Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

"Exame de Época Especial" 2012/13

Duration: **3 hours** **2013/07/19** — **9AM**, Room **C01**

- Please justify all your answers.
- This exam has THREE PAGES and SIX GROUPS. The total of points is 40.0.

Group 1 — Introduction to Stochastic Processes **2.5 points**

Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of i.i.d. Bernoulli r.v. with parameter $p = \frac{1}{2}$ and $V_n = \sum_{i=1}^{+\infty} X_{i+n} \times 2^{-i}$, $n \in \mathbb{N}_0$.[1]

(a) *Derive the expected value and the variance of V_n.* (1.0)

- **Stochastic processes**
  $\{X_i : i \in \mathbb{N}\} \overset{i.i.d.}{\sim} \text{Bernoulli}(p = \frac{1}{2})$
  $\{V_n = \sum_{i=1}^{+\infty} X_{i+n} \times 2^{-i} : n \in \mathbb{N}_0\}$ (sequence of binary expansions...)
- **Requested expected value**

$$\begin{aligned} E(V_n) &= \sum_{i=1}^{+\infty} E(X_{i+n}) \times 2^{-i} \\ &\overset{X_{i+n} \sim Ber(p)}{=} \sum_{i=1}^{+\infty} p \times 2^{-i} \\ &= p \times \sum_{i=1}^{+\infty} 2^{-i} \\ &\overset{p=\frac{1}{2}}{=} \frac{1}{2} \times \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

- **Requested variance**

$$\begin{aligned} V(V_n) &= \sum_{i=1}^{+\infty} V(X_{i+n}) \times \left(2^{-i}\right)^2 \\ &\overset{X_{i+n} \sim Ber(p)}{=} \sum_{i=1}^{+\infty} p(1-p) \times \left(2^2\right)^{-i} \\ &= p(1-p) \times \sum_{i=1}^{+\infty} 4^{-i} \\ &\overset{p=\frac{1}{2}}{=} \frac{1}{2} \times \left(1-\frac{1}{2}\right) \times \frac{\frac{1}{4}}{1-\frac{1}{4}} \\ &= \frac{1}{12}. \end{aligned}$$

[1] *Note that $\sum_{i=1}^{+\infty} X_i \times 2^{-i}$ is the binary expansion of a uniformly distributed on $[0,1]$.*

(b) *Is $\{V_n : n \in \mathbb{N}_0\}$ a (second order weakly) stationary process?* (1.5)

- **Investigating the 2nd. order weak stationarity**
  On one hand $E(V_n) \overset{(a)}{=} \frac{1}{2}$, hence, constant for all $n \in \mathbb{N}_0$.
  On the other hand, by the independence of the X_i ($i \in \mathbb{N}$), we get, for $n, h \in \mathbb{N}_0$:

$$\begin{aligned} cov(V_n, V_{n+h}) &= cov\left(\sum_{i=1}^{+\infty} X_{i+n} \times 2^{-i}, \sum_{j=1}^{+\infty} X_{j+n+h} \times 2^{-j}\right) \\ &= \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} 2^{-i-j} \times cov(X_{i+n}, X_{j+n+h}) \\ &\overset{i+n=j+n+h \Leftrightarrow i=j+h}{=} \sum_{j=1}^{+\infty} 2^{-(j+h)-j} \times cov(X_{j+h+n}, X_{j+h+n}) \\ &= \sum_{j=1}^{+\infty} 2^{-2j-h} \times V(X_{j+h+n}) \\ &\overset{X_{j+h+n} \sim Ber(p)}{=} 2^{-h} \times p(1-p) \times \sum_{i=1}^{+\infty} 4^{-i} \\ &= 2^{-h} \times \frac{1}{2} \times \left(1-\frac{1}{2}\right) \times \frac{\frac{1}{4}}{1-\frac{1}{4}} \\ &= 2^{-h} \times \frac{1}{12}, \end{aligned}$$

which does not depend on n and only depends on the (discrete) time lag h. Consequently, we are dealing with a second order weakly stationary process.

Group 2 — Poisson Processes **9.5 points**

1. *Assume that cars pass in a specific point of a highway at a rate of 10 per hour.*

(a) *Suppose two cars passed during the first 20 minutes. What is the probability that these two cars were the only cars that passed during the first hour?* (1.5)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda = 10)$
  $N(t)$ = number of cars pass in a specific point of a highway by time t (time in hours)
  $N(t) \sim \text{Poisson}(\lambda t)$
- **Requested probability**

$$\begin{aligned} P[N(1) = 2 \mid N(1/3) = 2] &= \frac{P[N(1) = 2,\, N(1/3) = 2]}{P[N(1/3) = 2]} \\ &= \frac{P[N(1/3) = 2,\, N(1) - N(1/3) = 2-2]}{P[N(1/3) = 2]} \\ &\overset{indep.\ incr.}{=} \frac{P[N(1/3) = 2] \times P[N(1) - N(1/3) = 2-2]}{P[N(1/3) = 2]} \\ &= P[N(1) - N(1/3) = 2-2] \\ &\overset{station.\ incr.}{=} P[N(1-1/3) = 0] \end{aligned}$$

$$
\begin{aligned}
P[N(1)=2 \mid N(1/3)=2] &\overset{N(t)\sim Poi(10t)}{=} e^{-10\times 2/3}\frac{(10\times 2/3)^0}{0!}\\
&= e^{-20/3}\\
&\simeq 0.001273.
\end{aligned}
$$

