 & 0 & \cdots & 0 \\ & & & & 0 \\ w_T & w_{T-1} & w_{T-2} & \cdots & w_1 \end{bmatrix} \qquad \Pi = \begin{bmatrix} \pi_1 & \cdots & \pi_T \end{bmatrix}$$

the set of predictive models can be written as

$$Y_{t+1}^{t+T} = WU_t^{t+T-1} + \Pi' s(t) + \mathrm{E}_{t+1}^{t+T}$$

Multistep Quadratic Cost:

$$J = E\left[\left\| Y_{t+1}^{t+T} \right\|^2 + \rho \left\| U_t^{t+T-1} \right\|^2 \middle| O^t \right]$$

Inserting the predictive models in the cost

$$J_T = E\left[\left\| WU_t^{t+T-1} + \Pi' s(t) + \mathrm{E}_{t+1}^{t+T} \right\|^2 + \rho \left\| U_t^{t+T-1} \right\|^2 \middle| O^t \right]$$

Since E_{t+1}^{t+T} is orthogonal to the other terms, and considering the conditioned mean, minimizing J is equivalent to minimize

$$\bar{J}_T = \left\| WU_t^{t+T-1} + \Pi' s(t) + \mathrm{E}_{t+1}^{t+T} \right\|^2 + \rho \left\| U_t^{t+T-1} \right\|^2$$

Expand $\bar{J}_T = \left\| W U_t^{t+T-1} + \Pi' s(t) + \mathrm{E}_{t+1}^{t+T} \right\|^2 + \rho \left\| U_t^{t+T-1} \right\|^2$ to get:

$$\bar{J}_T = (U_t^{t+T-1})' M U_t^{t+T-1} + 2(U_t^{t+T-1})' W' \Pi' s(t) + terms\,ind.of\ U_t^{t+T-1}$$

with the matrix *M* defined as

$$M = \rho I + W'W$$

Compare with a quadratic function with minimum at U^*

$$(U - U^*)' M (U - U^*) = U' M U - 2U' M U^* + U^{*}{}' M U^*$$

the optimum is seen to be

$$U_t^{t+T-1} = -M^{-1} W' \Pi' s(t)$$

$$U_t^{t+T-1} = -M^{-1}W'\Pi' s(t)$$

According to a receding horizon strategy, only the first element of this sequence is actually applied to the plant

$$u(t) = F' s(t)'$$

$$F' = -[1 \quad 0 \quad \cdots \quad 0] M^{-1} W' \Pi'$$

this formula for computing the gains is the basis for a version of the position version of GPC - Generalized Predictive Control *(Somewhat simplified)*.

With the gains computed for $T = \infty$, the **closed-loop poles** are the roots of the stable unit spectral factor of

$$\rho A(z)A(z^{-1}) + B(z)B(z^{-1})$$

where $\frac{B(z)}{A(z)}$ is the plant (open loop) transfer function.

Thus, for *T* large enough (how big should it be has been the subject of a major controversy!) the version of GPC presented stabilizes even open-loop non-minimum phase, unstable plants.

