Department of Mathematics, IST — Probability and Statistics Unit

Introduction to Stochastic Processes

2nd. Test 2nd. Semester — **2013/14**

Duration: **1h30m** **2014/06/11 — 3PM**, Room V1.14

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes **2.0 points**

The number of inspections by a supervisor to an industrial plant is governed by a delayed renewal process such that: **(2.0)**

- the first inspection time follows an exponential distribution with unit mean;
- the subsequent inter-inspection times follow a hyper-exponential distribution with parameters $(1, \frac{1}{2}; \; \frac{1}{2}, \frac{1}{2})$.

Derive the renewal function of this process.

Note/hint: Admit that the duration of any inspection is insignificant compared to the time between consecutive inspections; capitalize on the fact that $\frac{2(1+2s)}{s(3+4s)} = \frac{2}{3} \times \frac{1}{s} + \frac{1}{3} \times \frac{1}{\frac{3}{4}+s}$.

- **Delayed renewal process**
  $\{N_D(t) : t \geq 0\}$
  $N_D(t) =$ number of inspections done by time t
- **Inter-renewal times**
  X_i independent r.v., $i \in \mathbb{N}$
  $X_1 \sim \text{Exp}(1)$
  $X_i \stackrel{i.i.d.}{\sim} \text{Hyper-exp}(1, \frac{1}{2}; \; \frac{1}{2}, \frac{1}{2})$, $i \in \mathbb{N} \backslash \{1\}$
- **Important**
  $G(x) = P(X_1 \leq x)$
  $F(x) = P(X_i \leq x)$, $i \in \mathbb{N} \backslash \{1\}$
  $\frac{dF(x)}{dx} \stackrel{form.}{=} \frac{1}{2} \times f_{Exp(1)}(x) + \frac{1}{2} \times f_{Exp(1/2)}(x)$
- **Deriving the renewal function**
  Since the inter-renewal times are continuous r.v., the LST of the two inter-renewal distributions are given by

$$
\begin{aligned}
\tilde{G}(s) &= \int_{0^-}^{+\infty} e^{-sx} \, dG(x) \\
&= E(e^{-sX_1}) \\
&= M_{Exp(1)}(-s) \\
&\stackrel{form.}{=} \frac{1}{1+s}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx} \, dF(x) \\
&= \frac{1}{2} \times M_{Exp(1)}(-s) + \frac{1}{2} \times M_{Exp(1/2)}(-s) \\
&\stackrel{form.}{=} \frac{1}{2} \times \frac{1}{1+s} + \frac{1}{2} \times \frac{1/2}{1/2+s} \\
&= \frac{1}{2} \times \left(\frac{1}{1+s} + \frac{1}{1+2s} \right) \\
&= \frac{2+3s}{2(1+s)(1+2s)}.
\end{aligned}
$$

Moreover, the LST of the renewal function of a delayed renewal process, $m(t) = E[N_D(t)]$, can be obtained in terms of the LST of F and G:

$$
\begin{aligned}
\tilde{m}(s) &= \int_{0^-}^{+\infty} e^{-sx} \, dm(x) \\
&\stackrel{form.}{=} \frac{\tilde{G}(s)}{1 - \tilde{F}(s)} \\
&= \frac{\frac{1}{1+s}}{1 - \frac{2+3s}{2(1+s)(1+2s)}} \\
&= \frac{2(1+2s)}{2(1+s)(1+2s) - (2+3s)} \\
&= \frac{2(1+2s)}{2(1+3s+2s^2) - (2+3s)} \\
&= \frac{2(1+2s)}{3s+4s^2} \\
&= \frac{2(1+2s)}{s(3+4s)} \\
&\stackrel{hint}{=} \frac{2}{3} \times \frac{1}{s} + \frac{1}{3} \times \frac{1}{3/4+s}.
\end{aligned}
$$

