Laser pulses and dispersion
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- What is dispersion?
- Where does dispersion come from\(^1\)
- Resonances in a dispersive medium\(^1\)
- Monochromatic waves in dispersive media\(^1\)
- Pulse propagation in dispersive media: group velocity and GVD\(^2\)
- Dispersive medium as a chirp filter\(^3\)

*Fundamentals of Photonics:*

What is dispersion?

**Dispersive media**: the dielectric properties depend on the frequency (or wavelength):
refr. index $n$, susceptibility $\chi$, electric permittivity $\varepsilon$, speed of light $c$.

**Snell’s law depends on $n$**: optical components made from dispersive materials bend light with an angle depending on its wavelength.

\[
\sin \theta' = \frac{\sin \theta}{n(\lambda)}
\]

Blue light is more bent than red light.
Chromatic aberration
Dispersion makes transform-limited pulses become longer

Since the speed of light depends on the frequency, and a laser pulse is made of a range of frequencies, dispersion leads to **pulse broadening** and **chirping**.

This can be a big problem e.g. in optical fibers where short pulses travel over distances of many km.
Wavelength dependence of optical materials

Typically for optical materials $n(\lambda)$ decreases with wavelength.

For describing the propagation of laser pulses we will take into account:

$n(\lambda), \frac{dn}{d\lambda}, \frac{d^2n}{d\lambda^2}$
Dispersion is a property of light-matter interaction for a material.

It is strongly linked with the existence of absorption and resonances.
Linear, *nondispersive*, homogeneous, isotropic media

The relation at every \((r,t)\) is simply

\[ P = \varepsilon_0 \chi E \]

\(\chi\) is the *electric susceptibility*

D and \(E\) are also parallel

\(\varepsilon\) is the *electric permittivity*

\[ D = \varepsilon_0 E + P = \varepsilon_0 (1 + \chi) E = \varepsilon E \]

\[ B = \mu H \]

\[ \nabla \times \mathcal{H} = \varepsilon \frac{\partial \mathcal{E}}{\partial t} \]
\[ \nabla \cdot \mathcal{H} = 0 \]

\[ \nabla \times \mathcal{E} = -\mu \frac{\partial \mathcal{H}}{\partial t} \]
\[ \nabla \cdot \mathcal{E} = 0 \]
Linear, *dispersive*, homogeneous, isotropic media

The relation is dynamic rather than instantaneous.

The response depends on the values of $E(t')$ for all $t' \leq t$:

Since a correlation in the **time** domain is a product in the **frequency** domain:

$$P(t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi(t-t')E(t')dt'$$

$$P(v) = \varepsilon_0 \chi(v)\mathcal{E}(v)$$

**impulse response function**  
(time domain)

**transfer function**  
(frequency domain)

Note the similarities to the linear filter analysis that we have studied
Describing dispersion and absorption

Absorption and dispersion can be described in a similar way. Let’s introduce

- complex susceptibility:
  \[ \chi = \chi_r + i \chi_i \]
- complex permittivity:
  \[ \varepsilon = \varepsilon_0 (1 + \chi) = \varepsilon_0 (1 + \chi_r + i \chi_i) \]

We have seen that monochromatic wave propagation is described by the **Helmholtz equation**:

\[ \nabla^2 U + k^2 U = 0 \]

\[ k = \omega \sqrt{\varepsilon \mu_0} = k_0 \sqrt{1 + \chi} = k_0 \sqrt{1 + \chi_r + i \chi_i} \]

\[ (k_0 = \omega / c_0) \]

The wavenumber \( k \) also becomes complex.

