Formulae — Introduction to Stochastic Processes — 2nd. Semester 2018/19 — IST

Discrete and continuous univariate distributions

X (r.v.)	Values	$P(X=x)$ or $f_X(x)$	$E(X)$	$V(X)$	$M_X(t)$ or $E(X^k)$	$P_X(s)$
Binomial(n,p)	$\{0,\ldots,n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$[pe^t+(1-p)]^n$	$(1-p+ps)^n$
HyperG(N,M,n)	$\{\max\{0,n-N+M\},\ldots,\min\{n,M\}\}$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$n\frac{M}{N}$	$n\frac{M}{N}\frac{N-M}{N}\frac{N-n}{N-1}$	not interesting	not interesting
Geometric(p)	$\mathbb{N}$	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{ps}{1-(1-p)s}$
Geometric$^\star(p)$	$\mathbb{N}_0$	$(1-p)^x p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$	$\frac{p}{1-(1-p)s}$
NegativeBin(r,p)	$\{r,r+1,\ldots\}$	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\left[\frac{ps}{1-(1-p)s}\right]^r$
NegativeBin$^\star(r,p)$	$\{0,1,\ldots\}$	$\binom{y+r-1}{r-1}p^r(1-p)^y$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{p}{1-(1-p)e^t}\right]^r$	$\left[\frac{p}{1-(1-p)s}\right]^r$
Poisson(λ)	$\mathbb{N}_0$	$e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$	$e^{-\lambda(1-s)}$
Uniform$(\{1,\ldots,n\})$	$\{1,\ldots,n\}$	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$	$\frac{e^t(1-e^{tn})}{n(1-e^t)}$	$\frac{s(1-s^n)}{n(1-s)}$
Beta(α,β)	$[0,1]$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{+\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$	—
Cauchy(μ,σ)	$\mathbb{R}$	$\frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}$	nonexistent	nonexistent	nonexistent	—
$\chi^2_{(n)}$	$\mathbb{R}_0^+$	$\frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}e^{-\frac{x}{2}}$	n	$2n$	$\left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{n}{2}}, \quad t<\frac{1}{2}$	—
Exponential(λ)	$\mathbb{R}_0^+$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}, \quad t<\lambda$	—
Gamma(α,λ)	$\mathbb{R}_0^+$	$\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha, \quad t<\lambda$	—
LogNormal(μ,σ^2)	$\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}$	$e^{\mu+\frac{\sigma^2}{2}}$	$\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}$	$E(X^k)=e^{k\mu+\frac{k^2\sigma^2}{2}}$	—
Normal(μ,σ^2)	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2	$e^{\mu t+\frac{(t\sigma)^2}{2}}$	—
Rayleigh(σ)	$\mathbb{R}_0^+$	$\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$	$\sigma\sqrt{\frac{\pi}{2}}$	$\frac{4-\pi}{2}\sigma^2$	$E(X^k)=\left(\sqrt{2}\sigma\right)^k\Gamma\left(1+\frac{k}{2}\right)$	—
Uniform(a,β)	$[a,b]$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}, \ t\neq 0$	—
Weibull(α,β)	$\mathbb{R}_0^+$	$\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^\beta}$	$\alpha\Gamma\left(1+\frac{1}{\beta}\right)$	$\alpha^2\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma\left(1+\frac{1}{\beta}\right)^2\right]$	$E(X^k)=\alpha^k\Gamma\left(1+\frac{k}{\beta}\right)$	—

$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx, \ \alpha>0;$ $\quad \Gamma(n)=(n-1)!, \ n\in\mathbb{N};$ $\quad \Gamma(\alpha+1)=\alpha\Gamma(\alpha), \ \alpha>0;$ $\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Relating c.d.f.

$F_{NegativeBin(r,p)}(x) = 1 - F_{Binomial(x,p)}(r-1)$

$F_{Erlang(n,\lambda)}(x) = 1 - F_{Poisson(\lambda x)}(n-1)$

$F_{Gamma(\alpha,\beta)}(x) = F_{\chi^2_{(2\alpha)}}(2\beta x)$

$F_{Beta(\alpha,\beta)}(x) = 1 - F_{Binomial(\alpha+\beta-1,x)}(\alpha-1)$

Moment/probability generating function; moments

$M_X(t) = E\left(e^{tX}\right)$

$E(X^k) = \left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0}$

$P_X(s) = E(s^X); \quad P(X=k) = \frac{1}{k!} \times \left.\frac{d^k P_X(s)}{ds^k}\right|_{s=0}$

$E[X(X-1)\cdots(X-k+1)] = \left.\frac{d^k P_X(s)}{ds^k}\right|_{s=1}, \, k \in \mathbb{N}$

$E(X) = \int_0^{+\infty} [1 - F_X(x)]\, dx, \;$ for $X \geq 0$

$E(X^k) = \int_0^{+\infty} kx^{k-1}[1 - F_X(x)]\, dx, \;$ for $X \geq 0$

$SC(X) = \frac{E\{[X-E(X)]^3\}}{[SD(X)]^3}$

$KC(X) = \frac{E\{[X-E(X)]^4\}}{[SD(X)]^4} - 3$

Multinomial distribution

$P(N_1 = n_1, \ldots, N_d = n_d) = \frac{n!}{\prod_{i=1}^d n_i!} \times \prod_{i=1}^d p_i^{n_i}$