(b) *Assume that each car will pick up a hitchhiker with probability $\frac{1}{10}$. You are second in line. What is the probability that you will have to wait for more than 1 hour?* (1.5)

- **Split process**
  $N_{hitch}(t)$ = number of cars that picked up hitchhikers by time t
  $p = P(\text{car picking hitchhiker}) = \frac{1}{10}$
  $\{N_{hitch}(t) : t \geq 0\} \sim PP(\lambda p = 10 \times \frac{1}{10} = 1)$
  $N_{hitch}(t) \sim \text{Poisson}(\lambda p \times t = t)$

- **Requested probability**
  If you are second in line, you will have to wait for more than 1 hour iff the second car that picked a hitchhiker took more than 1 hour to pass in that specific point of the highway, i.e., iff the number of cars that picked hitchhikers and passed in one hour does not exceed 1. Thus, the requested probability is equal to:

$$
\begin{aligned}
P(S_2^{hitch} > 1) &= 1 - P(S_2^{hitch} \leq 1)\\
&= 1 - P[N_{hitch}(1) \geq 2]\\
&= P[N_{hitch}(1) < 2]\\
&= P[N_{hitch}(1) \leq 1]\\
&= F_{Poisson(1)}(1)\\
&\overset{table}{=} 0.7358.
\end{aligned}
$$

2. *Evaristo owns a vegetarian food stand that is open from 8:00 to 17:00 and admits that the customers arrive to it according to a non-homogeneous Poisson process with mean value function equal to*

$$
m(t) = \begin{cases} 10(t-8), & 8 \leq t \leq 11\\ 30 + 20(t-11), & 11 < t \leq 13\\ 70 + 15(t-13), & 13 < t \leq 17. \end{cases}
$$

(a) *Derive the time dependent rate of this process.* (1.0)

- **Stochastic process**
  $\{N(t) : 8 \leq t \leq 17\} \sim NHPP(\lambda(t))$
  $N(t)$ = number of arrivals to Evaristo's vegetarian food stand until time t
  $\lambda(t)$ = intensity function (to be determined!)

- **Mean value function**
  For $8 \leq t \leq 17$,

$$
m(t) = \begin{cases} 10(t-8), & 8 \leq t \leq 11\\ 30 + 20(t-11), & 11 < t \leq 13\\ 70 + 15(t-13), & 13 < t \leq 17. \end{cases}
$$

- **Intensity function or time dependent (arrival) rate**
  For $8 \leq t \leq 17$, $m(t) \overset{form.}{=} \int_8^t \lambda(z)\,dz$. As a consequence,

$$
\begin{aligned}
\lambda(t) &= \frac{d\,m(t)}{dt}\\
&= \begin{cases} \frac{d\,[10(t-8)]}{dt} = 10, & 8 \leq t \leq 11\\ \frac{d\,[30+20(t-11)]}{dt} = 20, & 11 < t \leq 13\\ \frac{d\,[70+15(t-13)]}{dt} = 15, & 13 < t \leq 17. \end{cases}
\end{aligned}
$$

(b) *Compute the probability that the time between the arrivals of the first and second customers exceeds one hour.* (2.0)

- **Auxiliary result**
  The time between the n^{th} and $(n+1)^{th}$ events X_{n+1} of a NHPP has survival function given by

$$
P(X_{n+1} = S_{n+1} - S_n > t) \overset{form}{=} \int_0^{+\infty} \lambda(s) e^{-m(t+s)} \frac{[m(s)]^{n-1}}{(n-1)!}\,ds
$$

- **Requested probability**
  For $n = 1$, $t = 1$ and our particular NHPP, $P(X_{1+1} = S_{1+1} - S_1 > 1)$ equals

$$
\begin{aligned}
\int_8^{17} \lambda(s) e^{-m(1+s)}\,ds &= \int_8^{10} 10 e^{-10(1+s-8)}\,ds + \int_{10}^{11} 10 e^{-[30+20(1+s-11)]}\,ds\\
&\quad + \int_{11}^{12} 20 e^{-[30+20(1+s-11)]}\,ds + \int_{12}^{13} 20 e^{-[70+15(1+s-13)]}\,ds\\
&\quad + \int_{13}^{16} 15 e^{-[70+15(1+s-13)]}\,ds + \int_{16}^{17} 15 e^{-[70+15(1+16-13)]}\,ds\\
&= e^{70} \int_8^{10} 10 e^{-10s}\,ds + e^{170} \int_{10}^{11} 10 e^{-20s}\,ds\\
&\quad + e^{170} \int_{11}^{12} 20 e^{-20s}\,ds + e^{110} \int_{12}^{13} 20 e^{-15s}\,ds\\
&\quad + e^{110} \int_{13}^{16} 15 e^{-15s}\,ds + e^{-130} \int_{16}^{17} 15\,ds\\
&\simeq 0.000045.
\end{aligned}
$$