Wavenumber in free space.
Absorption coefficient and propagation constant

Since $k$ is complex we can write

$$k = \beta - i \frac{1}{2} \alpha$$

Let us consider a plane wave propagating along $z$:

$$U(z) = \exp(-ikz) = \exp\left(-\frac{1}{2} \alpha z\right) \exp(-i\beta z)$$

\[\alpha = \text{absorption coefficient}\]

After a distance $z$ the intensity is attenuated by $\exp(-\alpha)$

$n$ and $\alpha$ are related by:

$$n - i \frac{1}{2} \frac{\alpha}{k_0} = \sqrt{1 + \chi_r + i \chi_i}$$

The wave travels with a phase velocity $c = c_0/n$.

$$\beta = nk_0 = \frac{\omega}{(c_0 / n)}$$
The Lorentz oscillator model

A dispersive medium can be described using a microscopic analog to a driven oscillator for the constitutive relation:

\[ \ddot{x} + \sigma \dot{x} + \omega_0^2 x = F/m \]

\[ \ddot{P} + \sigma \dot{P} + \omega_0^2 P = \omega_0^2 \varepsilon_0 \chi_0 \mathcal{E} \]

The time-varying electric field \( E \) applied to a Lorentz-oscillator atom induces a time-varying dipole moment \( P \) that contributes to the overall polarization density.

\[ \mathcal{E}(t) = \text{Re}\{E \exp(i\omega t)\} \]

\[ \mathcal{P}(t) = \text{Re}\{P \exp(i\omega t)\} \]
Susceptibility in a resonant medium

Solving this equation will allow us to find the impulse response (or transfer) function for a dispersive medium.

Replacing the expressions for $P$ and $E$:

$$\left(-\omega^2 + i\sigma\omega + \omega_0^2\right) P = \omega_0^2 \epsilon_0 \chi_0 \mathcal{E}$$

$$\Rightarrow P = \epsilon_0 \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2 + i\sigma\omega} \mathcal{E}$$

Writing the relation in the usual form and using the frequency $\nu$: we obtain the susceptibility for a resonant medium:

$$\chi(\nu) = \chi_0 \frac{\nu_0^2}{\nu_0^2 - \nu^2 + i\nu\Delta\nu}$$

$$\nu_0 = \frac{\omega_0}{2\pi} \quad \Delta\nu = \frac{\sigma}{2\pi}$$

Now we separate $\chi$ into the real and imaginary parts and obtain expressions similar to the amplitude (real) and phase (imaginary) of a driven oscillator.
Susceptibility near a resonance

\[ \chi_r(v) = \chi_0 \frac{v_0^2(v_0^2 - v^2)}{(v_0^2 - v^2) + (v\Delta v)^2} \]

\[ \chi_i(v) = -\chi_0 \frac{v_0^2 v\Delta v}{(v_0^2 - v^2) + (v\Delta v)^2} \]

\( \sim \chi_0, \text{ low } v \)
\( \sim 0, \text{ high } v \)

near the resonance:
**Lorentzian function** of FWHM \( \Delta v \)
Refractive index and absorption near a resonance ($\nu \approx \nu_0$)

For resonant atoms embedded in a nondispersive host of ref. index $n_0$ the following simplifications can be made:

\[
\alpha(\nu) \approx -\left( \frac{2\pi\nu}{n_0 c_0} \right) \chi_i(\nu)
\]

\[
n(\nu) \approx n_0 + \frac{\chi_r(\nu)}{2n_0}
\]
Media with multiple resonances

Generally a medium has several resonances. The overall susceptibility is the sum of all contributions at $\nu_1$, $\nu_2$, $\nu_3$ ...
Absorption and ref. index for an optical material in the visible

(Note that the abscissa is the wavelength instead of frequency)
Conclusions – monochromatic waves in dispersive media

- A dispersive medium shows a frequency-dependent susceptibility and refractive index.

- This leads to a frequency-dependent wavenumber $k$ and light speed $c$.

- A complex susceptibility can be used to describe dispersion (real) and absorption (imaginary).

- An oscillatory model for the constitutive relation leads to the appearance of resonances at specific frequencies.

- For a medium with multiple resonances, the overall susceptibility is the sum of each contribution.
Pulse propagation in dispersive media

The propagation of light pulses in dispersive media is of critical importance in fields such as optical fiber communications.

A pulse is composed of a range of frequencies – in a dispersive medium, they have different speeds and different attenuations.

