# Chapter 7

# Modelling thermal systems

— Mais qu'entends-tu par le vide ? demanda Michel, est-ce le vide absolu ?
— C'est le vide absolument privé d'air.
— Et dans lequel l'air n'est remplacé par rien ?
— Si. Par l'éther, répondit Barbicane.
— Ah ! Et qu'est-ce que l'éther ?
— L'éther, mon ami, c'est une agglomération d'atomes impondérables, qui, relativement à leurs dimensions, disent les ouvrages de physique moléculaire, sont aussi éloignés les uns des autres que les corps célestes le sont dans l'espace. Leur distance, cependant, est inférieure à un trois-millionièmes de millimètre. Ce sont ces atomes qui, par leur mouvement vibratoire, produisent la lumière et la chaleur, en faisant par seconde quatre cent trente trillions d'ondulations, n'ayant que quatre à six dix-millièmes de millimètre d'amplitude.

Jules VERNE (1828 — †1905), *Autour de la Lune*, 5

This chapter concerns thermal systems. You will study heat transfer in depth in another course, but simple cases can be modelled with a formulation similar to that employed to the systems in the previous chapters.

## 7.1 Energy, effort and flow

There is, however, an important difference. By convention:

- **temperature** $T$ is the effort variable; *Temperature*
- **heat flow rate** $q$ is the flow variable. *Heat flow rate*

However, heat is energy; it is in fact kinetic energy of molecules and atoms. (As you may know, an omnipresent æther was postulated for some centuries to explain the propagation of heat and especially of light, but this hypothesis, of which you may find a popular explanation at the beginning of this chapter, has been abandoned for about one century, having been by then contested for decades because of experimental results with which it could not reconciled.) *Heat is energy*

Consequently, it is not true in thermal systems that energy is the integral of the product of effort and flow — see (4.30) —, as the variable used for flow is an energy rate itself. And, as a result, the parallels with the other types of systems we have been studying are not perfect.

## 7.2 Basic components of a thermal system

*Heat accumulator*

There is only one type of accumulator, that of heat, i.e. of flow:

$$H(t) = m\, C_p(T(t) - T(0)) \tag{7.1}$$

*Specific heat*

Here $H(t) = \int_0^t q(t)\,\mathrm{d}t$ is the accumulated heat in mass $m$, which has a **specific heat** $C_p$ and is heated from temperature $T(0)$ to temperature $T(t)$.

Dissipation can take place in three different ways:

*Conduction*

- In **conduction** there is no macroscopic movement of the solid or fluid that undergoes the process. Heat is transmitted, or rather diffused, at the molecular and atomic levels. The heat flow, which we assume to be positive in the sense of increasing values of position $x$, is

$$q(t) = -kA\frac{\partial T(x,t)}{\partial x} \tag{7.2}$$

  where $A$ is the cross-sectional area and $k$ is the thermal conductivity. Assuming that the temperature distribution over $x$ is linear over a distance $L$, (7.2) becomes

$$q(t) = \underbrace{\frac{kA}{L}}_{\text{conduction heat transfer coefficient } h_c} (T(0,t) - T(L,t)) \tag{7.3}$$

  Notice that the minus sign is gone because $\frac{\partial T(x,t)}{\partial x} = \lim_{L\to 0} \frac{T(L,t)-T(0,t)}{L}$.

*Convection*

*Free convection*

*Forced convection*

- In **convection** there is macroscopic movement of the fluid where heat transfer is taking place. In solids matter cannot move and this way and consequently there can be no convection. If the fluid movement is due solely to the temperature gradients, this is called **free convection**; if fluid movement is due at least in part to some other reason (like fluid flow in pipes, or a blower), this is called **forced convection**. In any case, the heat flow, again assumed positive in the sense of increasing values of position $x$, is

$$q(t) = \underbrace{hA}_{\text{convection heat transfer coefficient } h_h} (T(0,t) - T(L,t)) \tag{7.4}$$

  Here $h$ is the convection heat transfer coefficient, and $A$ is the cross-sectional area over which heat transfer takes place.

