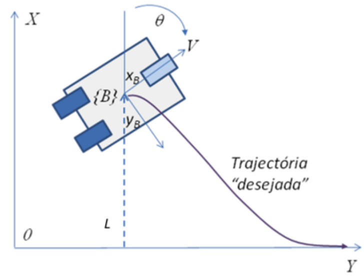
MODELAÇÃO E SIMULAÇÃO - 2019
MEEC – IST, TESTE N0.1 TIPO – V01
RESOLUÇÃO



(1) $\dot{x} = x(x+1)(x-1)$

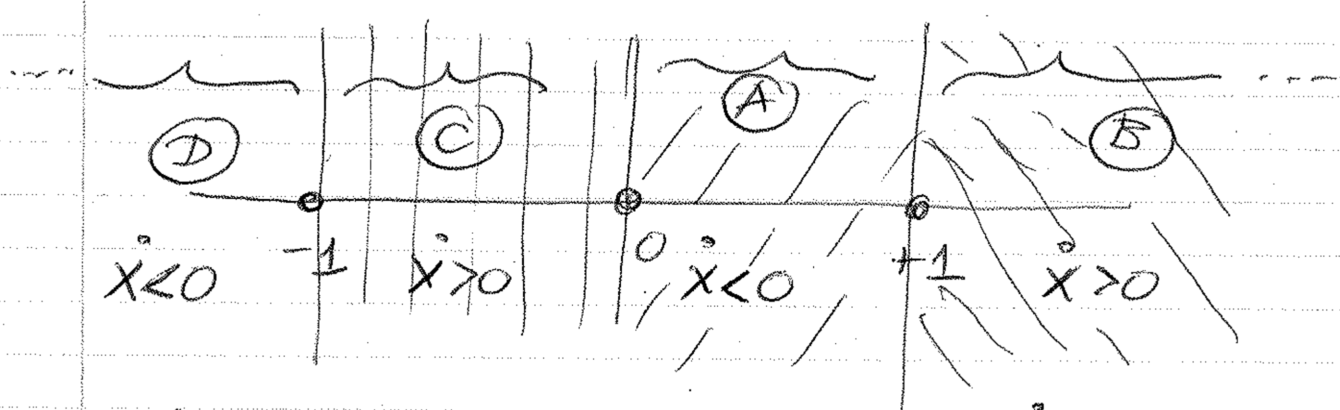
(1.1) $\{\bar{x} : (\bar{x})(\bar{x}+1)(\bar{x}-1) = 0\}$

$= \underbrace{\{0, -1, +1\}}$

EQUILIBRIUM POINTS

(1.2) "ROUGH SKETCH OF THE TRAJECTORIES)

STEP 1. COMPUTATION OF THE FIRST DERIVATIVE



REGION (A) : $x > 0$, $x > -1$, $x < +1$ $\Rightarrow$ $x > 0$, $x+1 > 0$, $x-1 < 0$ $\rightarrow$ $\dot{x} < 0$

REGION (B) : $x > 0$, $x > -1$, $x > +1$ $\Rightarrow$ $x > 0$, $x+1 > 0$, $x-1 > 0$ $\Rightarrow$ $\dot{x} > 0$

REGION (C) : $x < 0$, $x > -1$, $x < +1$ $\Rightarrow$ $x < 0$, $x+1 > 0$, $x-1 < 0$ $\Rightarrow$ $\dot{x} > 0$

REGION (D) : $x < 0$, $x < -1$, $x < +1$ $\Rightarrow$ $x < 0$, $x+1 < 0$, $x-1 < 0$ $\Rightarrow$ $\dot{x} < 0$

(1.3) LINEARIZATION

$$\dot{x} = f(x) = x(x^2-1) = x^3 - x$$

$\Rightarrow$ (LINEARIZATION ABOUT $\bar{x}$)