$\{(n_1, \ldots, n_d) \in I\!N_0^d : \sum_{i=1}^d n_i = n\}$

$M_{N_1,\ldots,N_{d-1}}(t_1, \ldots, t_{d-1}) = \left[\left(\sum_{i=1}^{d-1} p_i\, e^{t_i}\right) + p_d\right]^n$

$N_i \sim$ Binomial(n, p_i); $\quad Cov(N_i, N_j) = -n\, p_i\, p_j, \, i \neq j$

$M_{\underline{X}}(\underline{t}) = E\left[\exp\left(\sum_{i=1}^n t_i\, X_i\right)\right]$

$E\left(\prod_{i=1}^n X_i^{k_i}\right) = \left.\frac{\partial^{\sum_{i=1}^n k_i} M_{\underline{X}}(\underline{t})}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}\right|_{\underline{t}=\underline{0}}$

Functions of r.v.

$F_{Y=g(X)}(y) = P[X \in g^{-1}((-\infty, y])]$

$f_{Y=g(X)}(y) = f_X[g^{-1}(y)] \times \left|\frac{dg^{-1}(y)}{dy}\right|$

Hierarchical models resulting from mixtures

$P(X = x) = \sum_y P(X = x | Y = y) \times P(Y = y)$

$P(X = x) = \int_{I\!R_Y} P(X = x | Y = y) \times f_Y(y)\, dy$

$E[g(X)] = E\{E[g(X)|Y)]\}$

$V[g(X)] = V\{E[g(X)|Y]\} + E\{V[g(X)|Y]\}$

Functions of random vectors

$F_{\underline{Y}=\underline{g}(\underline{X})}(\underline{y}) = P\left[\underline{X} \in \underline{g}^{-1}\left(\prod_{i=1}^m (-\infty, y_i]\right)\right]$

$f_{\underline{Y}=\underline{g}(\underline{X})}(\underline{y}) = f_{\underline{X}}\left[\underline{g}^{-1}(\underline{y})\right] \times |J(\underline{y})|$

$J(\underline{y}) = \begin{vmatrix} \frac{\partial g_1^{-1}(\underline{y})}{\partial y_1} & \ldots & \frac{\partial g_1^{-1}(\underline{y})}{\partial y_n} \\ \vdots & \ldots & \vdots \\ \frac{\partial g_n^{-1}(\underline{y})}{\partial y_1} & \ldots & \frac{\partial g_n^{-1}(\underline{y})}{\partial y_n} \end{vmatrix}$

$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\, dy$

$f_{X-Y}(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u+y, y)\, dy$

$f_{XY}(v) = \int_{-\infty}^{+\infty} f_{X,Y}(v/y, y) \times \frac{1}{|y|}\, dy$

$f_{X/Y}(w) = \int_{-\infty}^{+\infty} f_{X,Y}(wy, y) \times |y|\, dy$

Order statistics

$P[X_{(n-k+1)} > x] = 1 - F_{Binomial(n, 1-F_X(x))}(k-1)$

$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)}, \ldots, x_{(n)}) = n! \times \prod_{i=1}^n f_X(x_{(i)})$

$F_{X_{(i)}}(x) = 1 - F_{Binomial(n, F_X(x))}(i-1)$

$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!}\, [F_X(x)]^{i-1}\, [1 - F_X(x)]^{n-i}\, f_X(x)$

$f_{\left(X_{(i)}, X_{(j)}\right)}(x, y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\, [F_X(x)]^{i-1}\, [F_X(y) - F_X(x)]^{j-i-1}\, [1 - F_X(y)]^{n-j}\, f_X(x) f_X(y), \, x < y$

Cap. 0

Bernoulli process

$\{X_n : n \in \mathbb{N}\} \overset{i.i.d.}{\sim}$ Bernoulli(p)

$S_n = \sum_{i=1}^n X_i, \quad T_k = \min\{n \in \mathbb{N} : S_n = k\}$

$\{U_k = T_k - T_{k-1} : k \in \mathbb{N}\} \overset{i.i.d.}{\sim}$ Geometric(p)

$S_m | S_n = k \sim$ HyperG(n, m, k), $0 \leq m \leq n$, $0 \leq k \leq n$

Cap. 1

A FEW PROPERTIES OF THE EXPONENTIAL DISTRIBUTION

$X_i \overset{indep}{\sim}$ Exponential(λ_i), $i = 1, \ldots, n \;\Rightarrow$ a) $\min_{i=1,\ldots,n}\{X_i\} \sim$ Exponential$(\sum_{i=1}^n \lambda_i)$

b) $P\left(X_j = \min_{i=1,\ldots,n}\{X_i\}\right) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$

c1) $\sum_{i=1}^n X_i \sim$ Hypo-exp.$(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \neq \lambda_j \ (i \neq j)$

c2) $f_{\sum_{i=1}^n X_i}(x) = \sum_{i=1}^n C_{i,n} \times \lambda_i \, e^{-\lambda_i x}$, $C_{i,n} = \prod_{j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$

d1) $X \sim$ Hyper-exp.$(\lambda_1, \ldots, \lambda_n; p_1, \ldots, p_n)$, $\lambda_i \neq \lambda_j \ (i \neq j)$ d2) $f_X(x) = \sum_{i=1}^n p_i \, f_{X_i}(x)$

