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F. Afonso, L. Amândio, A. Marta and A. Suleman  
Robust and Reliability Based Design Optimization  
May 22, 2015
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Uncertainty

Nikolaidis [1] defined uncertainty as:
"Certainty, in the context of decision theory, is the condition in which a decision maker knows everything needed to select the most desirable outcome. Uncertainty is the gap of what the decision maker presently knows and certainty."

- Every real engineering problem has an associated uncertainty, which can arise from many different sources and are present during design, manufacturing and operation [2].
- For aeronautic, Yu and Du [3] elaborated the following list:
  - **Uncertainties in operations** - e.g. aerodynamic loads, flight speed, altitude, angle of attack
  - **Uncertainties in material properties** - e.g. material tensile strength and Young’s modulus
  - **Uncertainties in manufacturing processes** - e.g. tolerances on dimensions and shapes
  - **Modeling uncertainties** - e.g. simplifications of computational models, experimental determination of parameters, fidelity appropriate to the design stage
Uncertainty Models

- Uncertainties can be represented in diverse ways and can be used in computational simulations or mathematical models.

- Several mathematical models have been proposed which can be divided in 3 main categories [4]:
  - Interval bound;
  - Membership function;
  - Probability density function.

- Historically uncertainty formulation have been done in terms of probability theory, although, recently its application is being questioned since several other distinct mathematical theories are being shown to perform well in designing with uncertainty [5], [6].

![Uncertainty Descriptions [4]](image-url)
According to Wojtkiewicz et al. [7] uncertainty can be classified in two categories:

- **Aleatory, stochastic or random** - if it is related to the inherent variability in natural phenomena. Hence, it cannot be reduced, short of changing the phenomenon itself. It is irreducible even if more sample data/information is collected.

- **Epistemic or subjective** - in which case the shortcomings of the models used to describe physical phenomena come into play. It stems from a lack of knowledge and is therefore reducible through obtaining additional information. It is also usually biased.
A deterministic optimization formulation does not account for the uncertainties in the design variables and parameters and simulation models.

Deterministic optimum solutions are usually associated to a high probability of failure (e.g. the optima can be very new constraint boundary).

Some of the advantages and disadvantages related to the deterministic optimization are summarized below:

**Pros**
- Easy to implement
- Relatively good results
- Industrial has years of experience with this kind of optimization

**Cons**
- Hard to apply to vehicles with novel configurations
- Difficulty when accounting for uncertainties
- Robustness and reliability inconsistency throughout the vehicle

Nowadays the markets competitiveness have impelled that the design is at the same time optimum and robust and reliable solutions.

Hence, it is of paramount importance that design optimizations take uncertainties into account [8].
In order to include uncertainties in design optimization process one has to change the standard deterministic problem appropriately.

Two main methodologies that incorporates uncertainty in the design optimization are:

- **RDO** - Robust Design Optimization [9] [10];
- **RBDO** - Reliability Based Design Optimization [10] [11] [12].

There have been developments in the way of combining the RDO and RBDO formulations:

- **R²BDO** - Robust and Reliability Based Design Optimization [10] [13].
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Random Variable

A random variable $X$ is a variable that can assume any value from a set of possible $x$ values, each associated with a given probability. Random variables can be either discrete or continuous.

Probability Density Functions

The Probability Density Function (PDF), $f_X(X)$, is the mathematical function that describes the distribution of the possible values $x$ of $X$ and their respective probabilities. Any PDF satisfies the following conditions:

$$f_X(X) \geq 0 \text{ for all values of } X \text{ and}$$

$$\int_{-\infty}^{+\infty} f_X(t) dt = 1$$

and two properties of the Cumulative Density Function (CDF):

$$P(-\infty < t < +\infty) = 1$$

$$P(t = a) = 0 \ \forall \ a \in \mathbb{R}$$
### Expected Value

The expected value, $E(X)$, or mean value $\mu_X$ of a random value $X$ defines the centre of its position and for a continuous random variable is given by:

$$E(X) = \mu_X = \int_{-\infty}^{+\infty} t \ f_X(t) \ dt$$

### Variance

The variance, $V(X)$, or the second moment $\sigma_X^2$ of a random value $X$ is a measure of dispersion of a distribution and is given by:

$$V(X) = \sigma_X^2 = E[X - \mu_X]^2 = E(X^2) - E(X)^2 = \int_{-\infty}^{+\infty} (t - \mu_X)^2 \ f_X(t) \ dt$$