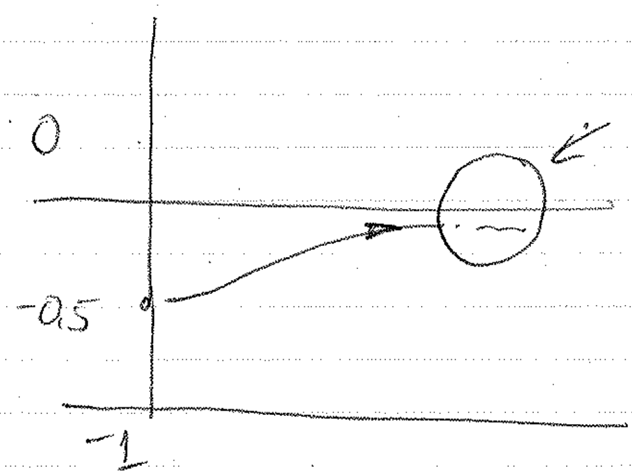
$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \delta x$$

$$\Rightarrow \delta\dot{x} = \left.(3x^2-1)\right|_{x=\bar{x}} \delta x$$

$$\boxed{\delta\dot{x} = a\,\delta x}$$

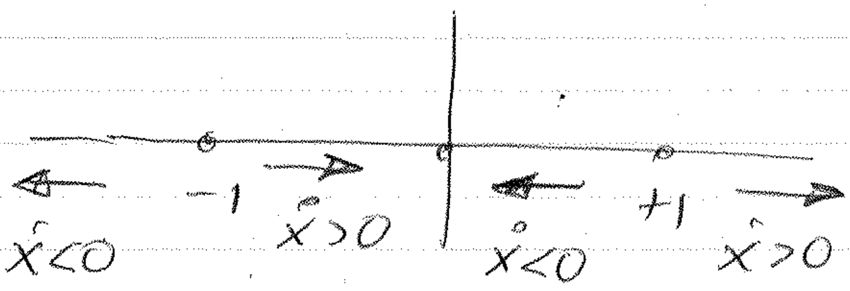
$\bar{x} = 0 \Rightarrow a = -1 \longrightarrow$ LOCALLY AS. STABLE
$\bar{x} = +1 \Rightarrow a = +2 \longrightarrow$ UNSTABLE
$\bar{x} = -1 \Rightarrow a = +2 \longrightarrow$ UNSTABLE

(1.4)



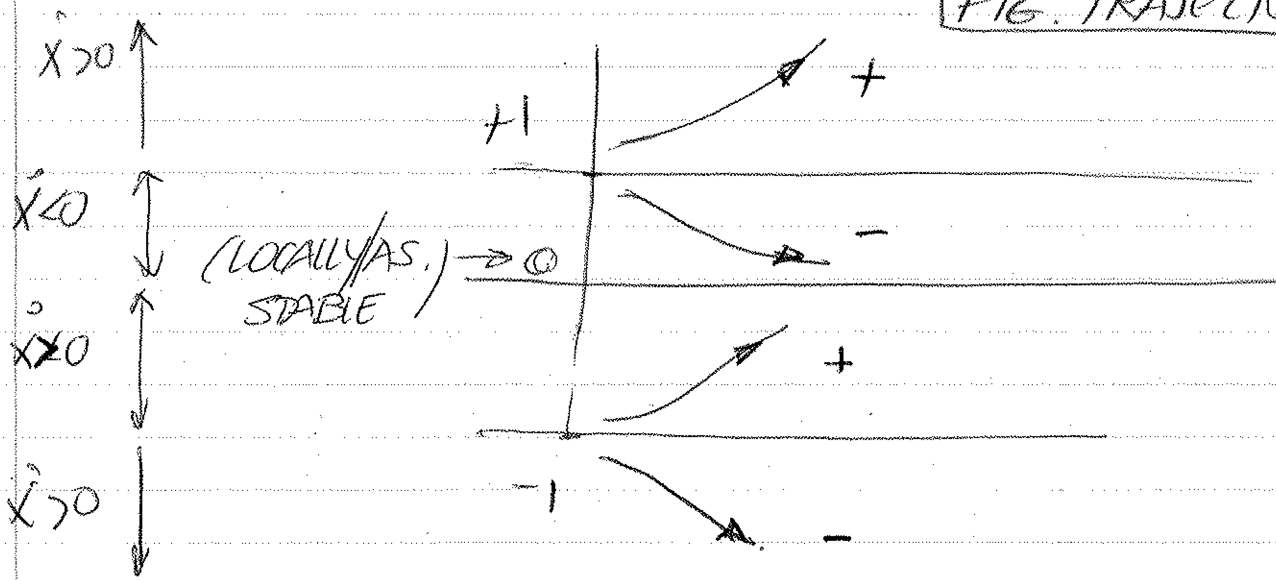
REACHES $\underline{0 \text{ at } t = \infty}$
(UNIQUENESS OF SOLUTIONS)