So what happens to the pulse?
Group velocity

Let us consider a pulsed plane wave travelling along the $z$ direction through a lossless ($\alpha = 0$) medium with ref. index $n(\omega)$

$$U(0,t) = A(t)e^{i\omega_0 t}$$

It can be shown that if $n(\omega)$ varies slowly over the spectral bandwidth of the pulse, then the wavefunction at a position $z$ is given by

$$U(z,t) = A(t)\exp\left[i\omega_0\left(t - \frac{z}{v}\right)\right]$$

where

- $v = c_0 / n(\omega_0)$ is the phase velocity
- $\beta' = \frac{d\beta}{d\omega}\bigg|_{\omega_0}$ is the group velocity
- $\beta = \frac{\omega}{(c_0/n)}$ is the carrier
- $c = c_0 / n(\omega_0)$ is the envelope
Group velocity vs. phase velocity

\[ v < c \quad v = c \quad v > c \]
Some remarks about group and phase velocity

- The group velocity is characteristic of each medium and varies with the central frequency $\omega_0$.
- For a nondispersive medium ($n = n_0 = \text{constant}$) we have

$$c = c_0 / n_0$$

$$\frac{1}{v} = \frac{d}{d\omega} \left( \frac{\omega}{c_0/n_0} \right) = n_0 / c_0 \quad \rightarrow v = c$$

- Group velocity is associated with a **time delay**:

$$\tau_d = z / v$$

- Normally the group velocity is expressed in terms of wavelength:

$$v = \frac{c_0}{n_0 - \lambda_0 \left( \frac{dn}{d\lambda} \right)}$$

(> demonstrate as an exercise)
The group velocity may also depend on the frequency.

In this case different frequency components will experience different time delays. This is called **group velocity dispersion (GVD)**.

Let us calculate the delay for two pulses with central frequencies separated by $\Delta \nu$:

$$\Delta \tau_d \approx \frac{d\tau_d}{d\nu} \Delta \nu = \frac{d}{d\nu} \left( \frac{z}{v(\nu)} \right) \Delta \nu = D_v \cdot z \cdot \Delta \nu$$

$$D_v = \frac{d}{d\nu} \left( \frac{1}{v} \right) = 2\pi \frac{d}{d\omega} \left( \frac{1}{v} \right) = 2\pi \beta''$$

dispersion coefficient (s/m.Hz)
The dispersion coefficient is a measure of pulse broadening.

A pulse with initial spectral width $\Delta \nu$ after a distance $z$ becomes enlarged by

$$\Delta \tau = |D_{\nu}| \cdot \Delta \nu \cdot z$$

Normally the dispersion coefficient is expressed in terms of wavelength:

(units = ps / km.nm)

$$D_{\lambda} = -\frac{\lambda_0}{c_0} \frac{d^2 n}{d\lambda^2}$$
Normal / anomalous and positive / negative dispersion

The sign of $D_\nu$ does not affect the pulse broadening, but it affects the phase (and the chirp) inside the pulse envelope.

$$\frac{dn}{dv} > 0 \left( \frac{dn}{d\lambda} < 0 \right)$$

normal dispersion

$$\frac{dn}{dv} < 0 \left( \frac{dn}{d\lambda} > 0 \right)$$

anomalous dispersion

$D_\nu > 0 \ (D_\lambda < 0)$

positive dispersion

$D_\nu < 0 \ (D_\lambda > 0)$

negative dispersion

Example: normal dispersion medium
Gaussian pulse in a normal dispersion medium

https://youtu.be/LfPuOnVL3ak
Gaussian pulse in a dispersion-managed medium

https://youtu.be/mMpCdKOrB-Y
Dispersion near a resonance

In this region the absorption and the refractive index can change very rapidly.

Points of inflexion of \( n(\lambda) = \frac{c_0}{c} \)

Maxima of \( N = \frac{c_0}{v} \)

\[
v = \frac{c_0}{n - \lambda_0 \left( \frac{dn}{d\lambda} \right)}
\]

Zeros of \( D_\lambda \)

\[
D_\lambda = -\frac{\lambda_0}{c_0} \frac{d^2 n}{d\lambda^2}
\]
Anomalous dispersion

Next to a resonance we may have some unusual conditions:

Slow / fast light propagation

We may have $v > c$ (fast light) or $v << c$ (slow light)

Negative group velocity

For $v < 0$ the pulse appears to exit the medium before entering it

R. Boyd & P. Narum, J. Mod. Optics 2007

Gain-assisted superluminal propagation

“The observed superluminal light pulse propagation is not at odds with causality, being a direct consequence of classical interference between its different frequency components in an anomalous dispersion \((dn/d\nu < 0)\) region.”