3. *Consider a maternity ward in a hospital and let $N(t)$ be the number of deliveries by time t. Admit that $\{N(t) : t \geq 0\}$ forms a conditional Poisson process with a geometrically distributed random rate (in deliveries per day), Λ, with parameter $p \in (0,1)$.*

(a) *Obtain the probability that there is at least one delivery on a given day.* (1.5)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim Conditional PP(\text{Geometric}(p))$
  $N(t)$ = number of deliveries until time t

- **Random arrival rate**
  $\Lambda \sim \text{Geometric}(p)$, $p \in (0,1)$

- **Requested probability**
Since
$$P[N(t+s)-N(s)=n] \stackrel{form}{=} \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\,dG(\lambda),$$
where G represents the c.d.f. of Λ, we get
$$\begin{aligned}P[N(1)\geq 1] &= 1-P[N(1)=0]\\ &= 1-\int_0^{+\infty} e^{-\lambda}\,dG(\lambda)\\ &= 1-E\left(e^{-\Lambda}\right)\\ &= 1-M_{Geometric(p)}(-1)]\\ &\stackrel{form.}{=} 1-\frac{pe^{-1}}{1-(1-p)e^{-1}}\\ &= \frac{1-e^{-1}}{1-(1-p)e^{-1}}.\end{aligned}$$

(b) *Calculate $P[\Lambda = 1 \mid N(1)=0]$ and compare this probability to $P(\Lambda=1)$.* (2.0)

- **Requested probability**
Since
$$\begin{aligned}P[N(t)=n \mid \Lambda=\lambda] &= e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ n\in\mathbb{N}_0\\ P(\Lambda=\lambda) &= (1-p)^{\lambda-1}p,\ \lambda\in\mathbb{N}\\ P[N(t)=n] &= \int_0^{+\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\,dG(\lambda),\end{aligned}$$
we obtain
$$\begin{aligned}P[\Lambda=1\mid N(1)=0] &\stackrel{T.\,Bayes}{=} \frac{P[N(1)=0\mid\Lambda=1]\times P(\Lambda=1)}{P[N(1)=0]}\\ &\stackrel{(a)}{=} \frac{e^{-1}\times p}{\frac{pe^{-1}}{1-(1-p)e^{-1}}}\\ &= 1-(1-p)e^{-1}\end{aligned}$$

- **Requested comparison**
$$\begin{aligned}P[\Lambda=1\mid N(1)=0] &= 1-(1-p)e^{-1}\\ &> 1-(1-p)\\ &= p\\ &= P(\Lambda=1),\end{aligned}$$
that is, knowing that there were no deliveries on a given day, i.e., that $N(1)$ takes its smallest possible value, expectedly increases the probability that the rate of deliveries Λ takes its smallest possible value 1.

Group 3 — Renewal Processes **8.0 points**

1. *Planes land at an airport according to a renewal process with inter-renewal times (in minutes) with distribution $\chi^2_{(2)}$.*

(a) *Obtain an approximate value to the probability that at least 50 planes land in the first 80 minutes.* (1.5)

- **Renewal process**
$\{N(t): t\geq 0\}$
$N(t)$ = number of planes that landed by time t
- **Inter-renewal times**
$X_i \stackrel{i.i.d.}{\sim} X,\ i\in\mathbb{N}$
$X\sim\chi^2_{(2)}$
$\mu = E(X) \stackrel{form}{=} 2$
$\sigma^2 = V(X) \stackrel{form}{=} 2\times = 4$
- **Requested approximate probability**
$$\begin{aligned}P[N(t)\geq n] &= 1-P[N(t)<n]\\ &\stackrel{form}{\simeq} 1-\Phi\left(\frac{n-t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right)\\ &\stackrel{t=80,\,n=50}{=} 1-\Phi\left(\frac{50-80/2}{\sqrt{80\times 4/2^3}}\right)\\ &\simeq 1-\Phi(1.58)\\ &\stackrel{table}{=} 1-0.9429\\ &= 0.0571.\end{aligned}$$

(b) *Derive the renewal function $m(t)$ of this renewal process, by using the Laplace-Stieltjes transform method and capitalizing on the table of important Laplace transforms in the formulae. Comment the result.* (2.0)