The standard deviation of $X$, $\sigma_X$, is the positive square root of the variance.
High Order Moments

For any positive integer $n$, $X$’s $n^{th}$ order central moment is defined by:

$$E[X - \mu_X]^n = \int_{-\infty}^{+\infty} (t - \mu_X)^n f_X(t) \, dt$$

The Skewness ($S(X)$) and the Kurtosis ($K(X)$) are respectively the third and fourth central moments.

Coefficient of Variance

The coefficient of variance, $c.o.v.$, is the standard deviation divided by the absolute value of the mean:

$$c.o.v. = \frac{\sigma}{|\mu|}$$
Multivariate Distributions

For independent random variables, the joint probability function is defined as the product of each individual PDF:

\[ f_{X_1, X_2, \ldots, X_N}(X_1, X_2, \ldots, X_N) = f_{X_1}(X_1) f_{X_2}(X_2) \ldots f_{X_N}(X_N) \]

If \( Z = g(X_1, \ldots, X_N) \) is another random variable, its mean is given by:

\[ E(Z) = E[g(X_1, \ldots, X_N)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(t_1, \ldots, t_N) f_{X_1, \ldots, X_N}(t_1, \ldots, t_N) \, dt_1, \ldots, dt_N \]

A generic formulation for the multivariate central moments is given by:

\[ M_{m_1, \ldots, m_N} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X_1, \ldots, X_N}(t_1, \ldots, t_N) \prod_{i=1}^{N} (x_i - \mu_x_i)^{m_i} \, dt_1, \ldots, dt_N \]
This is an important concept to help understanding the reliability targets.

In an uncertainty based optimization and in particularly in a Reliability Based Design Optimization the designer has to ensure that the probability of the design failing is within certain prescribed values.

Assuming that both objective and constraint functions have normal distributions, these probabilities are associated with different Sigma levels (or standard deviations $\sigma$).

<table>
<thead>
<tr>
<th>Sigma Level</th>
<th>Percent Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm 1\sigma$</td>
<td>68.26</td>
</tr>
<tr>
<td>$\pm 2\sigma$</td>
<td>95.46</td>
</tr>
<tr>
<td>$\pm 3\sigma$</td>
<td>99.73</td>
</tr>
<tr>
<td>$\pm 4\sigma$</td>
<td>99.9937</td>
</tr>
<tr>
<td>$\pm 5\sigma$</td>
<td>99.999943</td>
</tr>
<tr>
<td>$\pm 6\sigma$</td>
<td>99.9999998</td>
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Sigma level as percent variation
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Robust Design Optimization

Introduction

Robustness Definition

Noor [16] defined robustness as: “the degree of tolerance to variations (in either the components of a system or its environment). A robust ultra-fault-tolerant design of an engineering system is depicted. The performance of the system is relatively insensitive to variations in both the components and the environment. By contrast, a nonrobust design is sensitive to variations in either or both.”

Robust Design Optimization (RDO) methods:

- seek a design whose performance is insensitive to small changes in the uncertainty quantities;
- deal with everyday fluctuations;
- focus on the event distribution near the mean value.