HOMOGENOUS PP (independent and stationary increments)

$\{N(t) : t \geq 0\} \sim PP(\lambda)$

$S_n = \min\{t \geq 0 : N(t) = n\}$; $N(t) \geq n \Leftrightarrow S_n \leq t$

$N(t) \sim$ Poisson(λt)

$S_n \sim$ Erlang(n, λ); $F_{S_n}(t) = 1 - F_{Poisson(\lambda t)}(n-1)$

$(N(s) \mid N(t) = n) \sim$ Binomial$(n, s/t)$, $0 < s < t$

$\{N_i(t) : t \geq 0\} \overset{indep}{\sim} PP(\lambda_i)$, $i = 1, 2 \quad \Rightarrow \quad P\left[S_n^{(1)} < S_m^{(2)}\right] = 1 - F_{Binomial(n+m-1, \frac{\lambda_1}{\lambda_1+\lambda_2})}(n-1)$

$\{X_n = S_n - S_{n-1} : n \in \mathbb{N}\} \overset{i.i.d.}{\sim}$ Exp(λ)

$(S_1 \mid N(t) = 1) \sim$ Uniform$(0, t)$

$Y_i \overset{i.i.d.}{\sim}$ Uniform$(0, t)$, $i = 1, \ldots, n$

$(S_1, \ldots, S_n \mid N(t) = n) \sim (Y_{(1)}, \ldots, Y_{(n)})$

$N_1(t) = \#$registered events in $(0, t]$, under the non-homogeneous Bernoulli splitting mechanism associated with a $p : \mathbb{R}_0^+ \to [0, 1]$

$N_1(t) \sim$ Poisson $\left(\lambda \int_0^t p(s)\, ds\right)$

NON-HOMOGENOUS PP (independent increments)

$\{N(t) : t \geq 0\} \sim NHPP(\lambda(t))$

$m(t) = \int_0^t \lambda(z)\, dz$

$N(t+s) - N(s) \sim$ Poisson $(m(t+s) - m(s))$

$(N(s) \mid N(t) = n) \sim$ Binomial$(n, \frac{m(s)}{m(t)})$, $0 < s < t$, $n \in \mathbb{N}$

$f_{S_n}(t) = \lambda(t) e^{-m(t)} \frac{[m(t)]^{n-1}}{(n-1)!}$, $n \in \mathbb{N}$

$P(X_{n+1} = S_{n+1} - S_n > t) = \int_0^{+\infty} \lambda(s) e^{-m(t+s)} \frac{[m(s)]^{n-1}}{(n-1)!}\, ds$, $n \in \mathbb{N}$

$Y_i \overset{i.i.d.}{\sim} Y$, where $P(Y \leq u) = \frac{m(u)}{m(t)}$, for $0 \leq u \leq t$

$(S_1, \ldots, S_n \mid N(t) = n) \sim (Y_{(1)}, \ldots, Y_{(n)})$

CONDITIONAL PP (stationary increments)

$\{N(t) : t \geq 0\} \sim ConditionalPP(G)$

$P[N(t+s) - N(s) = n] = \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\, dG(\lambda)$

COMPOUND PP (independent and stationary increments)

$\{X(t) = \sum_{i=1}^{N(t)} Y_i : t \geq 0\} \sim CompoundPP(\lambda, Y)$

$E[X(t)] = \lambda t \times E(Y)$; $V[X(t)] = \lambda t \times E(Y^2)$

Cap. 2

RENEWAL PROCESSES

$\{N(t) : t \geq 0\} \sim RP$ with inter-renewal distribution F and mean μ; $F_n(t) = P(S_n \leq t)$

$P[N(t) = n] = F_n(t) - F_{n+1}(t)$

$m(t) = E[N(t)] = \sum_{n=1}^{+\infty} F_n(t)$

$m(t) = F(t) + \int_0^t m(t-x)\, dF(x)$

$\tilde{m}(s) = \int_{0^-}^{+\infty} e^{-st}\, dm(t) = \frac{\tilde{F}(s)}{1-\tilde{F}(s)}$

$H(t) = D(t) + \int_0^t H(t-x)\, dF(x)$

$|D(t)| < +\infty \;\Rightarrow\; H(t) = D(t) + \int_0^t D(t-x)\, dm(x)$

$H(t) = D(t) + \int_0^t D(t-x)\, dm(x)$

$\tilde{D}(s) = \int_{0^-}^{+\infty} e^{-st}\, dD(t)$ exists; $\tilde{H}(s) = \frac{\tilde{D}(s)}{1-\tilde{F}(s)}$

Important Laplace transforms

$f(t)$	$f^\star(s)=\int_0^\infty e^{-st}f(t)\,dt$	$g(t)$	$g^\star(s)=\int_0^\infty e^{-st}g(t)\,dt$
1	$1/s$	$af(t)+b\,h(t)$	$af^\star(s)+b\,h^\star(s)$
t^n	$n!/s^{n+1}$	$\frac{df(t)}{dt}$	$sf^\star(s)-f(0)$
$\frac{t^{n-1}e^{-at}}{(n-1)!}$	$1/(s+a)^n$	$e^{-at}f(t)$	$f^\star(s+a)$
$\sin(at)$	$a/(s^2+a^2)$	$\int_0^t f(u)du$	$f^\star(s)/s$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$1/[(s+a)(s+b)]$	$t^n f(t)$	$(-1)^n\frac{d^n}{ds^n}f^\star(s)$

Limit theorems et al.