- **Deriving the renewal function**
Since the inter-renewal times are continuous r.v., the LST of the inter-renewal distribution is given by
$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\,dF(x)\\ &= E(e^{-sX})\\ &= M_X(-s)\\ &\stackrel{form.}{=} \left(\frac{1/2}{1/2+s}\right)^{2/2}\\ &\stackrel{form.}{=} \frac{1}{1+2s}.\end{aligned}$$
Moreover, the LST of the renewal function can be obtained in terms of the one of F:
$$\begin{aligned}\tilde{m}(s) &\stackrel{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)}\\ &= \frac{1}{1+2s}\times\frac{1}{1-\frac{1}{1+2s}}\\ &= \frac{1}{2s}.\end{aligned}$$

Taking advantage of the LT in the formulae, we successively get:

$$\begin{aligned}
\frac{dm(t)}{dt} &= LT^{-1}[\tilde{m}(s), t] \\
&= LT^{-1}\left[\frac{1}{2s}, t\right] \\
&= \frac{1}{2} \times LT^{-1}\left[\frac{1}{s}, t\right] \\
&= \frac{1}{2} \\
m(t) &= \int_0^t \frac{1}{2}\, dx \\
&= \frac{t}{2},\ t \geq 0.
\end{aligned}$$

- **Comment**
  The inter-renewal times $X_i \stackrel{i.i.d.}{\sim} \chi^2_{(2)} \equiv \text{Gamma}(2/2, 1/2) \equiv \text{Exponential}(1/2)$, i.e., $\{N(t) : t \geq 0\} \sim PP(\lambda = 1/2)$, thus, the expected result $m(t) = \frac{t}{2}$.

2. *Admit that at time 0 we started to install a component of an electrical system. The duration U of this component is a r.v. with c.d.f. F_U. When the component breaks down it is replaced by a new/similar one and this replacement takes a random time D, independent of U and with c.d.f. F_D.*

   Consider the stochastic process $\{N(t) : t \geq 0\}$, where $N(t)$ represents the number of completed replacements by time t.

   (a) *Derive a renewal-type equation for $E[A(t)]$, the expected value of the age of the stochastic process at time t. (Do not try to solve it!)* (3.0)

   - **Renewal process**
     $\{N(t) : t \geq 0\}$
     $N(t) =$ number of completed replacements by time t
   - **Inter-renewal times**
     $X_i \stackrel{i.i.d.}{\sim} X \stackrel{st}{=} U + D$, $i \in \mathbb{N}$, where
     $U =$ duration of a component $\sim F_U$
     $D =$ time spent replacing a component $\sim F_D$
   - **Important r.v.**
     $A(t) =$ age of the process at time t
   - **Renewal-type equation**
     Applying the renewal argument, that is, conditioning on the time of the first renewal, $X_1 = x$ (which coincides with the time the first component broke down), we have
     – for $0 < x \leq t$,
     $$E[A(t) \mid X_1 = x] = E[A(t - x)]$$
     – for $x > t$,
     $$E[A(t) \mid X_1 = x] = t.$$

Consequently,

$$\begin{aligned}
E[A(t)] &= \int_0^{+\infty} E[A(t) \mid X_1 = x]\, dF(x) \\
&= \int_0^t E[A(t - x)]\, dF(x) + \int_t^{+\infty} t\, dF(x),
\end{aligned}$$

where $F(x) = P(X \leq x) = P(X_1 \leq x) = (F_U \star F_D)(x) = \int_{\mathbb{R}} F_U(x - y)\, dF_D(y)$.[2]

   (b) *Compute the limiting value of $E[A(t)]$ when $U \sim$ Exponential(ξ^{-1}) and $D \sim$ Exponential (λ^{-1}).* (1.5)

   - **R.v.**
     $U \sim \text{Exponential}(\xi^{-1})$
     $D \sim \text{Exponential}(\lambda^{-1})$
   - **Expected value and second moment of X**
     $$\begin{aligned}
     E(X) &= E(U) + E(D) \\
     &= \xi + \lambda \\
     E(X^2) &= V(X) + E^2(X) \\
     &\stackrel{U \perp\!\!\!\perp D}{=} V(U) + V(D) + (\xi + \lambda)^2 \\
     &= \xi^2 + \lambda^2 + (\xi + \lambda)^2 \\
     &= 2 \times (\xi^2 + \xi\lambda + \lambda^2)
     \end{aligned}$$
   - **Requested limit**
     $$\begin{aligned}
     \lim_{t \to +\infty} E[A(t)] &\stackrel{form}{=} \frac{E(X^2)}{2E(X)} \\
     &= \frac{2 \times (\xi^2 + \xi\lambda + \lambda^2)}{2(\xi + \lambda)} \\
     &= \xi + \frac{\lambda^2}{\xi + \lambda}.
     \end{aligned}$$

Group 4 — Renewal Processes (cont'd) **4.0 points**

A factory has an industrial oven which is inspected by an official team according to a renewal process whose inter-arrival times (in years) are i.i.d. r.v. with a Uniform$(1, 2.5)$ distribution. If the inspection team finds out that the maintenance of the industrial oven has been done more than a year ago, the company has to pay a fine of 1050 euros.