Taking advantage of the LT in the formulae, we successively get:

$$
\begin{aligned}
\frac{dm(t)}{dt} &= LT^{-1}\left[\tilde{m}(s), t\right] \\
&= LT^{-1}\left[\frac{2}{3} \times \frac{1}{s} + \frac{1}{3} \times \frac{1}{3/4+s}, t\right] \\
&= \frac{2}{3} \times LT^{-1}\left[\frac{1}{s}, t\right] + \frac{1}{3} \times LT^{-1}\left[\frac{1}{3/4+s}, t\right] \\
&= \frac{2}{3} + \frac{1}{3} \times e^{-3t/4}
\end{aligned}
$$

$$
\begin{aligned}
m(t) &= \int_0^t \left(\frac{2}{3} + \frac{1}{3} \times e^{-3x/4} \right) dx \\
&= \left(\frac{2x}{3} + \frac{1}{3} \times \frac{4}{3} e^{-3x/4} \right) \Big|_0^t \\
&= \frac{2t}{3} + \frac{4}{9} \left(1 - e^{-3t/4} \right), \; t \geq 0.
\end{aligned}
$$

Group 2 — Discrete time Markov chains **9.0 points**

1. Evaristo uses four expressions — *Duh!* (state 1), *Bummer!!* (state 2), *Dunno!!!* (state 3), and *Whoops!!!!* (state 4) — according to a DTMC with the following TPM:

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0 & 0.3 & 0.4 \\ 0.3 & 0.5 & 0 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.5 \end{bmatrix}.$$

(a) Find the long-run fraction of time Evaristo uses the expression *Bummer!!* (**2.0**)

Note: Recall that $\underline{\pi}\,\mathbf{P} = \underline{\pi}$ and check the footnote![1]

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  $X_n = n^{th}$ expression Evaristo used
- **State space**
  $\mathcal{S} = \{1, 2, 3, 4\}$
  $1 =$ *Duh!*
  $2 =$ *Bummer!!*
  $3 =$ *Dunno!!!*
  $4 =$ *Whoops!!!!*
- **TPM**

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0 & 0.3 & 0.4 \\ 0.3 & 0.5 & 0 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.5 \end{bmatrix}$$

- **Obs.**
  We are dealing with an irreducible DTMC with finite state space. Hence, all states are positive recurrent[, by Prop. 3.35]. Furthermore, the DTMC seems to be aperiodic.
- **Stationary distribution**
  Since the DTMC is irreducible, positive recurrent and aperiodic we can add that

$$\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i, j \in \mathcal{S},$$

where the row vector $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$ is the unique stationary distribution satisfying

$$\begin{cases} \pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j \in \mathcal{S} \\ \sum_{j\in\mathcal{S}} \pi_j = 1 \end{cases}$$

$$\begin{cases} \underline{\pi}\,\mathbf{P} = \underline{\pi} \\ \underline{\pi}\,\underline{1}^\top = 1, \end{cases}$$

where $\underline{1} = [1 \ \cdots \ 1]$ a row vector with $\#\mathcal{S}$ ones.
Using the first result in the footnote and considering $\pi_j = \frac{a_j}{\sum_{i\in\mathcal{S}} a_i}$ yields

[1] The following results may come handy in this and the next lines: $[253 \ 135 \ 117 \ 256] \times \mathbf{P} = [253 \ 135 \ 117 \ 256]$;
$\begin{bmatrix} 0.4 & -0.1 & -0.1 \\ -0.3 & 1 & -0.3 \\ -0.3 & -0.5 & 1 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 425 & 75 & 65 \\ 195 & 185 & 75 \\ 225 & 115 & 185 \end{bmatrix}$.

$$\begin{aligned}
[253 \ 135 \ 117 \ 256]\,\mathbf{P} &= [253 \ 135 \ 117 \ 256] \\
\frac{1}{253+135+117+256}[253 \ 135 \ 117 \ 256]\,\mathbf{P} &= \frac{1}{761}[253 \ 135 \ 117 \ 256] \\
\underline{\pi} &= \frac{1}{761}[253 \ 135 \ 117 \ 256] \\
\underline{\pi} &\simeq [0.333 \ 0.177 \ 0.154 \ 0.336]\,.
\end{aligned}$$

- **Requested probability**
  Thus, the long-run fraction of time Evaristo uses the expression *Bummer!!* is $\pi_2 \simeq 0.177$.
- **[Stationary distribution** (alternative!)
  The row vector denoting the stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$, is given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

where:
$\underline{1} = [1 \ \cdots \ 1]$ a row vector with $\#\mathcal{S}$ ones;
$\mathbf{I} =$ identity matrix with rank $\#\mathcal{S}$;
$\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}}$ is the TPM;
ONE is the $\#\mathcal{S} \times \#\mathcal{S}$ matrix all of whose entries are equal to 1.