$P\left[N(t)<n\right]\simeq\Phi\left(\frac{n-t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right)$

SLLN for renewal processes: $\frac{N(t)}{t}\overset{w.p.1}{\to}\frac{1}{\mu}$

Elementary renewal theorem: $\lim_{t\to+\infty}\frac{m(t)}{t}=\frac{1}{\mu}$

Key renewal th.: $D(t)$ dRi, F not lattice, $H(t)=D(t)+\int_0^t D(t-x)\,dm(x)\ \Rightarrow\ \lim_{t\to+\infty}H(t)=\frac{1}{\mu}\int_0^{+\infty}D(y)\,dy$

Blackwell's th.: F not lattice $\Rightarrow\ \lim_{t\to+\infty}[m(t+a)-m(t)]=\frac{a}{\mu}$

F lattice with period $d\ \Rightarrow\ \lim_{n\to+\infty}E[$number of renewals **at** $nd]=\frac{d}{\mu}$

Recurrence times

$A(t)=t-S_{N(t)},\quad Y(t)=S_{N(t)+1}-t,\quad X_{N(t)+1}=A(t)+Y(t)$

$E\left[S_{N(t)+1}\right]=\mu\times[m(t)+1]$

$X_{N(t)+1}\geq_{st}X_i,\ i\in\mathbb{N}$ (inspection paradox)

$\frac{A(t)}{t}\overset{w.p.1}{\to}0,\quad \lim_{t\to 0}\frac{E[Y(t)]}{t}=0$

$\lim_{t\to+\infty}E[Y(t)]=\frac{E(X^2)}{2\mu}$

$\lim_{t\to+\infty}E[A(t)]=\frac{E(X^2)}{2\mu}$

$\lim_{t\to+\infty}E[X_{N(t)+1}]=\frac{E(X^2)}{\mu}\geq E(X)$

$\lim_{t\to+\infty}P[Y(t)\leq x]=\lim_{t\to+\infty}P[A(t)\leq x]=F_e(x)=\frac{\int_0^x[1-F(u)]\,du}{\mu}$ (equilibrium distribution)

Reward renewal processes

$\{(X_n,R_n):n\in\mathbb{N}\}\overset{i.i.d.}{\sim}(X,R),\quad R(t)=\sum_{n=1}^{N(t)}R_n$

$E(X),E(R)<+\infty\ \Rightarrow$
a) $\frac{R(t)}{t}\overset{w.p.1}{\to}\frac{E(R)}{E(X)}$
b) $\lim_{t\to+\infty}\frac{E[R(t)]}{t}=\frac{E(R)}{E(X)}$

Alternating renewal processes

$\{(U_n,D_n):n\in\mathbb{N}\}\overset{i.i.d.}{\sim}(U,D),\quad Z(t)=\begin{cases}1, & \text{if }\exists n\in\mathbb{N}:S_n=\sum_{i=1}^n X_i=\sum_{i=1}^n(U_i+D_i)\leq t<S_n+U_{n+1}\\ 0, & \text{if }\exists n\in\mathbb{N}:S_n+U_{n+1}\leq t<S_{n+1}\end{cases}$

$E(U_n+D_n)<+\infty$, F not lattice $\Rightarrow\ \lim_{t\to+\infty}P[Z(t)=1]=\frac{E(U)}{E(U)+E(D)}$

Delayed renewal processes

$\{N_D(t):t\geq 0\}$, with $X_1\sim G$, $X_i\sim F$, $i=2,3,\dots$, and $(G\star F_{n-1})(t)=P(S_n\leq t)=\int_0^t G(t-x)\,dF_{n-1}(x)$

$P[N_D(t)=n]=P(S_n\leq t)-P(S_{n+1}\leq t)=(G\star F_{n-1})(t)-(G\star F_n)(t)$

$m_D(t)=\sum_{n=1}^{+\infty}(G\star F_{n-1})(t)$

$\tilde{m}_D(s)=\frac{\tilde{G}(s)}{1-\tilde{F}(s)}$

Regenerative processes

$\{X(t):t\geq 0\}$, with state space $\mathbb{N}_0$ and $S_1\sim F$

$U_j=$ time spent in state j during $[0,S_1)$

F not lattice, $E(S_1)<+\infty\ \Rightarrow\ \lim_{t\to+\infty}P[X(t)=j]=\frac{E(U_j)}{E(S_1)}=P_j$

$E(S_1)<+\infty\ \Rightarrow\ \lim_{t\to+\infty}\frac{\text{amount of time state } j \text{ during } (0,t)}{t}\overset{w.p.1}{=}P_j$

Cap. 3

Definitions and examples

$\{X_n : n \in \mathbb{N}_0\},\ P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i) = P_{ij}$

$\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}},\quad \underline{\alpha} = [\alpha_i]_{i\in\mathcal{S}} = [P(X_0 = i)]_{i\in\mathbb{S}}$