MUSMAR predictors (*Menga, Mosca, 1980)*

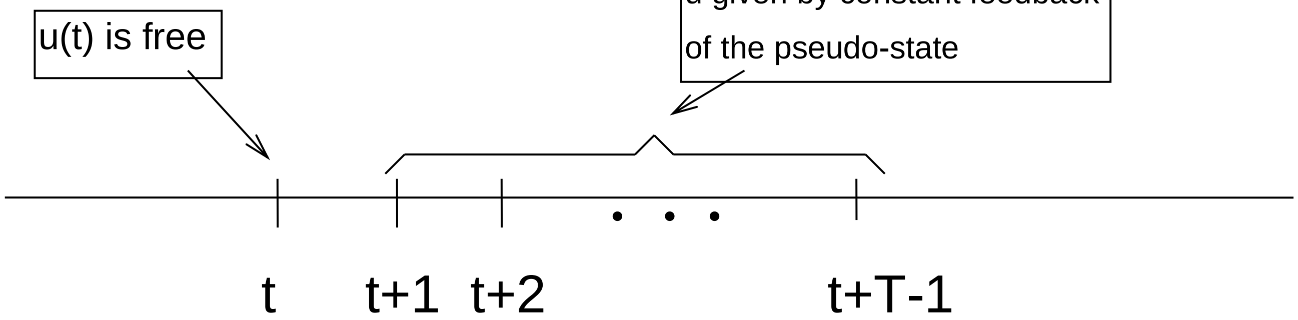
Assume a constant feedback law

$$u(t+k) = F_0{}' s(t+k) + \eta(t+k)$$

constant feedback

white dither noise uncorrelated with {e(t)}

is acting on the plant from time *t+1* up to time *t+T-1*



MUSMAR type predictors:

Assuming a constant feedback F_0 is acting on the plant from *t+1* up to *t+T-1*

$$\hat{y}(t+i|t) = \theta_i u(t) + \psi_i' s(t)$$

$$\hat{u}(t+i-1|t) = \mu_{i-1} u(t) + \phi_{i-1}' s(t)$$

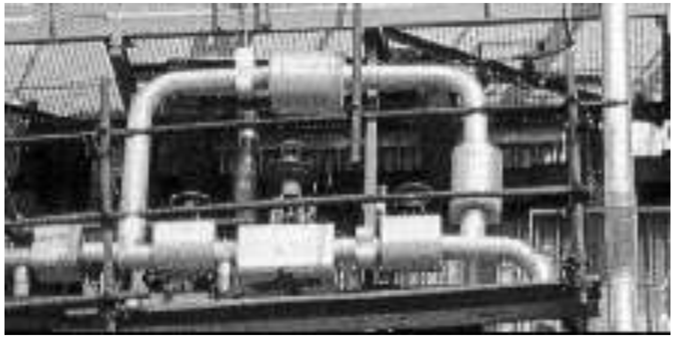
$$i = 1, \ldots, T$$

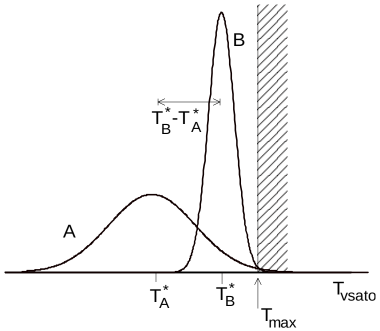
Note that: $\mu_0 = 1 \phi_0 = \overline{0}$

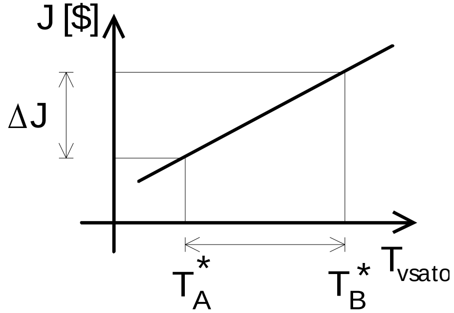
All the predictors depend on the feedback gain F_0, except θ_1 and ψ_1 which depend only on the parameters of the ARX plant model:

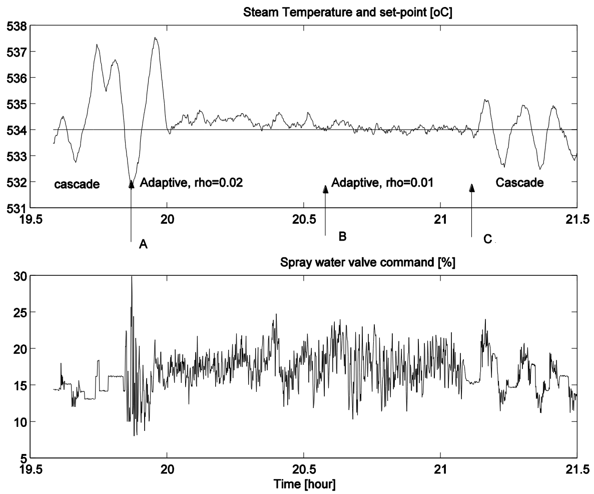
$$\theta_1 = b_0; \psi_1' = \begin{bmatrix} -a_1 & \ldots & -a_n & b_1 & \ldots & b_m \end{bmatrix}$$