In this way the impact of the uncertainties design is harder to perceive.
minimize \( F(\mu_f(x, r), \sigma_f(x, r)) \)
subject to \( G_i(\mu_{g_i}(x, r), \sigma_{g_i}(x, r)) \leq 0 \quad i = 1, \ldots, n_g \)
\[
P\left( x_{k, LB} \leq x_k \leq x_{k, UB} \right) \geq P_{bounds} \quad k = 1, \ldots, n_{DV}
\]

\( \mu \) mean
\( \sigma \) standard deviation
\( x \) deterministic design variable
\( r \) stochastic design variable
\( n_g \) number of constraints
\( n_{DV} \) number of design variables
\( P \) probability
The robust objective and constraints are now functions of the mean and standard deviation of objective and constraints, which in turn depend on the probabilistic distribution of variables.

The analytic integration of the mean and standard deviation is not possible at the majority of the cases.

\[
\begin{align*}
\mu_f(x) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) p_{x,r}(t) dt \\
\sigma_f(x) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [f(t) - \mu_f(x, r)]^2 p_{x,r}(t) dt
\end{align*}
\]

So a numerical technique is required:

- Monte Carlo Integration (MC);
- Taylor based Method of Moments (MM) [9];
- Sigma Point Method (SP) [17];
- Surrogate Approximation of \( \mu_f \) and \( \sigma_f \) directly [18];
- Gaussian Quadrature [19].
Monte Carlo (MC) Method

- Monte Carlo based methods are classical numerical approaches to solve an integral.
- Were first introduced in the 1940s and have been ever since widely used in uncertainty analysis.
- It is a stochastic method based on the probability distribution of the output of a process determined by the stochastic distribution of the inputs.
- The process is repeated \( n \) iterations by building a new sample with the desired characteristics at each iteration.

\[
\mu_f = \frac{1}{N} \sum_{i=1}^{n} f(x_i)
\]

\[
\sigma_f^2 = \frac{1}{n-1} \sum_{i=1}^{n} (f(x_i) - \mu_f)^2
\]
As the name implies, the Taylor Based Method of Moments employs a Taylor series expansion of the function $f$ of a random vector $x$ about the mean of $x$:

$$f(x) = f(\mu_x) + \sum_{i=1}^{N_{RV}} \left( \frac{\partial f}{\partial x_i} \right) (x_i - \mu_{x_i}) +$$

$$+ \left( \frac{1}{2} \right) \sum_{i=1}^{N_{RV}} \sum_{j=1}^{N_{RV}} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) (x_i - \mu_{x_i})(x_j - \mu_{x_j}) +$$

$$+ \left( \frac{1}{3!} \right) \sum_{i=1}^{N_{RV}} \sum_{j=1}^{N_{RV}} \sum_{k=1}^{N_{RV}} \left( \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \right) (x_i - \mu_{x_i})(x_j - \mu_{x_j})(x_k - \mu_{x_k}) +$$

$$+ \ldots$$
Taylor based Method of Moments (MM)

\[
\mu_f = f(\mu_x) + \left( \frac{1}{2} \right) \sum_{i=1}^{N_{RV}} \left( \frac{\partial^2 f}{\partial x_i^2} \right) \sigma_{x_i}^2 + \ldots
\]

\[
\sigma_f^2 = \sum_{i=1}^{N_{RV}} \left( \frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 +
\]

\[
+ \sum_{i=1}^{N_{RV}} \left[ \left( \frac{\partial^3 f}{\partial x_i^3} \right) \left( \frac{\partial f}{\partial x_i} \right) \left( \frac{\kappa_{x_i}}{3} \right) + \left( \frac{\partial^2 f}{\partial x_i^2} \right)^2 \left( \frac{\kappa_{x_i} - 1}{4} \right) \right] \sigma_{x_i}^4 +
\]

\[
+ \sum_{i=1}^{N_{RV}} \sum_{j=1, i \neq j}^{N_{RV}} \left[ \left( \frac{\partial^3 f}{\partial x_i^2 \partial x_j} \right) \left( \frac{\partial f}{\partial x_j} \right) + \left( \frac{1}{2} \right) + \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 \right] \sigma_{x_i}^2 \sigma_{x_j}^2 +
\]

\[
+ \ldots
\]