Since the footnote does not provide any inverse of a 4×4 matrix, we are bound to use a calculator and to obtain

$$\begin{aligned}
\underline{\pi} &= \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1} \\
&= \underline{1} \times \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0 & 0.3 & 0.4 \\ 0.3 & 0.5 & 0 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right)^{-1} \\
&= \underline{1} \times \begin{bmatrix} 1.4 & 0.9 & 0.9 & 0.8 \\ 0.7 & 2 & 0.7 & 0.6 \\ 0.7 & 0.5 & 2 & 0.8 \\ 0.9 & 0.8 & 0.8 & 1.5 \end{bmatrix}^{-1} \\
&\simeq [1 \ 1 \ 1 \ 1] \times \begin{bmatrix} 1.288765 & -0.349869 & -0.303219 & -0.385677 \\ -0.222405 & 0.661958 & -0.092970 & -0.096583 \\ -0.169842 & 0.018068 & 0.682326 & -0.280552 \\ -0.564060 & -0.152760 & -0.132392 & 1.099212 \end{bmatrix} \\
&\simeq [0.332457 \ 0.177398 \ 0.153745 \ 0.336399]\,.
\end{aligned}$$

Thus, the long-run fraction of time Evaristo uses the expression *Bummer!!* is equal to the sum of the entries of the 2nd. column of $(\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1}$: $\pi_2 \simeq 0.177397$.]

(b) Given that Evaristo just used the expression *Duh!*, determine the expected number of transitions until he says *Whoops!!!!* (**2.0**)

- **Initial/present state**
  $X_1 = i$

- **Important**
To obtain the expected number of transitions until Evaristo says *Whoops!!!!*, given that $X_1 = i$, we have to consider another DTMC where state 4 (*Whoops!!!!*) is absorbing. The associated TPM is

$$\mathbf{P}' = \begin{bmatrix} 0.6 & 0.1 & 0.1 & 0.2 \\ 0.3 & 0 & 0.3 & 0.4 \\ 0.3 & 0.5 & 0 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Requested expected value**
Let

$$\mathbf{Q} = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.3 & 0 & 0.3 \\ 0.3 & 0.5 & 0 \end{bmatrix}$$

be the substochastic matrix governing the transitions between the states in $T = \{1, 2, 3\}$, the class of transient states of this new DTMC, and

$$\tau = \inf\{n \in \mathbb{N} : X_n \notin T\}$$

be the number of transitions until Evaristo says *Whoops!!!!* Then [(see Prop. 3.116)] the 2nd. result in the footnote yields

$$\begin{aligned} [E(\tau \mid X_1 = i)]_{i \in T} &\overset{form.}{=} (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{1} \\ &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.3 & 0 & 0.3 \\ 0.3 & 0.5 & 0 \end{bmatrix} \right)^{-1} \times \underline{1} \\ &= \begin{bmatrix} 0.4 & -0.1 & -0.1 \\ -0.3 & 1 & -0.3 \\ -0.3 & -0.5 & 1 \end{bmatrix}^{-1} \times \underline{1} \\ &= \frac{1}{128} \begin{bmatrix} 425 & 75 & 65 \\ 195 & 185 & 75 \\ 225 & 115 & 185 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{565}{128} \\ \frac{455}{128} \\ \frac{525}{128} \end{bmatrix}. \end{aligned}$$

Finally, E(transitions until Evaristo says *Whoops!!!!* | he just said *Duh!*) is equal to

$$\begin{aligned} E(\tau \mid X_1 = 1) &= \frac{565}{128} \\ &\simeq 4.414063. \end{aligned}$$

(c) Find the probability that Evaristo will say *Bummer!!* before *Dunno!!!*, considering once again that he just used the expression *Duh!* **(2.0)**

Note: You may have to consider states 2 and 3 absorbing, eventually relabel the states, identify substochastic matrices $\mathbf{Q}$ and $\mathbf{R}$ and calculate $(\mathbf{I} - \mathbf{Q})^{-1} \times \mathbf{R}$.