$\Rightarrow$



$\Downarrow$

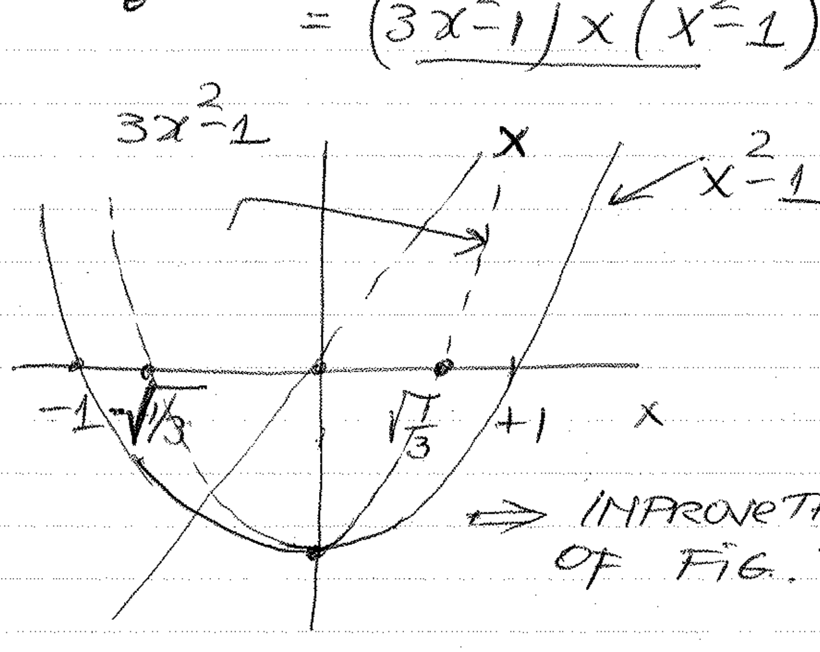
FIG. TRAJECTORY



STEP 2 - check 2nd DERIVATIVE

$$\ddot{x} = \frac{d}{dt}\dot{x} = (3x^2-1)(x^3-x)$$

$$= \underline{(3x^2-1)\,x\,(x^2-1)}$$



$\Rightarrow$ IMPROVE THE DETAILS OF FIG. TRAJECTORY

(3,A)

(2)

(2.1) Vehicle 1

$$\sum F_i^1 = m_1 \frac{dv_1}{dt} \; ; \; v_1 = \frac{dy_1}{dt}$$

$$-k(y_1 - y_2) - \beta(\dot{y}_1 - \dot{y}_2) - \gamma \dot{y}_1$$

Vehicle 2

$$\sum F_i^2 = m_2 \frac{dv_2}{dt} \; ; \; v_2 = \frac{dy_2}{dt}$$

$$-k(y_2 - y_1) - \beta(\dot{y}_2 - \dot{y}_1) - \gamma \dot{y}_2 = u$$

Let $[x_1, x_2, x_3, x_4]^T = [y_1, \dot{y}_1, y_2, \dot{y}_2]^T$

$$\Downarrow$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{\beta}{m_1} - \frac{\gamma}{m_1} & \frac{k}{m_1} & +\frac{\beta}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{\beta}{m_2} & -\frac{k}{m_2} & -\frac{\beta}{m_2} - \gamma \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{pmatrix}}_{B} u$$

$$y = (x_1, x_3) = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}}_{C} x$$