These estimates are third order accurate. The \( \kappa \) is the kurtosis of standard deviation and \( N_{RV} \) is the number of random variables.
Sigma Point Method (SP)

- The Sigma Point (SP) method is a derivative of the Taguchi method, used in statistical tolerance estimation since 1978 [17].
- The idea behind SP is that it is easier to match an input distribution (typically a normal distribution) than to linearise (or in general, approximate) a non-linear mapping.
- To compute the integrals in SP employs a procedure similar to Gaussian integration, but where the sample locations and respective weights are optimized to match the first moments of the input probability distribution.

\[
\hat{\mu}_f (\chi_0) = W_0 f (\chi_0) + \sum_{i=1}^{N_{RV}} W_i (f (\chi_+ + f (\chi_-))
\]

\[
2\sigma^2_f (\chi_0) = \sum_{i=1}^{N_{RV}} W_i (f (\chi_+ - f (\chi_-))^2 + \\
+ \sum_{i=1}^{N_{RV}} (W_i - 2W^2_i) ((\chi_+ + f (\chi_-) - 2f (\chi_0))^2
\]
Sigma Point Method (SP)

\[
\chi_0 = \mu_x
\]

\[
\chi_+ = \mu_x + \sqrt{(N_{RV} + K)} \left( \sqrt{\sum_{x}} \right)_i, i = 1, \ldots, N_{RV}
\]

\[
\chi_- = \mu_x - \sqrt{(N_{RV} + K)} \left( \sqrt{\sum_{x}} \right)_i, i = 1, \ldots, N_{RV}
\]

\[
W_0 = \frac{K}{N_{RV} + K}
\]

\[
W_i = W_{i+} = W_{i-} = \frac{1}{2(N_{RV} + K)}
\]

\[
(\sqrt{\sum_{x}})_i \quad \text{the } i^{th} \text{ row in the square root of the covariance matrix}
\]

\[
K \quad \text{real constant that should be so that } N_{RV} + K = 3, \text{ the kurtosis of the standard normal}
\]

- The number of evaluations required for compute sensitivities in a finite difference estimate for gradient based optimization is a huge problem in terms of computational time, if finite differences are used.
Gaussian Quadrature Method (GQM)

\[
\mu_f = \sum_{i_1=1}^{N} W_{i_1} (\sum_{i_2=1}^{N} W_{i_2} (\ldots (\sum_{i_n=1}^{N} W_{i_n} f(x_{i_1,i_2,\ldots,i_n}))))
\]

\[
\sigma_f^2 = \sum_{i_1=1}^{N} W_{i_1} (\sum_{i_2=1}^{N} W_{i_2} (\ldots (\sum_{i_n=1}^{N} W_{i_n} (f(x_{i_1,i_2,\ldots,i_n}) - \mu_f)^2))))
\]

This method provides an approximation of the mean and standard deviation of a function on a domain by a suitably weighted sum, where \(x_i\)'s are called nodes and are suitably selected in the domain.
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Reliability Based Design Optimization (RBDO) methods:

- Seeks a design whose probability of failure is less than a certain value
- Deals with extreme events that may lead to failure
- Focuses on the event distribution near the tails of the PDF
RBDO Formulation

\[ \text{minimize } \quad f(x, r) \]

subject to \[ g^r_{ic}(x, r) \leq 0 \quad i = 1, \ldots, n_{rc} \]
\[ g^d_j(x) \leq 0 \quad j = 1, \ldots, n_d \]
\[ x^L_k \leq x_k \leq x^U_k \quad k = 1, \ldots, n_{DV} \]