- **Important**
To calculate the requested probability, we have to consider once again another DTMC. In this case, states 2 (*Bummer!!*) and 3 (*Dunno!!!*) are absorbing and the associated TPM equals

$$\mathbf{P}^\star = \begin{bmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The substochastic matrices governing the transitions between the transient states of this DTMC and the transitions from the transient to the absorbing states are

$$\begin{aligned} \mathbf{Q} &= \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.5 \end{bmatrix} \\ \mathbf{R} &= \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \end{aligned}$$

respectively.

- **Requested probability**
Keeping in mind that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we get

$$\begin{aligned} \mathbf{U} &= [P(\text{reach absorbing state } k \mid X_1 = i)]_{i \in T, k \notin T} \\ &= (\mathbf{I} - \mathbf{Q})^{-1} \times \mathbf{R} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.5 \end{bmatrix} \right)^{-1} \times \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.5 \end{bmatrix}^{-1} \times \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \\ &= \frac{1}{0.4 \times 0.5 - (-0.2) \times (-0.1)} \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \\ &= \frac{1}{0.18} \begin{bmatrix} 0.09 & 0.09 \\ 0.09 & 0.09 \end{bmatrix} \\ &\simeq \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \end{aligned}$$

Thus, given that he just used the expression *Duh!*, the probability that Evaristo will say *Bummer!!* before *Dunno!!!* is 0.5.

2. Let $\{X_n : n \in \mathbb{N}_0\}$ be a branching process such that the number of offspring per individual has a Poisson distribution with parameter $\lambda = 2$.

(a) Starting with a single individual (i.e., $X_0 = 1$), verify that the extinction probability is $\pi \simeq 0.203188$. **(1.5)**

- **Branching process**
  $\{X_n : n \in \mathbb{N}_0\}$
  $X_n =$ size of generation n
- **Initial state**
  $X_0 = 1$ (single initial individual)
- **State space**
  $\mathcal{S} = \mathbb{N}_0$
- **Number of offspring per individual**
  $Z_l \equiv Z_{l,n} =$ number of offspring of the l^{th} individual of generation n
  $Z_l \stackrel{i.i.d.}{\sim} \text{Poisson}(\lambda),\ l \in \mathbb{N}$
  $P_j = P(Z_l = j) = e^{-\lambda}\frac{\lambda^j}{j!},\ j \in \mathbb{N}_0$
- **Obs.**
  $X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n \in \mathbb{N}$
- **Probability of extinction**
  Since $E(Z_l) = \lambda = 2 > 1$, the probability of extinction,

  $$\pi \stackrel{form.}{=} \lim_{n\to+\infty} P(X_n = 0 \mid X_0 = 1),$$

  is the smallest positive number satisfying

  $$\begin{aligned} \pi &\stackrel{form.}{=} \sum_{j=0}^{+\infty} \pi^j \times P_j \\ &= \sum_{j=0}^{+\infty} \pi^j \times e^{-\lambda}\frac{\lambda^j}{j!} \\ &= P_{Z_l}(\pi), \end{aligned}$$

  where $P_{Z_l}(s) = E(s^{Z_l}) \stackrel{form.}{=} e^{-\lambda(1-s)}$, $|s| \leq 1$, denotes the p.g.f. of the discrete r.v. $Z_l \sim \text{Poisson}(\lambda)$. Hence

  $$\pi = e^{-\lambda(1-\pi)}.$$

  Furthermore, $\pi = 0.203188$ satisfies $\pi = e^{-\lambda(1-\pi)}$, after all

  $$e^{-2\times(1-0.203188)} \simeq 0.203188.$$

(b) Suppose that, instead of starting with a single individual, $X_0 \sim \text{Poisson}(1)$. **(1.5)**
Obtain the extinction probability in this case in terms of π obtained in (a).