3 -

3.1 $G_1(s) = \dfrac{1}{s^2(s+10)} = \dfrac{Y_1(s)}{U(s)}$

$\Rightarrow \quad s^2(s+10)\,Y(s) = U(s)$

$\Rightarrow \quad (s^3 + 10s^2)\,Y(s) = U(s)$

TL^{-1} $\Rightarrow \quad \boxed{\dddot{y} + 10\ddot{y} = u}$

Let $x_1 = y$; $x_2 = \dot{y}$; $x_3 = \ddot{y}$

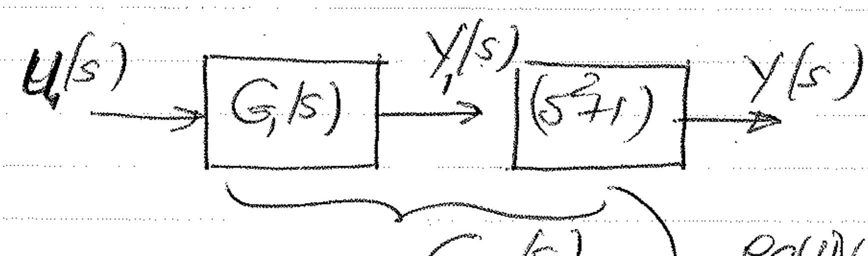
$$\Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -10x_3 + u \end{aligned}$$

$$\Rightarrow \quad \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -10 \end{pmatrix}}_{A} X + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{B} u$$

OUTPUT $Y = y \Rightarrow$

$$Y = \underbrace{(1 \quad 0 \quad 0)}_{C} X$$

3.2

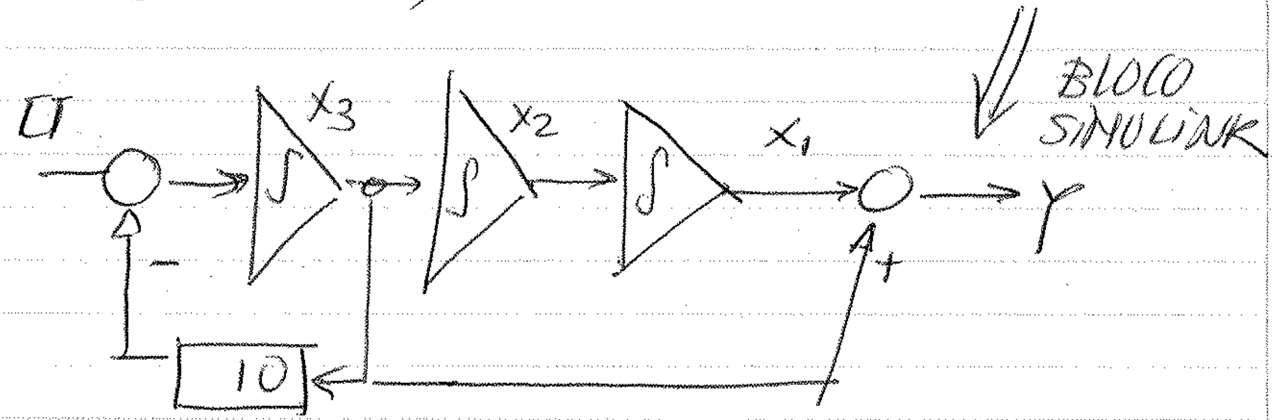


equiv.
$\ddot{y} + y$

$\Rightarrow \quad x_3 + x_1$

$$\Rightarrow \quad \dot{x} = Ax + Bu$$

$$y = Cx \quad ; \quad C = [1 \ 0 \ 1]$$



(3.3) Simply add "spurious" STATES

"Invent" x_4 that is NOT ACTUATED by u

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = (1 \ 0 \ 0 \ 0)\, x$$

NOT "SEEN" FROM A
I-O POINT OF VIEW

4 –

4.1 $\dot{L} = V\cos\theta$

$\dot{\theta} = \omega \leftarrow$ INPUT VARIABLE

$$\begin{pmatrix} \dot{L} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} V\cos\theta \\ \omega \end{pmatrix}, \text{ of the form } \dot{\xi} = f(\xi, u)$$