Alternatively,

\[ \text{minimize } \quad P(f(x, r) - \text{target} \geq 0) \quad \text{or} \quad P(\text{target} - f(x, r) \geq 0) \]

subject to \[ g^r_{ic}(x, r) \leq 0 \quad i = 1, \ldots, n_{rc} \]
\[ g^d_j(x) \leq 0 \quad j = 1, \ldots, n_d \]
\[ x^L_k \leq x_k \leq x^U_k \quad k = 1, \ldots, n_{DV} \]
Reliability Constraints

\[ g^{rc}_i = P_{fi} - P_{allow_i} = P(g(x, r) \geq 0) - P_{allow_i} \]

\[ P(g(x, r) \geq 0) = \int_{g(x,r)\geq0} p_{x,r}(t) dt \]

- \( x \) deterministic design variable
- \( r \) stochastic design variable
- \( g^{rc}_i \) reliability constraints
- \( g^d_j \) other design constraints
- \( n_{rc} \) number of reliability constraints
- \( n_d \) number of other constraints
- \( n_{DV} \) number of design variables
- \( P \) probability
- \( P_{fi} \) probability of failure
- \( P_{allow_i} \) allowable probability of failure
Determining the probability of failure, $P_{f_i}$, requires either sampling (again, Monte Carlo Method) or techniques such as the:

- **First Order Reliability Method** (FORM) [20] [21] [22];
- **Second Order Reliability Method** (SORM) [20] [21] [22];
- **Sequential Optimization and Reliability Assessment** (SORA) [23] [24];
- **Reliable Design Space** (RDS) [25].
In essence, FORM consists of creating a linear approximation to the limit state function \( g(r) \) (\( r \) now being a generalized set of random variables - which encompasses the uncertainties in both design variables and parameters). According to FORM the probability of failure is evaluated (approximately) as:

\[
P_{f_i} = \Phi(-\beta),
\]

where \( \Phi \) is the cumulative distribution function of the standard normal distribution and \( \beta \) is the distance from the Most Probable Point (MPP) of failure to the current iterate (also called reliability index), measured in the standard normal space - \( u \).

Two approaches to solve Most Probable Point problem was investigated:

- **Reliability Index Approach (RIA)**
- **Performance Measure Approach (PMA)**

A transformation from the \( r \)-space to the \( u \)-space is needed:

<table>
<thead>
<tr>
<th>Distribution type</th>
<th>Transformation, ( r_k = T^{-1}(u_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal ((\mu, \sigma))</td>
<td>( \mu + \sigma u_k )</td>
</tr>
<tr>
<td>Log-normal ((\mu, \sigma))</td>
<td>( e^{\mu+\sigma u_k} )</td>
</tr>
<tr>
<td>Uniform ((a, b))</td>
<td>( a + (b - a) \left(0.5 + 0.5 \text{erf} \left(\frac{u_k}{\sqrt{2}}\right)\right) )</td>
</tr>
<tr>
<td>Gamma ((a, b))</td>
<td>( ab \left(\frac{u_k}{\sqrt{9a}} + 1 - \frac{1}{9a}\right)^2 )</td>
</tr>
</tbody>
</table>
The MPP problem using RIA can be defined as:

\[
\text{minimize} \quad (u^T u)^{\frac{1}{2}} \\
\text{subject to} \quad g(r(u)) = 0
\]

The reliability constraint in the RBDO problem may be written in terms of the reliability index \( \beta \):

\[
g_{rc}^i = \beta_{reqd} - \beta_i,
\]

where \( \beta_{reqd} \) is the specific reliability index. This approach has drawbacks, when for a particular set of design variables failure does not occur, or when the limit state surface is far from the origin.
To suppress the RIA draw-back the PMA was devise [26]. In the PMA approach the inverse problem of that one stated in RIA is solved instead with:

\[
\minimize_u \quad -g(u)
\]

subject to \( (u^T u)^{\frac{1}{2}} - \beta_{reqd} = 0 \)

This is not only a more robust formulation than RIA, it also immediately returns the required value of the reliability constraint:

\[ g_{i}^{rc} = g^* (r(u)) \]

Another important advantage is that the MPP subproblem may be formulated in a minimax approach, effectively handling multiple constraints simultaneously, something that is not possible with RIA.