- **New initial state**
  $X_0 = Z_0 \sim \text{Poisson}(1)$
- **New probability of extinction**
  Then using the total probability law and the fact that the offspring are produced independently

  $$\begin{aligned} P(\text{extinction}) &= \sum_{j=0}^{+\infty} P(\text{extinction} \mid X_0 = j) \times P(X_0 = j) \\ &= \sum_{j=0}^{+\infty} [P(\text{extinction} \mid X_0 = 1)]^j \times P(X_0 = j) \end{aligned}$$

$$\begin{aligned} P(\text{extinction}) &= \sum_{j=0}^{+\infty} \pi^j \times P(X_0 = j) \\ &= P_{X_0}(\pi) \\ &\stackrel{form.}{=} e^{-(1-\pi)} \\ &\simeq e^{-(1-0.203188)} \\ &\simeq 0.450764. \end{aligned}$$

- **[Obs.**
  Interestingly, if $X_0 \sim \text{Poisson}(\lambda = 2)$ then we would get $P(\text{extinction}) = e^{-\lambda(1-\pi)} = \pi \simeq 0.203188$.]

Group 3 — Continuous time Markov chains **9.0 points**

1. Consider an assembly line where parts arrive according to a Poisson process with rate λ parts per minute. A part is immediately processed upon arrival and it takes an exponentially distributed time with a rate of μ parts per minute to process it.

   (a) Write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0\,j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \mathbb{N}_0$, where $X(t)$ represents the number of parts being processed at time t. (Do not try to solve the differential equations!) **(1.5)**

   - **CTMC**
     $\{X(t) : t \geq 0\}$
     $X(t) =$ number of parts being processed at time t
   - **Birth and death rates**
     Since the inter-arrival times are i.i.d., exponentially distributed r.v. which we assume to be independent of the processing times, which are also i.i.d. exponentially distributed r.v., $\{X(t) : t \geq 0\}$ is indeed a birth-death process with rates

     $$\begin{aligned} \lambda_j &= \lambda,\ j \in \mathbb{N}_0 \\ \mu_j &= j\mu,\ j \in \mathbb{N}. \end{aligned}$$

   - **Kolmogorov's forward differential equations**
     Note that

     $$\begin{aligned} P_j(t) &\equiv P_{0\,j}(t) = P[X(t) = j \mid X(0) = 0],\ j \in \mathbb{N}_0 \\ P_{-1}(t) &= 0 \\ \lambda_{-1} &= 0 \\ \mu_0 &= 0 \end{aligned}$$

     therefore the Kolmogorov's forward differential equations

     $$\frac{d\,P_j(t)}{dt} \stackrel{form.}{=} P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j + \mu_j),\ j \in \mathbb{N}_0,$$

     reads as follows:

     $$\begin{aligned} \frac{d\,P_0(t)}{dt} &= P_1(t)\,\mu - P_0(t)\,\lambda \\ \frac{d\,P_j(t)}{dt} &= P_{j-1}(t)\,\lambda + P_{j+1}(t)\,j\mu - P_j(t)\,(\lambda + j\mu),\ j \in \mathbb{N}. \end{aligned}$$

(b) Show that the p.g.f. of $(X(t) \mid X(0) = 0)$, $P(z,t) = E\left[z^{X(t)} \mid X(0) = 0\right]$, satisfies the following partial differential equation **(2.0)**

$$\frac{\partial P(z,t)}{\partial t} + \lambda\,(1-z)\,P(z,t) - \mu\,(1-z)\frac{\partial P(z,t)}{\partial z} = 0.$$

- **R.v.**
  $(X(t) \mid X(0) = 0)$
- **Rewriting the Kolmogorov's forward differential equations**
  By multiplying the j^{th} Kolmogorov's forward differential equation by z^j, summing in $j \in \mathcal{S}$ and noting that

$$\sum_{j\in\mathcal{S}} z^j \times \frac{d\,P_j(t)}{dt} \stackrel{form.}{=} \frac{\partial P(z,t)}{\partial t}$$

$$\frac{\partial P(z,t)}{\partial z} \stackrel{form.}{=} \sum_{j\in\mathcal{S}} j z^{j-1} \times P_j(t) = \sum_{j\in\mathcal{S}} (j+1) z^j \times P_{j+1}(t),$$

we can reduce the set of Kolmogorov's forward differential equations derived in (a) to a SINGLE PARTIAL DIFFERENTIAL EQUATION, whose solution is the p.g.f. of the r.v. $(X(t) \mid X(0) = i)$:

$$\begin{aligned}
\sum_{j\in\mathcal{S}} z^j \times \frac{d\,P_j(t)}{dt} &= \sum_{j\in\mathcal{S}} z^j \times \left[P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j + \mu_j)\right] \\
\frac{\partial P(z,t)}{\partial t} &\stackrel{\lambda_j=\lambda,\,\mu_j=j\mu}{=} \sum_{j\in\mathcal{S}} z^j \times P_{j-1}(t)\,\lambda + \sum_{j\in\mathcal{S}} z^j \times P_{j+1}(t)\,(j+1)\mu \\
&\quad - \sum_{j\in\mathcal{S}} z^j \times P_j(t)\,\lambda - \sum_{j\in\mathcal{S}} z^j \times P_j(t)\,j\mu \\
&= \lambda z \sum_{j\in\mathcal{S}} z^{j-1} \times P_{j-1}(t) + \mu \sum_{j\in\mathcal{S}} (j+1) z^j \times P_{j+1}(t) \\
&\quad -\lambda \sum_{j\in\mathcal{S}} z^j \times P_j(t) - \mu z \sum_{j\in\mathcal{S}} j z^{j-1} \times P_j(t) \\
&= \lambda z\,P(z,t) + \mu \frac{\partial P(z,t)}{\partial z} - \lambda\,P(z,t) - \mu z \frac{\partial P(z,t)}{\partial z},
\end{aligned}$$

i.e.,

$$\frac{\partial P(z,t)}{\partial t} + \lambda\,(1-z)\,P(z,t) - \mu\,(1-z)\frac{\partial P(z,t)}{\partial z} = 0.$$

(c) Using the *Mathematica* commands **(1.5)**

- pde = $D[P[z,t],t] + \lambda\,(1-z)\,P[z,t] - \mu\,(1-z)\,D[P[z,t],z] == 0$;
- soln = DSolve[{pde, $P[z,0] == 1$}, $P[z,t]$, $\{z,t\}$];
- soln = $P[z,t]$/.Dispatch[soln];
- Simplify[soln[[1]]]

led to the solution $e^{\frac{\lambda(z-1)e^{\mu(-t)}\left(e^{\mu t}-1\right)}{\mu}}$ (for $t \geq 0$ and $|z| \leq 1$).
After making brief comments about the first two commands, identify $P_j(t)$ and calculate $\lim_{t\to+\infty} P_j(t)$.

- **Brief comments**
  The 1st. command sets the partial differential equation.
  The purpose of the 2nd. command is to solve the partial differential equation in terms of $P(z,t)$ considering an initial condition reflecting the fact that $X(0) = 0$, which is indeed equivalent to $P(z,0) = E[z^{X(0)}] = z^0 \times P[X(0) = 0] = 1$.
- **Solution of the partial differential equation**
  $P(z,t) = E\left[z^{X(t)} \mid X(0) = 0\right] = e^{\frac{\lambda(z-1)e^{\mu(-t)}\left(e^{\mu t}-1\right)}{\mu}}$, for $t \geq 0$ and $|z| \leq 1$
- **Identifying** $P_j(t)$
  Consulting the table with p.g.f. we conclude that

$$\begin{aligned}
P(z,t) &= \exp\left[-\frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right) \times (1-z)\right] \\
&\equiv \text{p.g.f. of a Poisson with parameter } \alpha(t),
\end{aligned}$$

where $\alpha(t) = \frac{\lambda}{\mu}\,(1 - e^{-\mu t})$, that is,

$$P_j(t) = e^{-\alpha(t)} \frac{[\alpha(t)]^j}{j!},\ j \in \mathbb{N}_0,$$

- **Requested limit**
  Since $\lim_{t\to+\infty} \alpha(t) = \frac{\lambda}{\mu}$, we get

$$\lim_{t\to+\infty} P_j(t) = e^{-\lambda/\mu} \frac{(\lambda/\mu)^j}{j!},\ j \in \mathbb{N}_0.$$

[This p.f. coincides expectedly with the one of $L_s^{M/M/\infty}$.]

2. Consider a drive-in banking service modeled as an $M/M/1$ queueing system in equilibrium, with arrival (resp. service) rate equal to λ (resp. μ) customers per minute (where $\lambda/\mu < 1$).

   (a) When $\lambda = 2$, it is desired to have fewer than 5 customers in the system 99% (or more) of the time. **(1.5)**
   How large should the service rate be?

