Photon optics

• Photons and modes

• Photon properties
  Energy, polarization, position, momentum, interference, time

• Photon streams
  Mean photon flux
  Randomness of photon flow
  Photon-number statistics

References:
Fundamentals of Photonics, Ch. 12
From ray optics to quantum optics

- Ray optics
- Wave optics
- Electromagnetic optics
- Quantum optics
In quantum optics, light is described by photons

- **Ray optics**
  - Rays

- **Wave optics**
  - Scalar wave function \( u(r,t) \)

- **EM optics**
  - Vectorial wavefunction \( E(r,t), H(r,t) \)

- **Quantum optics**
  - Photon
    - Carries EM energy, momentum and angular momentum (spin)
    - Can carry orbital angular momentum
    - Zero rest mass: travels at \( c_0 \) (or \( c \) in matter)
    - Particle *and* wavelike character

**Photons and modes**  **Photon properties**  **Photon streams and statistics**
EM optics theory of light in a resonator

Light
EM field consisting in a superposition of discrete orthogonal modes

A combination of
- frequency
- spatial distribution
- polarization

One possibility for the expansion functions are monochromatic waves

\[ E(r,t) = Re[E(r,t)] \]
\[ E(r,t) = \sum_{q} A_q U_q(r) \exp(i2\pi \nu_q t) \hat{\epsilon}_q \]

The (complex) spatial distribution function is normalized:

\[ \int_{V} |U_q(r)|^2 = 1 \]
A convenient choice for the $U_q(r)$ is a set of standing waves:

$$U_q(r) = \left(\frac{2}{d}\right)^{\frac{3}{2}} \sin\left(q_x \frac{\pi}{d} x\right) \sin\left(q_y \frac{\pi}{d} y\right) \sin\left(q_z \frac{\pi}{d} z\right)$$

$q = (q_x, q_y, q_z)$, $q_i = 1, 2, 3…$

The energy density $\mathcal{W}$ of each mode and the total energy contained in the mode are:

$$\mathcal{W} \equiv \frac{1}{2} \varepsilon E^2 = \frac{1}{2} \varepsilon |A_q|^2 |U_q(r)|^2$$

$$E_q = \frac{1}{2} \varepsilon \int_V |A_q|^2 |U_q(r)|^2 \, dr = \frac{1}{2} \varepsilon |A_q|^2$$

In classic EM any positive value of $E_q$ is possible.
Photon optics theory of light in a resonator

Mode energy is restricted to discrete values separated by a fixed energy. Only multiples of this unit are allowed: the energy is quantized. Each unit of energy is carried by one photon.

Light
Set of modes, each containing an integral number of identical photons.

Mode frequency, spatial distribution, direction of propagation and polarization are assigned to the photon.
The photon is the energy unit of an electromagnetic mode

Energy of a photon in a mode of frequency $\nu$: $E = \hbar \nu = h\omega$
(only added or taken in multiples of $h\nu$)

Energy of mode with $n$ photons:

$E_n = (n + \frac{1}{2}) h\nu, \ n = 0, 1, 2, \ldots$

Example: Photon with $\lambda = 1 \ \mu m$

$E = h\nu = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{1 \times 10^{-6}}$

$\approx 1.99 \times 10^{-19} \ J \approx 1.24 \ eV$

zero-point energy:

$E_0 = \frac{1}{2} h\nu$

(in most experiments, only energy differences are measured)
The particle/wave nature of a photon depends on its frequency.
A photon may have linear or circular polarization

For each monochromatic wave traveling in some direction, there are always two polarization modes:

- $x$ and $y$ directions
- $x'$ and $y'$ directions, etc
- right- and left circular

the photon polarization is that of the mode.

How to describe the polarization of a photon?

In classical EM theory, polarization is described by a Jones vector $(A_x, A_y)$:

- LP (linear pol.) along x-axis
- LP along 45 ° from x-axis
- RCP (right hand circular pol.)
The components of the Jones vector become probabilities

\[ A_x, A_y \text{ complex probability amplitudes} \]

\[ |A_x|^2, |A_y|^2 \text{ probability that the photon is observed in the } x, y \text{ polarization modes} \]
The position of a photon is also a probabilistic function

A photon of frequency $\nu$ is associated with a wavefunction $U(r)\exp(i2\pi \nu t)$ of the mode and extends over the entire mode.

But when it hits a detector of area $dA$, it is either detected or not detected.