Superheated steam temperature control *(Lemos, 1997)*

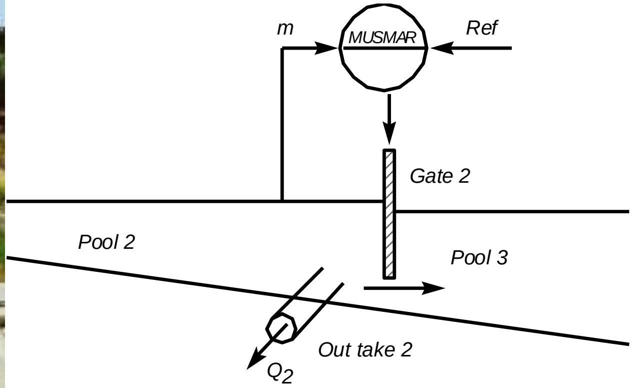


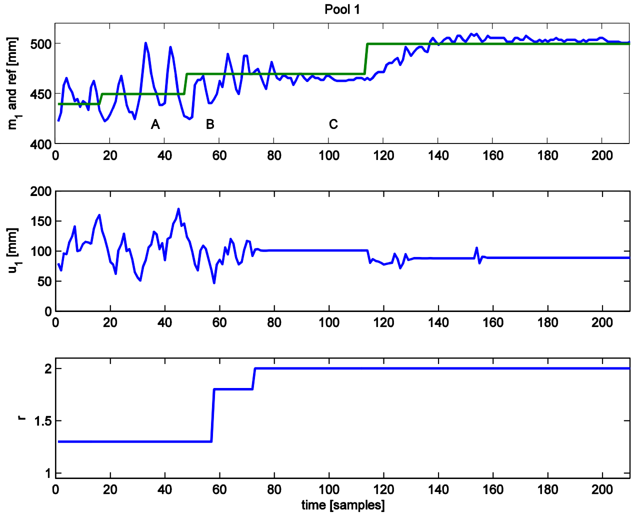




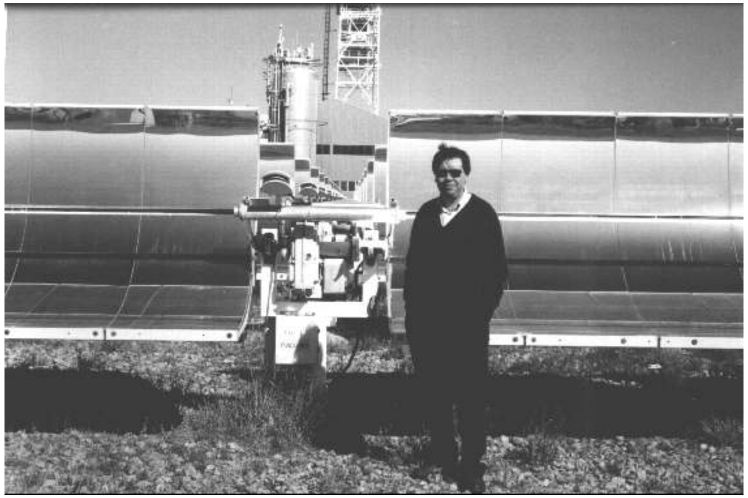


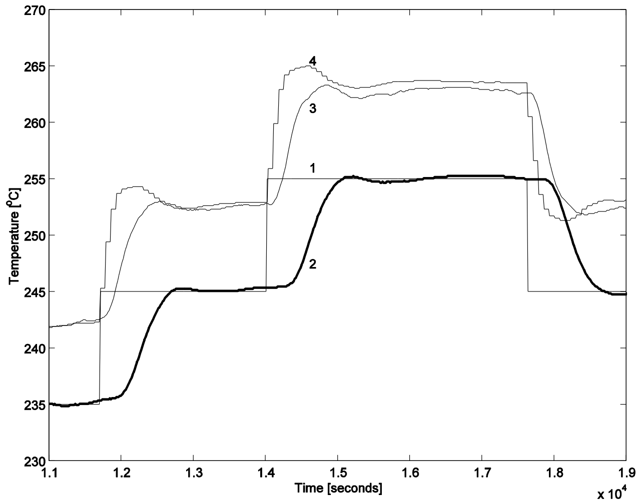
Water delivery canal level control – Exponential weight (Lemos 2006)

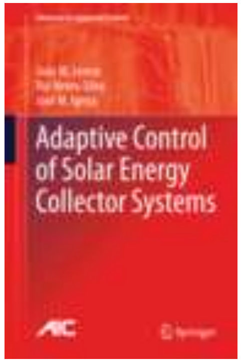




Parabolic trough solar field temperature control







J. M. Lemos, R. Neves-Silva and J. M. Igreja.
Adaptive Control of Solar Energy Collector Systems,
Springer (Advances in Industrial Control), 2014
(*Copy available at the IST library - DEEC*)

Dynamic Programming in discrete time

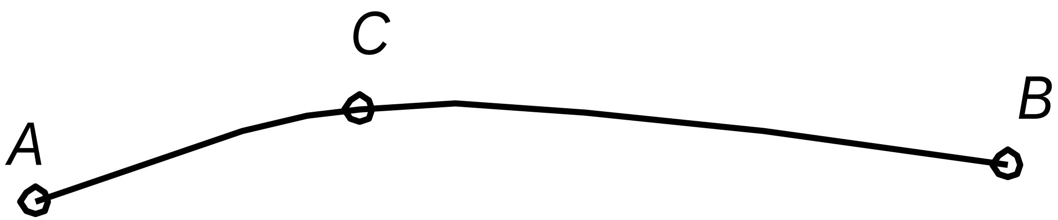