(a) *Assume the maintenance of the oven is scheduled exactly $T = 2$ years after each visit of the inspection team. How much does this company spend in fines per year in the long-run?* (2.0)

- **Renewal process**
  $\{N(t) : t \geq 0\}$
  $N(t) =$ number of inspections by time t (time in years)

[2] Let us remind the reader that $\star$ stands for the convolution of the c.d.f. of two independent r.v.

- **Obs.**
  We are going to assume that inspection and maintenance times are negligible.
- **Inter-renewal times**
  $X_n \overset{i.i.d.}{\sim} X,\ n \in \mathbb{N}$
  $X \sim \text{Uniform}(1, 2.5)$
  $f_X(x) = \begin{cases} \frac{1}{2.5-1} = \frac{2}{3}, & 1 < x < 2.5 \\ 0, & \text{otherwise} \end{cases}$
- **Reward renewal process**
  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \geq 0\}$
  $R(t)$ = total amount spent in fines until time t
  R_n = amount spent in fines due to the n^{th} inspection
  $= \begin{cases} 1050, & 1 < X_n < 2 \text{ (i.e., inspection occurred before maintenance)} \\ 0, & 2 \leq X_n < 2.5 \end{cases}$
  $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R),\ n \in \mathbb{N}$
- **Expected inter-renewal time**
  $E(X) \overset{form}{=} \frac{1+2.5}{2} = \frac{7}{4}$
- **Expected amount spent per inspection**

$$
\begin{aligned}
E(R) &= 1050 \times P(1 < X \leq 2) + 0 \times P(2 < X < 2.5) \\
&= 1050 \times \int_1^2 \frac{2}{3}\, dx \\
&= 1050 \times \frac{2}{3} \\
&= 700
\end{aligned}
$$

- **Long-run amount spent in fines per year**
  Since $E(X), E(R) < +\infty$, we can add that

$$
\frac{R(t)}{t} \overset{w.p.1}{\rightarrow} \frac{E(R)}{E(X)},
$$

where $\frac{E(R)}{E(X)}$ represents the amount spent in fines per year in the long-run. Moreover,

$$
\begin{aligned}
\frac{E(R)}{E(X)} &= \frac{700}{\frac{7}{4}} \\
&= 400.
\end{aligned}
$$

(b) *Find the value of $T \in (1, 2.5)$ that minimizes the amount spent in fines per year in the long-run? Comment.* (2.0)

- **(Re)defining R**
  For $T \in (1, 2.5)$,

$$
R = \begin{cases} 1050, & 1 < X \leq T \\ 0, & T < X < 2.5. \end{cases}
$$

Hence,

$$
\begin{aligned}
E(R) &= 1050 \times P(1 < X \leq T) + 0 \times P(T < X < 2.5)
\end{aligned}
$$

$$
\begin{aligned}
E(R) &= 1050 \times \int_1^T \frac{2}{3}\, dx \\
&= 1050 \times \frac{2}{3} \times (T-1) \\
&= 700T - 700.
\end{aligned}
$$

- **Minimizing the long-run amount spent in fines per year**
  Since

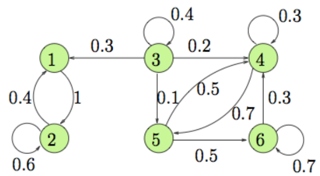
$$
\begin{aligned}
\frac{E(R)}{E(X)} &= \frac{700T - 700}{\frac{7}{4}} \\
&= 400T - 400
\end{aligned}
$$

is an increasing function of $T \in (1, 2.5)$, the long-run amount spent in fines per year takes its minimum value at $T = 1$.

- **Comment**
  The company should adopt $T = 1$ instead of $T = 2$ because for $T = 1$ we get $\frac{E(R)}{E(X)} = 0$, which is a smaller amount than the long-run amount spent by the company in fines per year if the maintenance is scheduled exactly 2 years after each inspection.

Group 5 — Discrete time Markov chains **9.0 points**

1. *Let $\{X_n : n \in \mathbb{N}_0\}$ be a discrete time Markov chain (DTMC), with state space $\mathcal{S} = \{1, \ldots, 6\}$ and the following transition diagram:*



(a) *Obtain the associated transition probability matrix (TPM).* (1.0)

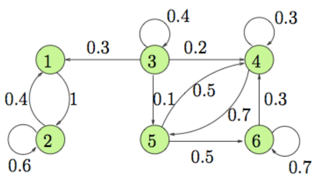
- **DTMC**
  $\{X_n : n \in \mathbb{N}_0\}$
- **State space**
  $\mathcal{S} = \{1, \ldots, 6\}$
- **TPM**
  Follows from the transition diagram above:

$$
\mathbf{P} = [P_{ij}]_{i,j \in \mathcal{S}}
$$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.4 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$

(b) *Classify the states of this DTMC. Is it irreducible? Is it periodic?* (2.0)