Chapman-Kolmogorov equations; marginal and joint distributions

$P^{n+m}_{ij} = P(X_{n+m} = j \mid X_0 = i) = \sum_{k\in\mathcal{S}} P^n_{ik} P^m_{kj}$ $\qquad \mathbf{P}^n = \left[P^n_{ij}\right]_{i,j\in\mathcal{S}},\quad \underline{\alpha}^n = [P(X_n = j)]_{j\in\mathcal{S}} = \underline{\alpha}\,\mathbf{P}^n$

$P(X_{n_1} = i_{n_1}, \ldots, X_{n_k} = i_{n_k}) = \left(\sum_{i\in\mathcal{S}} \alpha_i \times P^{n_1}_{i,i_{n_1}}\right) \times \prod_{j=2}^{k} P^{n_j - n_{j-1}}_{i_{n_{j-1}},i_{n_j}},\quad 0 \le n_1 < n_2 < \cdots < n_k,\ i_{n_1}, \ldots, i_{n_k} \in \mathcal{S}$

Classification of states; recurrent and transient states

$f^n_{ij} = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$ $\qquad T_i = \min\{n \in \mathbb{N} : X_n = i \mid X_0 = i\}$

$f_{ij} = \sum_{n=1}^{+\infty} f^n_{ij},\quad f_i \equiv f_{ii} = \sum_{n=1}^{+\infty} f^n_{ii} = P(T_i < +\infty)$ $\qquad$ recurrent if $f_i = 1$, transient if $f_i < 1$

$\mu_{ii} = E(T_i) = \sum_{n=1}^{+\infty} n \times f^n_{ii}$ (i recurrent) $\qquad$ positive rec. if $\mu_{ii} < +\infty$, null rec. if $\mu_{ii} = +\infty$

Limit behavior of irreducible (aperiodic) Markov chains

$P_j = \sum_{i\in\mathcal{S}} P_i P_{ij},\ j \in \mathcal{S}$ (stationary dist.); $\quad \lim_{n\to+\infty} P^n_{ij} = \pi_j > 0,\ i, j \in \mathcal{S}$ (if all states are posit. rec. aper.)

$\pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j \in \mathcal{S},\quad \sum_{j\in\mathcal{S}} \pi_j = 1;\qquad \mu_{jj} = \frac{1}{\pi_j};\qquad \underline{\pi} = \underline{\pi}\mathbf{P},\ \underline{\pi}\,\underline{1} = 1;\quad \underline{\pi} = [\pi_j]_{j\in\mathcal{S}} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1}$

Markov chains with costs/rewards

$\lim_{N\to+\infty} \frac{1}{N+1} E\left[\sum_{n=0}^{N} c(X_n) \mid X_0 = i\right] = \sum_{j\in\mathcal{S}} \pi_j c(j) = \underline{\pi} \times \underline{c}$ (long-run expected cost per time unit)

$\phi(i) = E\left[\sum_{n=0}^{+\infty} \alpha^n c(X_n) \mid X_0 = i\right]$ (expected total discounted cost incurred over..., starting at state i)

$\phi(i) = c(i) + \alpha \sum_{j\in\mathcal{S}} P_{ij}\phi(j),\ i \in \mathcal{S}$ $\qquad \underline{\phi} = [\phi(i)]_{i\in\mathcal{S}} = (\mathbf{I} - \alpha\mathbf{P})^{-1} \times \underline{c}$

Time reversible Markov chains

$\{X_{n-m} : m \in \mathbb{Z}\},\quad Q_{ij} = \frac{\pi_j \times P_{ji}}{\pi_i}$ $\qquad \pi_i \times P_{ij} = \pi_j \times P_{ji},\quad i, j \in \mathcal{S}$

$P_{i,i_1} \times P_{i_1,i_2} \times \cdots \times P_{i_k,i} = P_{i,i_k} \times \cdots \times P_{i_2,i_1} \times P_{i_1,i}$, for any $i, i_1, i_2, \ldots, i_k$, $k \in \mathbb{N}$ (Kolmogorov's criterion)

Branching processes

$\{X_n : n \in \mathbb{N}_0\},\quad X_0 = 1,\quad X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n \in \mathbb{N},\quad \{Z_l : l \in \mathbb{N}\}$ i.i.d., non-negat. r.v., $\quad P_j = P(Z_l = j),\ j \in \mathbb{N}_0$

$P_n(s) = E(s^{X_n}),\quad P(s) = E(s^{Z_l}),\quad s \in [0, 1]$ $\qquad P_{n+1}(s) = P_n[P(s)] = P[P_n(s)],\ n \in \mathbb{N},\ s \in [0, 1]$

$\mu = E(Z_l),\quad \sigma^2 = V(Z_l),\quad E(X_n \mid X_0 = 1) = \mu^n$ $\qquad V(X_n \mid X_0 = 1) = \begin{cases} \sigma^2\mu^{n-1} \times \frac{\mu^n - 1}{\mu - 1}, & \text{if } \mu \neq 1 \\ n\sigma^2, & \text{if } \mu = 1, \end{cases}$

$\pi = \lim_{n\to+\infty} P(X_n = 0 \mid X_0 = 1)$ $\qquad$ if $\mu \le 1$, $\pi = 1$; if $\mu > 1$, $\pi = \sum_{j=0}^{+\infty} \pi^j \times P_j$