Bellman, years 1950
Actually older roots, starting from the XVII century (Jacob Bernouilli solution of the Brachistochrone problem).

Bellman's Optimum Principle

Traveling in state space from A to B.

Find the optimal path.

If you start at an intermediate point C on the optimal path, the optimal path between C and B is the same as the section of the optimal path between A and B that lies between C and B.



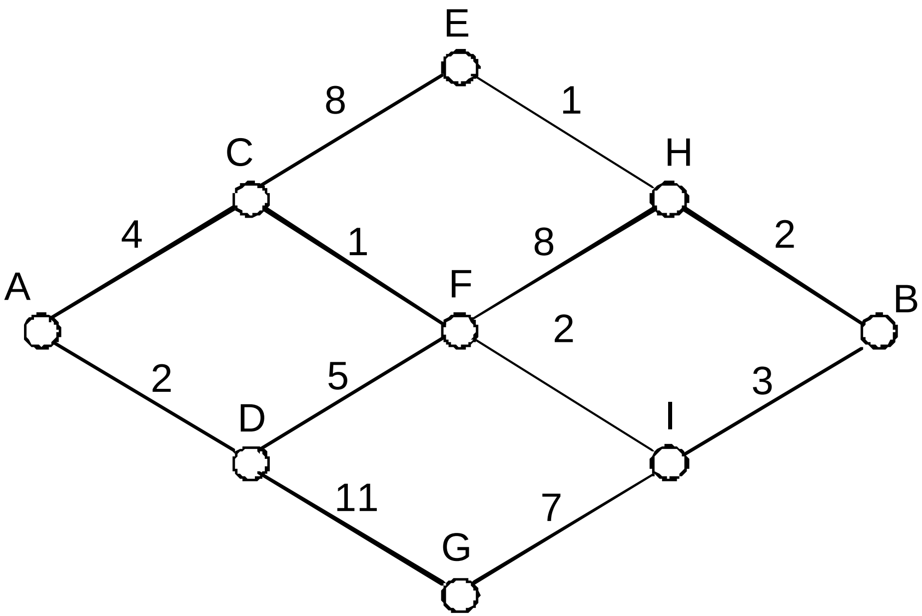
Dynamic Programming algorithm

1. Characterize each state by:
   - Its value: The value of the objective function when you follow the optimal path from it to the end.
   - The optimal decision to take when you are on it.