*Radiation*

- In **radiation** heat is propagated by the emission of photons. It is the only heat transmission that can take place in a vacuum. It is also the only one corresponding to a nonlinear law:

$$q(t) = C_r A \left(T_1^4 - T_2^4\right) \tag{7.5}$$

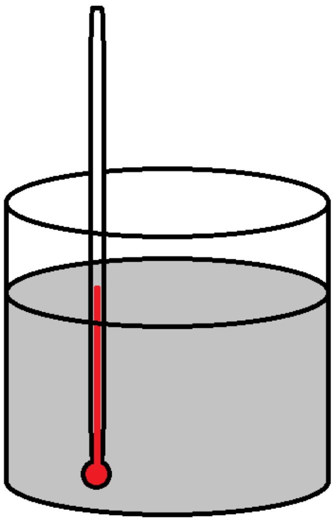

Figure 7.1: A thermometer immersed in a fluid.

Here $C_r$ is a proportionality constant that we need not delve into. This law, of course, can be linearised around some point, and the result will be approximately valid in a vicinity thereof; for instance:

$$q(t) = \underbrace{C_r A \left(T_1 + T_2\right) \left(T_1^2 + T_2^2\right)}_{\text{radiation heat transfer coefficient } h_r} \left(T_1 - T_2\right) \tag{7.6}$$

Notice that all cases — conduction, convection, and linearised radiation — can be reduced to the following form:

$$\Delta T = Rq \tag{7.7}$$

Here $R$ is thermal resistance and $\Delta T$ is the temperature difference. *Thermal resistance*

The relations are summed up in Table 7.1.

**Example 7.1.** The reading of a mercury or alcohol thermometer at time $t = 0$ is $T_t(0)$. At that instant, it is immersed in a fluid at temperature $T_f$ (see Figure 7.1). Let the thermal resistance of the glass be $R$, and the specific heat of mass $m$ of mercury or alcohol be $C_p$. How does $T_t(t)$ evolve?

This can be seen using an equivalent electrical circuit. Both temperatures become tensions; the thermal resistance of the glass becomes a resistance $R$ and the heat accumulator becomes a capacitor $C$. So the equivalent circuit is the one in Figure 7.2. The corresponding transfer function — remember (5.15) — is

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs} \tag{7.8}$$

Temperature $T_f$ is constant and is applied from $t = 0$ on, so the Laplace transform of the input is

$$\mathscr{L}\left[\left(T_f - T_t(0)\right) H(t)\right] = \frac{T_f}{s} \tag{7.9}$$

Table 7.1: Effort, flow, accumulators and dissipators in thermal systems

| | Thermal system | SI |
|---|---|---|
| effort $e$ | temperature $T$ | K or °C |
| flow $f$ | heat flow rate $q$ | W |
| effort accumulator | — | — |
| accumulated effort $e_a = \int e \, \mathrm{d}t$ | — | — |
| relation between accumulated effort and flow $e_a = \varphi(f)$ | — | — |
| accumulated energy $E_e = \int e_a \, \mathrm{d}f$ | — | — |
| flow accumulator | heat accumulator with mass $m$ and specific heat $C_p$ | kg × J kg$^{-1}$ K$^{-1}$ |
| accumulated flow $f_a = \int f \, \mathrm{d}t$ | heat $H = \int q \, \mathrm{d}t$ | J |
| relation between accumulated flow and effort $f_a = \varphi(e)$ | heat $H = m \, CpT$ | |
| accumulated energy $E_f = \int f_a \, \mathrm{d}e$ | — | — |
| dissipator | thermal resistance with resistance $R$ | K/W |
| relation between effort and flow $e = \varphi(f)$ | $T = Rq$ | |
| dissipated energy $E_d = \int f \, \mathrm{d}e$ | — | — |

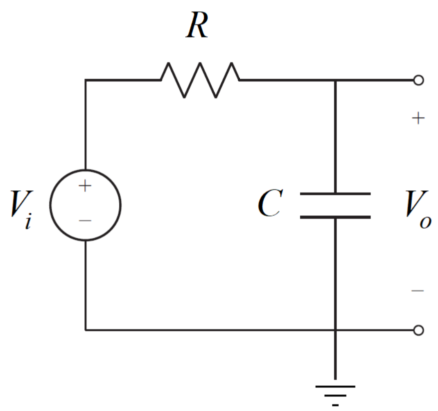


Figure 7.2: Electrical circuit equivalent to the thermal system in Figure 7.1.