First passage times; absorption probabilities

$f^n_{ij} = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$ $\qquad \underline{f}^n_j = [f^n_{ij}]_{i\in\mathcal{S}}$

$f^n_{ij} = \begin{cases} P_{ij}, & n = 1 \\ \sum_{k\neq j} P_{ik} f^{n-1}_{kj}, & n = 2, 3, \ldots \end{cases}$ $\qquad \underline{f}^n_j = \begin{cases} \underline{f}^1_j = [P_{ij}]_{i\in\mathcal{S}}, & n = 1 \\ {}^{(j)}\mathbf{P} \times \underline{f}^{n-1}_j = \left[{}^{(j)}\mathbf{P}\right]^{n-1} \times \underline{f}^1_j, & n = 2, 3, \ldots \end{cases}$

$\mathcal{S} = T \cup C_1 \cup C_2 \cup \cdots$, $\quad \mathbf{Q} = [Q_{ij}]_{i,j \in T}$, $\quad \mathbf{R} = [P_{kl}]_{k \in T,\, l \in \overline{T}}$, $\quad \tau = \inf\{n \in \mathbb{N}_0 : X_n \notin T\}$, $\quad u_{ik} = P(X_\tau = k \mid X_0 = i)$

$\mathbf{U} = [u_{ik}]_{i \in T,\, k \in \overline{T}} = (\mathbf{I} - \mathbf{Q})^{-1} \times \mathbf{R}$ $\qquad [E(\tau \mid X_0 = i)]_{i \in T} = (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{1}$

$\left[E[\sum_{n=0}^{\tau-1} g(X_n) \mid X_0 = i]\right]_{i \in T} = (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{g}$

Cap. 4

DEFINITIONS AND EXAMPLES

$\{X(t) : t \geq 0\}$, $\quad P[X(t+s) = j \mid X(s) = i,\, X(u) = x(u),\, 0 \leq u < s] = P[X(t+s) = j \mid X(s) = i]$

$P[X(t+s) = j \mid X(s) = i] = P[X(t) = j \mid X(0) = i] = P_{ij}(t)$, $\quad \mathbf{P}(t) = [P_{ij}(t)]_{i,j \in \mathcal{S}}$, $\quad \underline{\alpha} = [\alpha_i]_{i \in \mathcal{S}} = [P[X(0) = i]]_{i \in \mathcal{S}}$

PROPERTIES OF THE TRANSITION MATRIX; CHAPMAN-KOLMOGOROV EQUATIONS

$P_{ij}(t+s) = \sum_{k \in \mathcal{S}} P_{ik}(t) \times P_{kj}(s)$ $\qquad \mathbf{P}(t+s) = \mathbf{P}(t) \times \mathbf{P}(s) = \mathbf{P}(s) \times \mathbf{P}(t)$

$P[X(t) = j] = \sum_{i \in \mathcal{S}} \alpha_i \times P_{ij}(t),\ j \in \mathcal{S}$ $\qquad [P[X(t) = j)]_{j \in \mathcal{S}} = \underline{\alpha} \times \mathbf{P}(t)$

$q_{ij} = \nu_i \times P_{ij},\ i \neq j$ $\qquad \lim_{h \to 0^+} \frac{P_{ij}(h)}{h} = q_{ij},\ i \neq j; \quad \lim_{h \to 0^+} \frac{1 - P_{ii}(h)}{h} = \nu_i$

$\frac{d\,P_{ij}(t)}{dt} = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$ (backward eq.) $\qquad \frac{d\,P_{ij}(t)}{dt} = \sum_{k \neq j} P_{ik}(t) q_{kj} - P_{ij}(t)\nu_j$ (forward eq.)

$r_{ij} = \begin{array}{ll} q_{ij}, & i \neq j \\ -\nu_i, & i = j, \end{array}$ $\qquad \mathbf{R} = [r_{ij}]_{i,j \in \mathcal{S}}$ $\qquad \frac{d\,\mathbf{P}(t)}{dt} = \left[\frac{d\,P_{ij}(t)}{dt}\right]_{i,j \in \mathcal{S}} = \mathbf{R} \times \mathbf{P}(t) = \mathbf{P}(t) \times \mathbf{R}$

COMPUTING THE TRANSITION MATRIX: FINITE STATE SPACE

$\mathbf{P}(t) = e^{\mathbf{R}\,t} = \sum_{n=0}^{+\infty} \frac{\mathbf{R}^n\, t^n}{n!} = \lim_{n \to +\infty} \left(\mathbf{I} + \frac{\mathbf{R}\,t}{n}\right)^n$

COMPUTING THE TRANSITION MATRIX: INFINITE STATE SPACE

$P^\star_{ij}(s) = \int_0^{+\infty} e^{-st} P_{ij}(t) dt, \quad i, j \in \mathcal{S}$

$\int_0^{+\infty} e^{-st} \frac{d\,P_{ij}(t)}{dt} dt = s \times P^\star_{ij}(s) - P_{ij}(0) = \sum_{k \neq j} P^\star_{ik}(s) \times q_{kj} - P^\star_{ij}(s) \times \nu_j$

BIRTH AND DEATH PROCESSES

$\frac{d\,P_{0j}(t)}{dt} = \lambda_0\, P_{1j}(t) - \lambda_0\, P_{0j}(t),\ j \in \mathbb{N}_0$; $\ \frac{d\,P_{ij}(t)}{dt} = \lambda_i\, P_{i+1,j}(t) + \mu_i\, P_{i-1,j}(t) - (\lambda_i + \mu_i)\, P_{ij}(t),\ i \in \mathbb{N},\ j \in \mathbb{N}_0$ (b. eq.)