The probability $p(r)dA$ of observing a photon at a point $r$ within $dA$, at any time, is proportional to the local optical intensity $I(r)$:

$$p(r)dA \propto I(r)dA,$$

$$I(r) \propto |U(r)|^2.$$
Photons behave as extended AND localized entities

Example: standing wave in cavity of length $d$

$$I(x, y, z) \propto \sin^2 \left( \frac{\pi z}{d} \right)$$

We apply the **photon position rule** to find out the probability of finding the photon

<table>
<thead>
<tr>
<th>Waves</th>
<th>extended in space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particles</td>
<td>localized in space</td>
</tr>
<tr>
<td>Photons</td>
<td>wavelike (probability) + particlelike (when detected)</td>
</tr>
<tr>
<td></td>
<td>= wave-particle duality</td>
</tr>
</tbody>
</table>

![Graph showing max and zero probability](image)
Transmission of a single photon through a beamsplitter

\( T \) intensity transmittance
\( R = 1 - T \) intensity reflectance

\( I_t = T I \)
\( I_r = R I = (1 - T) I \)

What is the probability of finding a transmitted/reflected photon (it can only choose one direction)?
Applying the photon position rule:

\[ p_t = \frac{I_t}{I} = T \]
\[ p_r = \frac{I_r}{I} = 1 - T \]
A photon carries linear and spin angular momentum

An EM plane wave carries a linear momentum density (per unit volume):

\[ (\mathcal{W}/c)\hat{k} \]

(check *Fund. Photon.*, 5.4)

A photon carries a linear momentum:

\[ p = (E/c)\hat{k} \]
\[ E = \hbar \omega = \hbar c k \]

Linear momentum associated with a photon in a plane-wave mode \( k \):

\[ p = \hbar k \]

Magnitude:

\[ p = |p| = \hbar k = \hbar \frac{\omega}{c} = \frac{E}{c} = \frac{h}{\lambda} \]

Photon spin:

\[ S = \pm \hbar \]
A single photon can exhibit interference effects

Performing Young’s experiment with a single photon:

\[ I(x) \approx 2I_0 \left( 1 + \cos \frac{2\pi \theta x}{\lambda} \right) \]

G. I. Taylor (1909) did this experiment and obtained the classical interference pattern by acquiring many individual photons. The extended nature of each photon allows it to pass through both holes at the same time, but be detected individually.
Describing photons using non-monochromatic modes

In monochromatic (single frequency) modes, with constant intensity, a photon is likely to be detected at any time.

More generally, light can be expanded in polychromatic modes, of time-varying intensity.

The probability $p(r) \cdot dA dt$ of observing a photon at a point $r$ within an area $dA$ and a time $dt$, is proportional to the optical intensity $I(r, t)$:

$$p(r, t) dA dt \propto I(r, t) dA dt, \quad I(r, t) \propto |U(r, t)|^2$$

**photon position and time**
A photon in a wavepacket described by $U(t)$ with a duration $\sigma_t$ has an uncertainty in its frequency (and energy) given by $\sigma_v$ such that

$$\sigma_v \sigma_t \geq \frac{1}{4\pi}$$

$$\sigma_\omega \sigma_t \geq \frac{1}{2}$$

$$\sigma_E \sigma_t \geq \frac{\hbar}{2}$$
What happens when there is a large number of photons?

- The number of photons in each mode is generally random.
- It is described by a **probability distribution** that depends on the quantum state of light.

**Temporal pattern:**
A detector that integrates spatially registers photons at random times.

**Spatial pattern:**
A detector that integrates temporally registers photons at random positions.
Mean variables for photon streams

**Monochromatic light**

**mean photon-flux density**

$$\phi(r) = \frac{l(r)}{h\nu}$$ (photons/s · cm²)

**Quasi-monochromatic light**

**mean photon-flux density**

$$\phi(r) \approx \frac{l(r)}{h\nu}$$ (photons/s · cm²)

### Table: Mean Photon-Flux Density

<table>
<thead>
<tr>
<th>Source</th>
<th>Mean Photon-Flux Density (photons/s-cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Starlight</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Moonlight</td>
<td>$10^8$</td>
</tr>
<tr>
<td>Twilight</td>
<td>$10^{10}$</td>
</tr>
<tr>
<td>Indoor light</td>
<td>$10^{12}$</td>
</tr>
<tr>
<td>Sunlight</td>
<td>$10^{14}$</td>
</tr>
<tr>
<td>Laser light(^a)</td>
<td>$10^{22}$</td>
</tr>
</tbody>
</table>