- **Transition diagram**



- **Classification of the states**
  The inspection of the transition diagram leads to the following conclusions:
  - states 1 and 2 are accessible from state 3, but no states in $\mathcal{S}\backslash\{1,2\}$ are accessible from 1 and 2; as a result $\{1,2\}$ constitutes a FINITE CLOSED COMMUNICATION CLASS, thus, all its states are POSITIVE RECURRENT [(see Prop. 3.55)];
  - state 3 is not accessible from any states except itself but states 1 and 2 are accessible from state 3; consequently $\{3\}$ is FINITE COMMUNICATING CLASS that is NOT CLOSED, therefore its only state is TRANSIENT [(see Prop. 3.55)];
  - states 4, 5 and 6 communicate with each other, they are accessible from state 3, however, no states in $\mathcal{S}\backslash\{4,5,6\}$ are accessible from 4, 5 and 6; thus, $\{4,5,6\}$ is a FINITE CLOSED COMMUNICATION CLASS, and for this reason all its states are POSITIVE RECURRENT [(see Prop. 3.55)].

- **Irreducibility of the MC?**
  $\mathcal{S}$ does not constitute a sole communicating class, thus, this DTMC is NOT IRREDUCIBLE.

- **Periodicity of the states?**
  This DTMC is NOT PERIODIC. In fact, a close inspection of the transition diagram leads to the conclusion that we can return to: state 1 in $2,3,\ldots$ steps; state 2 in $1,2,3,\ldots$ steps; etc.

2. *Let $\{X_n : n \in \mathbb{N}_0\}$ be the associated DTMC, where X_n represents the weather conditions (sunny, state 1; cloudy, state 2; rainy, state 3) on day n. According to a very simple weather model this DTMC is governed by the following TPM:*

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix}.$$

(a) *Consider the initial distribution $\underline{\alpha} = [\frac{1}{3} \;\; \frac{1}{3} \;\; \frac{1}{3}]$ and find $P(X_2 = 3)$.* (1.5)

- **DTMC**
  $\{X_n : n \in \mathbb{N}_0\}$
  $X_0 =$ initial weather
  $X_n =$ weather on day n

- **State space**
  $\mathcal{S} = \{1,2,3\}$
  $1 =$ sunny
  $2 =$ cloudy
  $3 =$ rainy

- **TPM**

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix}$$

- **Requested probability**
  Since

$$\begin{aligned} \underline{\alpha} &= [P(X_0 = i)]_{i\in\mathcal{S}} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\ \underline{\alpha}^n &= [P(X_n = i)]_{i\in\mathcal{S}} \\ &\overset{form}{=} \underline{\alpha} \times \mathbf{P}^n, \end{aligned}$$

we get

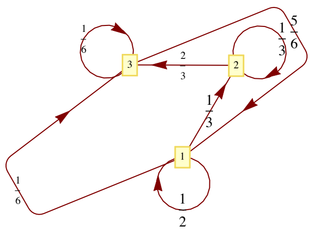
$$\begin{aligned} \underline{\alpha}^2 &= [P(X_2 = i)]_{i\in\mathcal{S}} \\ &= \underline{\alpha} \times \mathbf{P}^2 \\ P(X_2 = 3) &= \underline{\alpha} \times 3^{rd} \text{ column of } \mathbf{P}^2 \\ &= \underline{\alpha} \times \mathbf{P} \times 3^{rd} \text{ column of } \mathbf{P} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix} \times \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix} \\ &= \frac{5}{18}. \end{aligned}$$

(b) *What is the long-run proportion of time the weather is cloudy?* (1.5)

Note: *The following result may come handy:*

$$\begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{5}{6} \\ 1 & \frac{5}{3} & \frac{1}{3} \\ \frac{1}{6} & 1 & \frac{11}{6} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{9} & -\frac{1}{9} & -\frac{1}{3} \\ -\frac{32}{63} & \frac{47}{63} & \frac{2}{21} \\ \frac{13}{63} & -\frac{25}{63} & \frac{11}{21} \end{bmatrix}.$$

- **Important**
  Judging by the transition diagram



  we are dealing with an irreducible DTMC with a finite state space. Hence, all states are positive recurrent[, by Prop. 3.35]. Furthermore, the DTMC is aperiodic: after all, we can return to state 1 in 1, 2, 3, etc. steps.

- **Stationary distribution**
  Since the DTMC is irreducible positive recurrent and aperiodic we can add that

$$\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i,j \in \mathcal{S},$$

  where $\{\pi_j : j \in \mathcal{S}\}$ is the unique stationary distribution and satisfies the following system of equations:

$$\begin{cases} \pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j \in \mathcal{S} \\ \sum_{j\in\mathcal{S}} \pi_j = 1. \end{cases}$$

  Equivalently [(see Prop. 3.68)], the row vector denoting the stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$, is given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

  where:
  $\underline{1} = [1 \quad \cdots \quad 1]$ a row vector with $\#\mathcal{S}$ ones;
  $\mathbf{I}$ = identity matrix with rank $\#\mathcal{S}$;
  $\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}}$ is the TPM;
  $\mathbf{ONE}$ is the $\#\mathcal{S} \times \#\mathcal{S}$ matrix all of whose entries are equal to 1.