   Do this characterization by progressing backwards, from the last state to the others that precede it.

2. Start from the beginning and follow the optimal decision at each state.

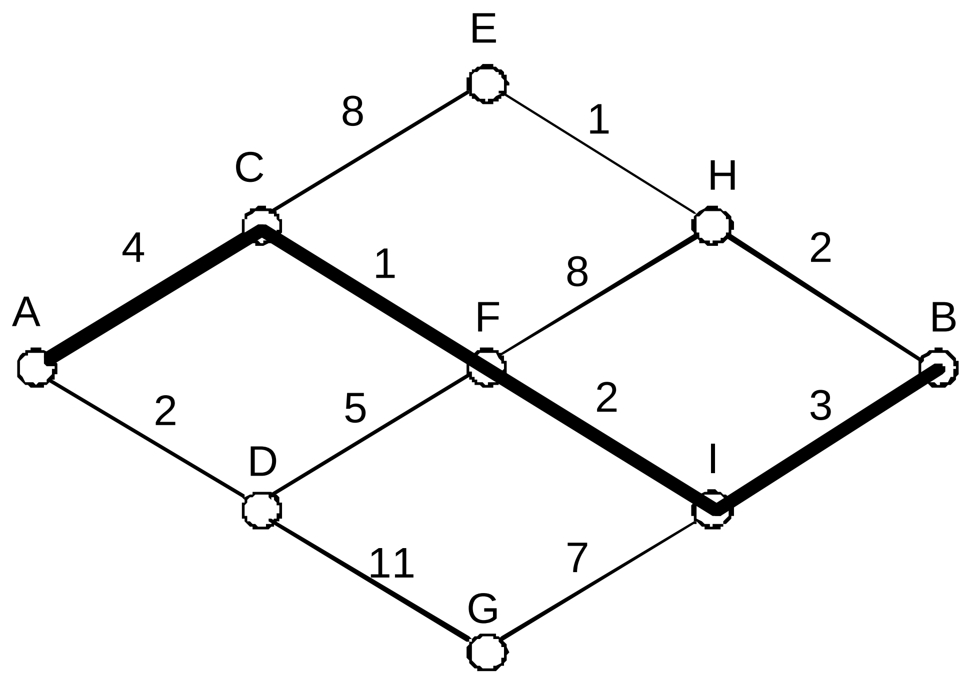
Example: Find the optimal path between A and B



Use the index of the levels of the network as “time”.

Start from B and go backwards, labeling each node with its value and the optimal decision to take whenm you are on it.

Solution for the optimal path:



Dynamic optimization problem with continuous state

Discrete time nonlinear state model

$$x(k+1) = f(x(k, u(k)), \qquad x(0) = x_0$$

Objective function (*e. g.*, a cost to minimize)

$$J\left(u|_{[0,N-1]}\right) = \psi\left(x(N)\right) + \sum_{k=0}^{N-1} L(x(k), u(k))$$

Find the sequence of control values that optimizes the objective function.