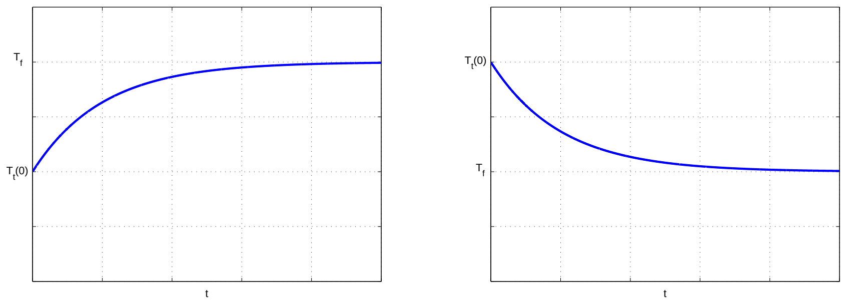


Figure 7.3: Evolution of temperature $T_t(t)$ in the system Figure 7.1 when $T_t(t) < T_f$ (left) and when $T_t(t) > T_f$ (right).

Notice that the amplitude of the temperature change is $T_f - T_t(0)$; since the Heaviside function begins at 0 and ends in 1, we must multiply it by $T_f - T_t(0)$ and take into account the different initial value also when calculating the output we want to know:

$$T_t(t) - T_t(0) = \mathscr{L}^{-1}\left[\frac{T_f}{s}\frac{1}{1+RmC_p s}\right]$$
$$= T_f \mathscr{L}^{-1}\left[\frac{\frac{1}{RmC_p}}{s(\frac{1}{RmC_p}+s)}\right] = T_f\left(1 - e^{-\frac{1}{RmC_p}t}\right) \quad (7.10)$$

We conclude that temperature $T_t(t)$ begins at $T_t(0)$, ends at $T_f$, and changes exponentially with time. This is illustrated in Figure 7.3. □

# Glossary

> Aber in Phantásien waren fast alle Wesen, auch die Tiere, mindestens zweier Sprachen mächtig: Erstens der eigenen, die sie nur mit ihresgleichen redeten und die kein Außenstehender verstand, und zweitens einer allgemeinen, die man Hochphantäsisch oder auch die Große Sprache nannte. Sie beherrschte jeder, wenngleich manche sie in etwas eigentümlicher Weise benützten.
>
> Michael Ende (1929 — †1995), *Die unendliche Geschichte von A biz Z*, 2

**conduction** condução
**convection** convecção
**forced convection** convecção forçada
**free convection** convecção livre
**heat accumulator** acumulador de calor
**heat flow rate** fluxo de calor
**radiation** radiação
**specific heat** calor específico
**thermal resistance** resistência térmica
**temperature** temperatura

# Exercises

1. Consider the Wave Energy Converter of Figure 3.2, when submerged in sea water at constant temperature $T_{sea}$. Assume that the air inside the device has a homogeneous temperature $T_{air}(t)$ and is heated by the device's Power Take-Off (PTO) mechanism, at temperature $T_{PTO}(s)$, that delivers an electrical power $P(t)$ to the electrical grid with an efficiency $\eta$. (See Figure 7.4.) Heat transfer from the PTO to the air inside the device takes place by convection with a constant coefficient $h$ and for area $A$; neglect the thermal capacity of the metallic WEC itself; consider that $C_{air}$ and $C_{PTO}$ are the thermal capacities of the air and the PTO, that have masses $m_{air}$ and $m_{PTO}$. Let $T(t) = T_{air}(t) - T_{PTO}(t)$. Find transfer function $\frac{T(s)}{P(s)}$.

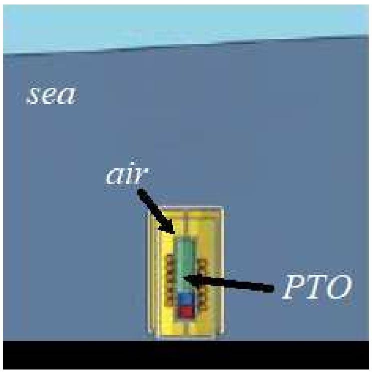


Figure 7.4: The Wave Energy Converter of Exercise 1.

2. Explain which of the electrical systems in Figure 5.19 have a thermal equivalent and which do not, and why.