  By capitalizing on the inverse in the footnote, we obtain

$$\begin{aligned} \underline{\pi} &= \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1} \\ &= \underline{1} \times \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right)^{-1} \\ &= \underline{1} \times \begin{bmatrix} \frac{3}{2} & \frac{2}{3} & \frac{5}{6} \\ 1 & \frac{5}{3} & \frac{1}{3} \\ \frac{1}{6} & 1 & \frac{11}{6} \end{bmatrix}^{-1} \\ &\simeq [1 \quad 1 \quad 1] \times \begin{bmatrix} \frac{7}{9} & -\frac{1}{9} & -\frac{1}{3} \\ -\frac{32}{63} & \frac{47}{63} & \frac{2}{21} \\ \frac{13}{63} & -\frac{25}{63} & \frac{11}{21} \end{bmatrix} \\ &= \begin{bmatrix} \frac{10}{21} & \frac{5}{21} & \frac{2}{7} \end{bmatrix}. \end{aligned}$$

  Thus, the long-run proportion of days which are cloudy is equal to

$$\pi_2 = \frac{5}{21}.$$

(c) *What is the expected return time to state 1?* (1.0)

- **Expected return time to state 1**
  Since the DTMC is aperiodic π_1 can be is the long-run fraction of time that the chain spends in state 1. As a consequence the mean recurrence time at state 1 is given by

$$\begin{aligned} \mu_{11} &= \frac{1}{\pi_1} \\ &= \frac{21}{10}. \end{aligned}$$

(d) *Determine the expected number of days until it rains, given that the weather is now cloudy.* (2.0)

- **Initial/present state**
  $X_0 = i$

- **Important**
  To obtain the expected number of days until it rains, given $X_0 = i$, we have to consider another CTMC where state 3 (rainy) is absorbing. The associated TPM is

$$\mathbf{P}' = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

- **Requested expected value**
  Let

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

be the substochastic matrix governing the transitions between the states in $T = \{1,2\}$, the class of transient states of this new DTMC, and

$$\tau = \inf\{n \in \mathbb{N}_0 : X_n \notin T\}$$

be the number of transitions/days until it rains. Then, by capitalizing on the fact that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we obtain

$$\begin{aligned}
[E(\tau \mid X_0 = i)]_{i\in T} &= (\mathbf{I}-\mathbf{Q})^{-1} \times \underline{1} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}\right)^{-1} \times \underline{1} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ 0 & \frac{2}{3} \end{bmatrix}^{-1} \times \underline{1} \\
&= \frac{1}{\frac{1}{2}\times\frac{2}{3} - 0}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix} \times \underline{1} \\
&= \begin{bmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
E(\text{days until it rains} \mid \text{the weather is now cloudy}) &= E(\tau \mid X_0 = 2) \\
&= \frac{3}{2}.
\end{aligned}$$

Group 6 — Continuous time Markov chains **7.0 points**

1. *Admit that the number of customers in a self-service system at time t, $X(t)$, is governed by a birth and death process $\{X(t) : t \geq 0\}$ with rates equal to: $\lambda_j = \lambda$, for $j \in \mathbb{N}_0$; and $\mu_j = j\mu$, for $j \in \mathbb{N}$.*

   (a) *Write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \mathbb{N}_0$. (Do not try to solve them!)* (1.5)

   - **Birth and death process**
     $\{X(t) : t \geq 0\}$
     $X(t) =$ number of customers in the queueing system at time t
   - **Birth and death rates**
     $\lambda_j = \lambda$, $j \in \mathbb{N}_0$
     $\mu_j = j\mu$, for $j \in \mathbb{N}$

   - **State space**
     $\mathcal{S} = \mathbb{N}_0$
   - **Kolmogorov's forward differential equations**
     Note that
     $$P_j(t) \;\equiv\; P_{0j}(t) = P[X(t) = j \mid X(0) = 0],\ j \in \mathcal{S}$$
     and since $\mathcal{S} = \mathbb{N}_0$, we are dealing with
     $$\begin{aligned} P_{-1}(t) &= 0 \\ \lambda_{-1} &= 0 \\ \mu_0 &= 0. \end{aligned}$$
     Consequently, Kolmogorov's forward differential equations
     $$\frac{d\,P_j(t)}{dt} = P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j + \mu_j),\ j \in \mathcal{S}$$
     read as follows:
     $$\begin{aligned} \frac{d\,P_0(t)}{dt} &= P_1(t)\,\mu - P_0(t)\,\lambda; \\ \frac{d\,P_j(t)}{dt} &= P_{j-1}(t)\,\lambda + P_{j+1}(t)\,(j+1)\mu - P_j(t)\,(\lambda + j\mu),\ j \in \mathbb{N}. \end{aligned}$$