$\frac{d\,P_{i0}(t)}{dt} = P_{i1}(t)\, \mu_1 - P_{i0}(t)\, \lambda_0,\ i \in \mathbb{N}_0$; $\ \frac{d\,P_{ij}(t)}{dt} = P_{i,j-1}(t)\, \lambda_{j-1} + P_{i,j+1}(t)\, \mu_{j+1} - P_{ij}(t)\, (\lambda_j + \mu_j),\ i \in \mathbb{N}_0,\ j \in \mathbb{N}$ (f. eq.)

$P_j(t) \equiv P[X(t) = j \mid X(0) = i]$ $\qquad P(z,t) = E\left[z^{X(t)} \mid X(0) = i\right],\ |z| \leq 1$

$\sum_{j \in \mathcal{S}} z^j \times \frac{d\,P_j(t)}{dt} = \frac{\partial P(z,t)}{\partial t}$ $\qquad \frac{\partial P(z,t)}{\partial z} = \sum_{j \in \mathcal{S}} j z^{j-1} \times P_j(t) = \sum_{j \in \mathcal{S}} (j+1) z^j \times P_{j+1}(t)$

$\sum_{j \in \mathcal{S}} z^j \times \frac{d\,P_j(t)}{dt} = \sum_{j \in \mathcal{S}} z^j \times [P_{j-1}(t)\, \lambda_{j-1} + P_{j+1}(t)\, \mu_{j+1} - P_j(t)\, (\lambda_j + \mu_j)]$ (forward eq.)

LIMIT BEHAVIOR OF CTMC

$\{X_n : n \in \mathbb{N}_0\}$ (embedded DTMC), $\quad \underline{\pi} = [\pi_j]_{j \in \mathcal{S}}$ (stationary distribution of the embedded DTMC)

$P_j = \lim_{t \to +\infty} P_{ij}(t)$ $\qquad P_j = \frac{\frac{\pi_j}{\nu_j}}{\sum_{k \in \mathcal{S}} \frac{\pi_k}{\nu_k}}, \quad j \in \mathcal{S}$

$\underline{P} = [P_j]_{j \in \mathcal{S}}, \quad \underline{P} \times \mathbf{R} = \underline{0}, \quad \sum_{j \in \mathcal{S}} P_j = 1$ $\qquad P_j \times \nu_j = \sum_{i \in \mathcal{S}} P_i \times q_{ij}, \quad j \in \mathcal{S}$

$P_0 = \left[1 + \sum_{n=1}^{+\infty} \frac{\lambda_0\, \lambda_1 \ldots \lambda_{n-1}}{\mu_1\, \mu_2 \ldots \mu_n}\right]^{-1}, \quad P_j = \frac{\lambda_{j-1}}{\mu_j} P_{j-1} = P_0 \times \frac{\lambda_0\, \lambda_1 \ldots \lambda_{j-1}}{\mu_1\, \mu_2 \ldots \mu_j},\ j \in \mathbb{N}$ (birth and death proc.)

Queues

L_s	$E(L_s) = \lambda_e E(W_s) = \lambda \times (1 - P_b) \times E(W_s)$
	$E(L_s) = \frac{\lambda_e}{\mu} + E(L_q)$
L_q	$E(L_q) = \lambda_e E(W_q)$
W_s	$E(W_s) = \frac{1}{\mu} + E(W_q)$

M/M/1

Rates	$\lambda_k = \lambda, \quad k \in \mathbb{N}_0$ $\mu_k = \mu, \quad k \in \mathbb{N} \qquad \left(\rho = \frac{\lambda}{\mu} < 1\right)$
L_s	$P(L_s = k) = \rho^k\,(1-\rho), \quad k \in \mathbb{N}_0$ $E(L_s) = \frac{\rho}{(1-\rho)}$
L_q	$P(L_q = k) = \begin{cases} 1-\rho^2, & k = 0 \\ \rho^{k+1}\,(1-\rho), & k \in \mathbb{N} \end{cases}$ $E(L_q) = \frac{\rho^2}{(1-\rho)}$
W_s	$(W_s \mid L_s = k) \sim \text{Gamma}(k+1, \mu),\ k \in \mathbb{N}_0$ $W_s \sim \text{Exponential}(\mu(1-\rho))$ $E(W_s) = \frac{1}{\mu(1-\rho)}$
W_q	$(W_q \mid L_s = k) \sim \text{Gamma}(k, \mu),\ k \in \mathbb{N}$ $F_{W_q}(t) = \begin{cases} 0, & t < 0 \\ 1-\rho, & t = 0 \\ (1-\rho) + \rho \times F_{Exp(\mu(1-\rho))}(t), & t > 0 \end{cases}$ $(W_q \mid W_q > 0) \sim \text{Exponential}(\mu(1-\rho))$ $E(W_q) = \frac{\rho}{\mu(1-\rho)}$