   - **Birth-death queueing system**
     $M/M/1$
   - **Birth/death rates**
     $\lambda_k = \lambda = 2,\ k \in \mathbb{N}_0$
     $\mu_k = \mu,\ k \in \mathbb{N}$
   - **Traffic intensity/ergodicity condition**
     $\rho = \frac{\lambda}{\mu} = \frac{2}{\mu} < 1$
   - **Performance measure (in the long-run)**
     L_s = number of customers in the drive-in banking service
     $P(L_s = k) = \rho^k\,(1-\rho),\ k \in \mathbb{N}_0$

- **Requested service rate**
  We have to deal with $\mu > \lambda = 2$ and

$$\begin{aligned}
\mu \ : \ & P(L_s < 5) \geq 0.99 \\
& \sum_{k=0}^{4} \rho^k (1-\rho) \geq 0.99 \\
& (1-\rho)\,\frac{1-\rho^5}{1-\rho} \geq 0.99 \\
& 1-\rho^5 \geq 0.99 \\
& \frac{\lambda}{\mu} \leq (1-0.99)^{1/5} \\
& \mu \geq \frac{2}{0.01^{1/5}} \\
& \mu \geq 5.023773.
\end{aligned}$$

(b) Admit that the service rate is equal to $\mu = \frac{1}{2}$ customers per minute. It is the policy of the company to add another server if an arriving customer waits an average of 3 or more minutes for the server. (**1.0**)

Find the arrival rate needed to justify a second server.

- **Birth-death queueing system**
  $M/M/1$ with $\mu = \frac{1}{2}$
- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{\mu} = 2\lambda < 1$
- **Performance measure (in the long-run)**
  W_q = time (in hours) an arriving customer waits for the server
  $E(W_q) \stackrel{form}{=} \frac{\rho}{\mu(1-\rho)}$
- **Requested arrival rate**
  We have to deal with $\lambda < \frac{1}{2}$ and

$$\begin{aligned}
\lambda \ : \ & E(W_q) \geq 3 \\
& \frac{\rho}{\mu(1-\rho)} \geq 3 \\
& \frac{2\lambda}{\frac{1}{2}(1-2\lambda)} \geq 3 \\
& 4\lambda \geq 3 - 6\lambda \\
& \lambda \geq 0.3.
\end{aligned}$$

(c) Now, consider the system has two servers and that the arrival (resp. service) rate is equal to $\lambda = 2$ (resp. $\mu = 1.5$) customers per minute. (**1.5**)

What are the probabilities that an arriving customer will:

(i) find both servers busy?

(ii) spend more than 5 minutes in the system?

- **New birth-death queueing system**
  $M/M/m$, where $m = 2$ servers.

- **Birth/death rates**
  $\lambda_k = \lambda = 2,\ k \in \mathbb{N}_0$
  $\mu_k = \begin{cases} \mu = 1.5, & k = 1 \\ 2\mu = 3, & k = 2, 3, \ldots \end{cases}$
- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{m\mu} = \frac{2}{3} < 1$
- **Performance measure (in the long-run)**
  L_s = number of customers in the drive-in banking service
- **1st. requested probability**

$$\begin{aligned}
P(L_s \geq m) & \stackrel{form.}{=} C(m, m\rho) \\
& \stackrel{form., m=2}{=} \frac{2\rho^2}{1+\rho} \\
& = \frac{2\left(\frac{2}{3}\right)^2}{1+\frac{2}{3}} \\
& = \frac{8}{15} \\
& \simeq 0.5(3).
\end{aligned}$$

- **Another performance measure (in the long-run)**
  W_s = time an arriving customer spends in the system
- **2nd. requested probability**

$$\begin{aligned}
P(W_s > t) & \stackrel{form.}{=} \left[1 + \frac{e^{\mu[1-m(1-\rho)]t}}{1-m(1-\rho)} \times C(m, m\rho)\right] e^{-\mu t} \\
& \stackrel{t=5}{=} \left[1 + \frac{e^{1.5\times[1-2(1-2/3)]\times 5}}{1-2(1-2/3)} \times \frac{8}{15}\right] e^{-1.5\times 5} \\
& \simeq 0.011334.
\end{aligned}$$