WITH $\xi = \begin{pmatrix} L \\ \theta \end{pmatrix}$; $u = \omega$

EQ. POINT $\boxed{L_0 = 0 \; ; \; \theta_0 = \pi/2 \; , \; \omega_0 = u_0 = 0}$

Compute the linearization

$$\delta\dot{\xi} = \frac{\partial f}{\partial \xi}\Big|_0 \delta\xi + \frac{\partial f}{\partial u}\Big|_0 \delta u$$

$$= \underbrace{\begin{pmatrix} 0 & -V\sin\theta|_{\theta_0} \\ 0 & 0 \end{pmatrix}}_{A} \delta\xi + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} \delta u$$

$$\Rightarrow \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta u$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \delta\xi_1 \\ \delta\xi_2 \end{pmatrix} \Rightarrow \boxed{\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= u \end{aligned}}$$

(4.2) $\dot{x}_1 = x_2$; $u = +k_1 x_1 - k_2 x_2$

$\dot{x}_2 = u$

$\Rightarrow \dot{x}_1 = -x_2$

$\dot{x}_2 = k_1 x_1 - k_2 x_2$

$$\Rightarrow \boxed{\dot{x} = \underbrace{\begin{pmatrix} 0 & -1 \\ k_1 & -k_2 \end{pmatrix}}_{A} x}$$

The system is of the form $\dot{x} = Ax$

To analyze the stability of $\bar{x} = 0$ compute the eigenvalues of A

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & +1 \\ -k_1 & \lambda + k_2 \end{vmatrix} = \underline{\lambda^2 + \lambda k_2 + k_1}$$

$k_1 > 0;\ k_2 > 0$

This is a second order system with positive coefficients $\Rightarrow$ eigenvalues have negative real part $\Rightarrow$ system is <u>STABLE</u>!

(4.3) $\dot{x} = \begin{pmatrix} 0 & -1 \\ 10 & -11 \end{pmatrix} x$

$\Rightarrow \lambda^2 + 11\lambda + 10 = 0 \Rightarrow$

$$\boxed{\lambda_1 = -1;\ \lambda_2 = -10}$$

CLOSED-LOOP EIGENVALUES

Comput the eigenvectors $Av = \lambda v$

$\boxed{\lambda_1 = -1}$ $A v_1 = \lambda_1 v_1$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 10 & -11 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = -\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

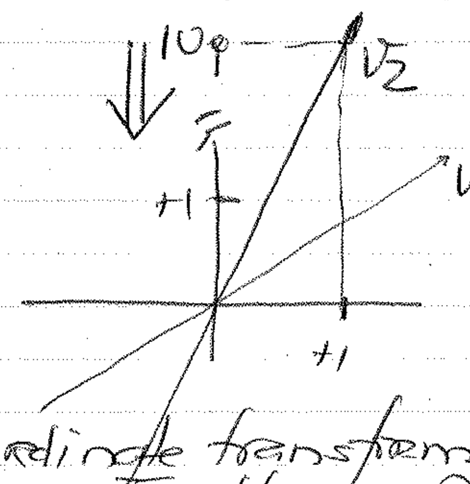
$$\Rightarrow -v_{12} = -v_{11} \Rightarrow v_{11} = v_{12} \Rightarrow \boxed{v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\boxed{\lambda_2 = -10}$ $A v_2 = \lambda_2 v_2$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 10 & -11 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = -10\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$\Rightarrow -v_{22} = -10 v_{21} \Rightarrow v_{22} = 10 v_{21} \Rightarrow \boxed{v_2 = \begin{bmatrix} 10 \\ 1 \end{bmatrix}}$$