M/M/∞

Rates	$\lambda_k = \lambda, \quad k \in \mathbb{N}_0$ $\mu_k = k\mu, \quad k \in \mathbb{N} \qquad \left(\rho = \frac{\lambda}{\mu} < +\infty\right)$
L_s	$L_s \sim \text{Poisson}(\lambda/\mu)$

$L_q \overset{st}{=} 0, \quad W_s \sim \text{Exp}(\mu), \quad W_q \overset{st}{=} 0$

$X(t)$ = number of customers in the system at time t

$(X(t) \mid X(0) = 0) \sim \text{Poisson}(\lambda\,(1 - e^{-\mu t})/\mu)$

M/G/∞

$(X(t) \mid X(0) = 0) \sim \text{Poisson}\left(\lambda \int_0^t [1 - G(t-s)]\, ds\right)$

$\lim_{t \to +\infty} (X(t) \mid X(0) = 0) \sim \text{Poisson}(\lambda/\mu)$

M/M/m

Rates

$$\lambda_k = \lambda, \quad k \in \mathbb{N}_0$$

$$\mu_k = \begin{matrix} k\mu, & k = 1, \ldots, m \\ m\mu, & k = m+1, m+2, \ldots \end{matrix} \qquad \left(\rho = \frac{\lambda}{m\mu} < 1\right)$$

L_s

$$P(L_s = k) = \begin{matrix} \frac{m!}{k!}\,(1-\rho)(m\rho)^{k-m} C(m, m\rho), & k = 0, 1, \ldots, m-1 \\ (1-\rho)\,\rho^{k-m}\,C(m, m\rho), & k = m, m+1, \ldots \end{matrix}$$

$$C(m, m\rho) = P\left(L_s \geq m\right) = \frac{\frac{(m\rho)^m}{m!(1-\rho)}}{\sum_{j=0}^{m-1} \frac{(m\rho)^j}{j!} + \frac{(m\rho)^m}{m!(1-\rho)}}$$

$$C(1, \rho) = \rho$$

$$C(2, 2\rho) = \frac{2\rho^2}{1+\rho}$$

$$E(L_s) = m\rho + \frac{\rho}{1-\rho} C(m, m\rho)$$

L_q

$$P(L_q = k) = \begin{matrix} 1 - \rho\, C(m, m\rho), & k = 0 \\ (1-\rho)\,\rho^k\, C(m, m\rho), & k \in \mathbb{N} \end{matrix}$$

$$E(L_q) = \frac{\rho}{1-\rho} C(m, m\rho)$$

W_s

$$(W_s \mid L_s = k) \sim \begin{matrix} \text{Exp}(\mu),\ k = 0, \ldots, m-1, \\ \text{Exp}(\mu) \star \text{Gamma}(k-m+1, m\mu),\ k = m, m+1, \ldots, \end{matrix}$$

$$1 - F_{W_s}(t) = \begin{matrix} \left[1 + \mu t\, C(m, m\rho)\right] e^{-\mu t}, & t \geq 0, & \rho = \frac{m-1}{m} \\ \left[1 + \frac{e^{\mu[1-m(1-\rho)]t}}{1-m(1-\rho)} \times C(m, m\rho)\right] e^{-\mu t}, & t \geq 0, & \rho \neq \frac{m-1}{m} \end{matrix}$$

$$E(W_s) = \frac{1}{\mu} + \frac{C(m, m\rho)}{m\mu(1-\rho)}$$

W_q

$$(W_q \mid L_s = k) \sim \text{Gamma}(k-m+1, m\mu),\ k = m, m+1, \ldots$$

$$(W_q \mid W_q > 0) \sim \text{Exponential}(m\mu(1-\rho))$$

$$1 - F_{W_q}(t) = \begin{matrix} 1, & t < 0 \\ C(m, m\rho), & t = 0 \\ C(m, m\rho) \times [1 - F_{Exp(m\mu(1-\rho))}(t)], & t > 0 \end{matrix}$$

$$E(W_q) = \frac{C(m, m\rho)}{m\mu(1-\rho)}$$

M/M/m/m

Rates

$$\lambda_k = \begin{matrix} \lambda, & k = 0, 1, \ldots, m-1 \\ 0, & k = m, m+1, \ldots \end{matrix}$$

$$\mu_k = \begin{matrix} k\mu, & k = 1, \ldots, m \\ 0, & k = m+1, m+2, \ldots \end{matrix} \qquad \left(\rho = \frac{\lambda}{m\mu} < +\infty\right)$$

L_s

$$P(L_s = k) = \begin{matrix} \frac{\frac{(m\rho)^k}{k!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}} = \frac{m!}{k!\,(m\rho)^{m-k}} \times B(m, m\rho), & k = 0, 1, \ldots, m \\ 0, & k = m+1, m+2, \ldots \end{matrix}$$

$$B(m, m\rho) = P(L_s = m) = \frac{\frac{(m\rho)^m}{m!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}}$$

$$E(L_s) = m\rho\,[1 - B(m, m\rho)]$$

$$L_q \overset{st}{=} 0, \quad W_s \sim \text{Exp}(\mu), \quad W_q \overset{st}{=} 0$$