\(^a\) A 10-mW He–Ne laser beam at $\lambda_o = 633$ nm focused to a 20-μm-diameter spot.
Mean variables for photon streams

**Mean photon-flux**

\[
\Phi(r) = \int_A \phi(r) = \frac{P}{h\nu},
\]

\[
P = \int_A I(r) \, dA
\]

(photons/s)

(J/s = W)

Example:

\[
P = 1 \text{ nW} @ 0.2 \mu\text{m}
\]

\[
E \approx 10^{-18} \text{ J} = 6.24 \text{ eV}
\]

\[
\Phi \approx 10^{-9} / 10^{-18} \approx 10^9 \text{ photons/s} = 1 \text{ photon/ns}
\]

**Mean photon number**

detected in an area \( A \) and in a time \( T \)

\[
\bar{n} = \Phi T = \frac{E}{h\nu}
\]

\[
E = PT
\]

(J)
Randomness applied to photon streams

- Single photons: $l(r,t) \propto p(r,t)$
- Photon streams: $l(r,t) \propto \phi(r,t)$

Probability at $(r,t)$
Mean photon flux density at $(r,t)$

Constant intensity, but random detection times (depends on source)
Density of photon detections follows $P(t)$
Photon number statistics

Example: \( P = 1 \text{ nW} \@ 1.0 \mu m \) \( E \approx 0.2 \cdot 10^{-18} \text{ J} = 1.25 \text{ eV} \)
\( \Phi \approx 10^{-9} / 0.2 \cdot 10^{-18} = 5 \text{ photon/ns or 0.005 ps}^{-1} \) on average

Question: If we divide 100 ns in \( 10^5 \) intervals of \( T = 1 \) ps each
- how many will have \( n = 0 \) photons?
- how many will have \( n = 1 \) photon?
- how many will have \( n = 2 \) photons?
- etc…
The statistical distribution of the photons depends on the source.

**Coherent light**
- Photon arrivals are independent
- **Poisson distribution**
  \[ p(n) = \frac{n^n \exp(-n)}{n!}, \quad n = 0, 1, 2, \ldots \]

**Thermal light**
- Photon arrivals are not independent
- **Boltzmann distribution**
  \[ p(E_n) \propto \exp\left(-\frac{E_n}{kT}\right), \quad k = 1.38 \times 10^{-23} \text{ J/K} \]
Photon statistics for coherent light

Constant optical power but random arrival times:

Poisson distribution $p(n)$ for photon number $n$

Photon properties
Photon statistics for coherent light

The mean value, variance and signal-to-noise ratio are defined as

\[ \bar{n} = \sum_{n=0}^{\infty} np(n) \]
\[ \sigma_n^2 = \sum_{n=0}^{\infty} (n - \bar{n})^2 p(n) \]
\[ \text{SNR} = \frac{\bar{n}^2}{\sigma_n^2} \]

For the Poisson distribution:

\[ \sigma_n^2 = \bar{n} = \text{SNR} \]

i.e. the SNR improves with the number of photons. This is very important for the elimination of errors in optical communications.

<table>
<thead>
<tr>
<th>SNR = 1000</th>
<th>SNR = 100</th>
<th>SNR = 10</th>
<th>SNR = 3.1623</th>
<th>SNR = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
</tr>
</tbody>
</table>
Photon statistics for thermal light

Photon number depends on their energy and the temperature. At 300 K, $kT = 0.026$ eV

Using

$$E_n = (n + \frac{1}{2})h\nu$$

$$\sum_{n=0}^{\infty} p(n) = 1$$

the probability of finding $n$ photons is

$$p(n) = \frac{1 - \exp\left(-\frac{h\nu}{kT}\right)}{\exp\left(\frac{nh\nu}{kT}\right)}$$
This probability distribution may be written more simply as

\[ p(n) = \frac{1}{\bar{n} + 1} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^n \]

Where the mean value is

\[ \bar{n} = \frac{1}{1 - \exp \left( \frac{\hbar \nu}{kT} \right)} \]
Photon statistics for thermal light

For the Bose-Einstein distribution:

\[
\sigma_n^2 = \bar{n} + \bar{n}^2
\]

\[
\text{SNR} = \frac{\bar{n}}{\bar{n} + 1}
\]

Thermal light has a larger variance than coherent light (more uncertainty)
The SNR is always <1, no matter the power
This form of light is too noisy for high-data-rate information transmission