$\Downarrow$

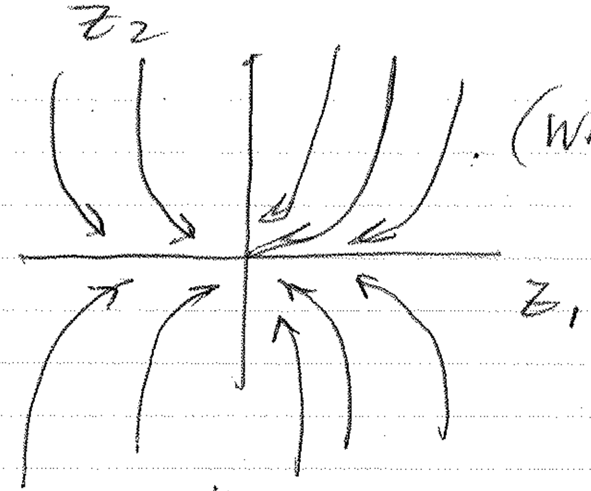


Do a coordinate transformation to diagonalize the system. In the new coordinates

$$\dot{z} = \bar{A} z; \quad \bar{A} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix}}_{\text{DIAGONAL}}$$

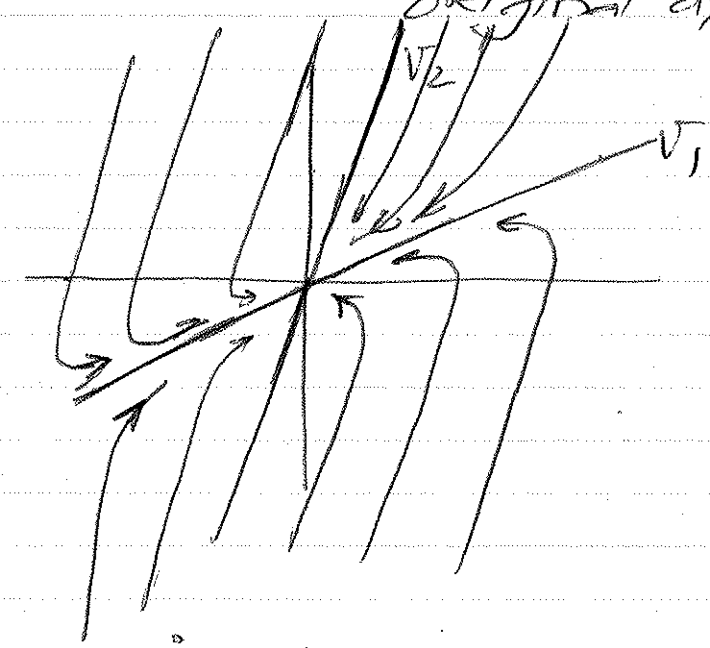
PLOT THE TRAJECTORIES in THE (z_1, z_2) PLANE

$\Rightarrow$



(WHY? CHECK!)

$\Downarrow$ "GO BACK TO THE (x_1, x_2) PLANE" and "move along" the original axis



(4.4) — $\dot{x}_1 = x_2$

$$\dot{x}_2 = k_1 x_1 - k_2 x_2 - k_1 r$$

$$\Rightarrow \ddot{x}_1 = -\dot{x}_2 = -k_1 x_1 + k_2 x_2 + k_1 r$$

$$= -k_1 x_1 - k_2 \dot{x}_1 + k_1 r$$

$$\Rightarrow \ddot{x}_1 + k_2 \dot{x}_1 + k_1 x_1 = k_1 r \Rightarrow \frac{X_1(s)}{R(s)} = \frac{k_1}{s^2 + k_2 s + k_1}$$