   (b) *Consider $\rho = \frac{\lambda}{\mu} < +\infty$ and derive the equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t)$.* (2.5)

   - **Ergodicity condition**
     $\rho = \frac{\lambda}{\mu} < +\infty$.[3]
   - **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_j(t)$
     Firstly,
     $$\begin{aligned} P_0 &= \left(1 + \sum_{n=1}^{+\infty} \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\ldots\,\mu_n}\right)^{-1} \\ &= \left(1 + \sum_{n=1}^{+\infty} \frac{\lambda\times\lambda\times\cdots\times\lambda}{\mu\times 2\mu\times\cdots\times n\mu}\right)^{-1} \\ &= \left(1 + \sum_{n=1}^{+\infty} \frac{\lambda^n}{n!\mu^n}\right)^{-1} \\ &= \left(\sum_{n=0}^{+\infty} \frac{\rho^n}{n!}\right)^{-1} \\ &= e^{-\rho}. \end{aligned}$$
     Secondly,
     $$\begin{aligned} P_j &= P_0 \times \frac{\lambda_0\,\lambda_1\,\ldots\,\lambda_{j-1}}{\mu_1\,\mu_2\,\ldots\,\mu_j} \\ &= P_0 \times \frac{\lambda\times\lambda\times\cdots\times\lambda}{\mu\times 2\mu\times\cdots\times j\mu} \\ &= e^{-\rho}\frac{\rho^j}{j!},\ j \in \mathbb{N}. \end{aligned}$$

[3] The traffic intensity ρ does not need to be smaller than 1 because we are essentially dealing with a self-service system.

Finally, we can state that

$$P_j \equiv P_{Poisson(\rho)}(j).$$

2. *Consider a queueing system in which each customer always finds a free server. Admit the system is initially empty, the customers arrive to the system according to a Poisson process with a rate equal to $\lambda = 3$ customers per hour and that the service times are i.i.d. r.v. exponentially distributed with parameter $\mu = 2^{-1}$.*

(a) *Obtain the probability that there are more than 2 customers in the system at time $t = 12$.* (2.0)

- **Birth and death queueing system**
  $M/M/\infty$
- **Arrival process/rate**
  $PP(\lambda)$
  $\lambda = 3$ customers per hour
- **Service times/rate**
  $S_i \stackrel{i.i.d.}{\sim}$ Exponential$(\mu^{-1} = 2)$
  $\mu = \frac{1}{2}$ (1 customer every 2 hours)
- **Servers**
  $m = \infty$ because we are dealing with a queueing system in which each customer always finds a free server
- **R.v.**
  $(X(t) \mid X(0) = 0)$ = number of customers in the system at time t, given that the system is initially empty
- **Requested probability**
  According to the formulae, we have for the $M/M/\infty$

$$(X(t) \mid X(0) = 0) \sim \text{Poisson}(\lambda\,(1 - e^{-\mu t})/\mu),$$

therefore

$$\begin{aligned} P[X(12) > 2 \mid X(0) = 0] &= 1 - P[X(12) \leq 2 \mid X(0) = 0] \\ &= 1 - F_{Poisson(3\times(1-e^{-\frac{1}{2}\times 12})/\frac{1}{2})}(2) \\ &\simeq 1 - F_{Poisson(5.985127)}(2) \\ &\simeq 1 - F_{Poisson(6)}(2) \\ &\stackrel{table}{=} 1 - 0.0620 \\ &= 0.9380. \end{aligned}$$

(b) *Admit we aim at a decrease of the expected value of customers in the system in the long-run. Is there any advantage in having service times that are i.i.d. r.v. with a $\chi^2_{(3)}$ distribution?* (1.5)

- **Performance measure of the original queueing system**

$L_s^{M/M/\infty}$ = number of customers in the $M/M/\infty$ queueing system (in the long-run)
$L_s^{M/M/\infty} \sim$ Poisson$(\lambda/\mu = 3/\frac{1}{2} = 6)$ (see formulae)
$E(L_s^{M/M/\infty}) = 6$

- **Alternative queueing system**
  $M/G/\infty$
  $\lambda = 3$
  $\mu_G = \frac{1}{E(service\,time)} = \frac{1}{E[\chi^2_{(3)}]} = \frac{1}{3}$
- **Performance measure of the alternative queueing system**
  $L_s^{M/G/\infty} = \lim_{t\to+\infty}(X(t) \mid X(0) = 0)$ = number of customers in the $M/G/\infty$ queueing system (in the long-run)
  $L_s^{M/G/\infty} \sim$ Poisson$(\lambda/\mu_G = 3/\frac{1}{3} = 9)$ (see formulae)
  $E(L_s^{M/G/\infty}) = 9$
- **Comment**
  Since $E(L_s^{M/G/\infty}) = 9 > 6 = E(L_s^{M/M/\infty})$, there in no advantage in having service times that are i.i.d. r.v. with a $\chi^2_{(3)}$ distribution.