Value function

The value function represents the minimum cost that can be obtained when starting from state x, at instant k, up to the final instant N:

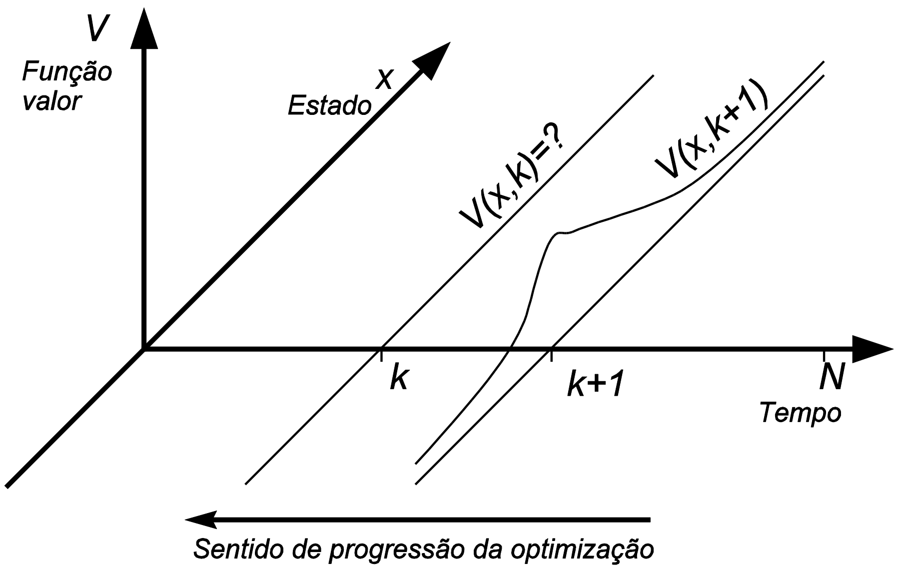
$$V(x,k) = \min_{u|_{[k,N-1]}} \left(\psi(x(N) + \sum_{i=k}^{N-1} L(x(i),u(i)) \right)$$

Hamilton-Jocobi-Bellman (HJB) equation in discrete time

Relates the value function in successive instants of time

$$V(x,k) = \min_{u \in U} \left(L(x,u) + V(f(x,u), k+1) \right)$$

Terminal condition $V(x,N) = \psi(x)$



Linear Quadratic Problem

Linear dynamics

$$x(k+1) = \Phi x(k) + \Gamma u(k), \qquad x(0) = x_0$$

Quadratic cost

$$J = x^T(N) Q_0 x(N) + \sum_{k=0}^{N-1} x^T(k) Q x(k) + u^T(k) R u(k)$$

HJB equation

$$V(x,k) = \min_{u} \left[x^T Q x + u^T R u + V(\Phi x + \Gamma u, k+1) \right]$$

$$V(x,N) = x^T(N) Q_0 x(N)$$

$$V(x,k) = x^T Qx + \min_u \left[u^T Ru + V(\Phi x + \Gamma u, k+1)\right]$$

Assume a solution of the form

$$V(x,k) = x^T P(k)x$$

The HJB equation reads

$$x^T P(k)x = x^T Qx + \min_u \left[u^T(R + \Gamma^T P(k+1)\Gamma)u + 2x^T \Phi^T P(k+1)\Gamma u\right] +$$
$$+ x^T \Phi^T P(k+1)\Phi x$$

$$x^T P(k)x = x^T Qx + \min_u [u^T(R + \Gamma^T P(k+1)\Gamma)u + 2x^T\Phi^T P(k+1)\Gamma u] +$$
$$+x^T\Phi^T P(k+1)\Phi x$$

Comparing with the quadratic form

$$(u - u^*)^T M(u - u^*) = u^T Mu - 2u^{*T} Mu + u^{*T} Mu^*$$

yields

$$M = R + \Gamma^T P(k+1)\Gamma \qquad -u^{*T} M = x^T\Phi^T P(k+1)\Gamma$$

$$u^* = F(k)x \qquad F(k) = -[R + \Gamma^T P(k+1)\Gamma]^{-1}\Gamma^T P(k+1)\Phi$$

Riccati equation

$$P(k) = Q - \Phi^T P(k+1)\Gamma[R + \Gamma^T P(k+1)\Gamma]^{-1}\Gamma^T P(k+1)\Phi + \Phi^T P(k+1)\Phi$$