A dispersive medium can be represented as a chirp filter

Let us consider a linear, lossless, dispersive medium where a monochromatic plane wave is propagating along $z$.

We are now going to show that the dispersion can be represented by a chirp filter, and determine the characteristics of the filter.
Transfer function $H(\omega)$ for a monochromatic wave

Propagation along $z$ introduces a phase shift

In fact, *propagation* is equal to multiplying the wavefunction by a complex exp:

\[
\beta(\omega)z = \frac{\omega n(\omega)}{c_0} z
\]

\[
U(z,t) = U(0,t)e^{-i\beta(\omega)z}
\]

\[
V(z,\omega) = \tilde{H}(\omega)V(0,\omega)
\]

\[
\tilde{H}(\omega) = e^{-i\beta(\omega)z}
\]

transfer function for a monochromatic wave
Transfer function $H(\omega)$ for a monochromatic wave

For a pulse travelling along $z$ the wavefunction must include a time-varying amplitude:

$$U(z,t) = A(z,t) \exp[-i(\beta_0 z - \omega_0 t)]$$

Fourier-transforming this, we obtain the general expression for $V(z,\omega)$ at a position $z$:

$$V(z,\omega) = \tilde{A}(z,\omega - \omega_0) e^{-i\beta_0 z}$$  \hspace{1cm} (2)

At $z = 0$ we have

$$V(0,\omega) = \tilde{A}(0,\omega - \omega_0)$$

Inserting in (1)
Envelope transfer function $H_e(\delta \omega)$ for a pulse

Making $\ (1) = (2)\ 

$$\tilde{A}(z, \omega - \omega_0) = \tilde{A}(0, \omega - \omega_0) \exp\left[-i(\beta(v) - \beta_0)\right]$$

Since we are working around $\omega_0$, let us find the envelope transfer function for $\tilde{A}(\delta \omega)$:

$$\tilde{A}(z, \delta \omega) = \tilde{A}(0, \delta \omega) \tilde{H}_e(\delta \omega)$$

$\delta \omega = \omega - \omega_0$

$$\tilde{H}_e(\delta \omega) = \exp\left\{-i\left[\beta(\omega_0 + \delta \omega) - \beta(\omega_0)\right]z\right\}$$
The envelope transfer function as a chirp filter

The envelope transfer function is of the type:

\[ \tilde{H}_e(\delta \omega) = \exp[-i\psi(\delta \omega)] \]

\[ \psi(\delta \omega) = [\beta(\omega_0 + \delta \omega) - \beta(\omega_0)]z \]

Let’s expand this around \( \omega_0 \):

\[ \psi(\delta \omega) \approx [\beta'\delta \omega + \frac{1}{2} \beta'' \delta \omega^2]z \]

This has the form of an ideal + chirp filter, so let’s compare with the general expression:

\[ \exp(-i\tau_d \delta \omega) \exp\left(-i \frac{b}{4} \delta \omega^2\right) \]

We obtain expressions for the **time delay** and the **chirp coefficient** (both proportional to the distance \( z \)):

\[ \tau_d = \beta' z = \frac{z}{\nu} \]

\[ b = 2\beta'' z = \frac{D}{\nu} \frac{z}{\pi} \]
Conclusions

The propagation of a pulse in a dispersive medium can be described by two filters:

• an **ideal filter** with a time delay $\tau_d = z/v$ associated to the group velocity $v = 1/\beta'$

• a **chirp filter** with chirp coefficient $b = 2\beta''z$

• the output pulses will have a chirp sign depending on the sign of $\beta''$

• for optical media, material dispersion makes $\beta'' > 0$ and the pulses become **up-chirped**

![Diagram showing pulse propagation through dispersive medium with ideal and chirp filters, showing original pulse, dispersive medium, and delayed, broadened pulse.](image)