There are other alternative approaches to the RIA and PMA presented here.
- SORM has seen little practical use in the main reliability problems since it requires higher order information on the objective function and constraints [27] [12] [22].
- Thus FORM is the most widely used.
Huang [23] define the Sequential Optimization and Reliability Assessment (SORA) method as:

"a single-loop method containing a serial of cycles of decoupled deterministic optimization and reliability assessment for improving the efficiency of probabilistic optimization."

The original SORA approach does not take into account the effect of changing variance in design problems.

Currently some efforts are being made in order to improve SORA efficiency to solve problems with changing variance.

The goal of SORA is to use serial single loops to efficiently optimize the objective function and assess its reliability.

The difference between this and other similar methods is the way it uses inverse MPP to establish deterministic constraints that are equivalent to the probabilistic ones.
The Reliable Design Space (RDS) method was introduced by Shan and Wang [25] and aims to convert the original deterministic constraint into a probabilistic one.

By this approach, only one single optimization loop needs to be done to reach the solution.

As a consequence, the number of required function evaluations is expected to drastically reduce, thus making this the most efficient method in theory.

However, it is important to notice that this method requires partial derivatives, which are not always obtainable and thus reducing its efficiency on those cases.
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<td>RBDO approach</td>
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<td></td>
<td>RBDO Approach</td>
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<tr>
<td>10</td>
<td>References</td>
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</table>
The Robust and Reliability Based Design Optimization (R²BDO) is a method that combines the RDO and RBDO characteristics. R²BDO uses the robust objective function and the reliable constraints. It was introduced by Paiva [10].

According to the author, this change outperforms the original RBDO formulation, because this last has frequent problems in dealing with the probabilistic objective function targets. A bad choice of the starting point can lead to probabilities which are either too close to zero or to one, varying slowly. What leads to insensitive objective functions and most likely the optimizer stops before reaching a true candidate to local/global minimum.

Also the RBO formulation is not the best in terms of treating with constraint since the user is left with the choice of weights, from which it is defined how far from the failure surface should the average optimum lie. These can be solve by calibrating weights in order to mimic a probabilistic constraint. The way the RBDO deals with constraints is better suited for optimization with uncertainties.
**R2BDO Formulation**

\[
\begin{align*}
\text{minimize} \quad & F (\mu_f(x, r), \sigma_f(x, r)) \\
\text{subject to} \quad & g_{i rc}^r (x, r) \leq 0 \quad i = 1, \ldots, n_{rc} \\
& g_{j d}^d (x) \leq 0 \quad j = 1, \ldots, n_d \\
& x_{LB k} \leq x_k \leq x_{UB k} \quad k = 1, \ldots, n_{DV}
\end{align*}
\]
### Mean and Standard Deviation Computation

The numeric methods to compute the mean and standard deviation of the objective function are the same ones used in RDO:

- Monte Carlo Integration (MC);
- Taylor based Method of Moments (MM);
- Sigma Point Method (SP);
- Surrogate Approximation of $\mu_f$ and $\sigma_f$ directly;
- Gaussian Quadrature.

### Probability of Failure Computation

The reliability constraint is also treated by:

- First Order Reliability Method (FORM);
- Second Order Reliability Method (SORM);
- Sequential optimization and reliability assessment (SORA).
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The Rosenbrock function is a difficult objective function and in this test its application is mainly focus on assessing how the accuracy of each method is affected by different uncertainty levels and target reliabilities.

To the classic Rosenbrock function, it can be added a constraint to study the robustness and reliability of one of the design parameters.

The constraint is basically an unitary circle.

The standard formulation of the Rosenbrock function is given by:

\[
\begin{align*}
\text{minimize} & \quad f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\
\text{subject to} & \quad g(x) = x_1^2 + x_2^2 - 1 \leq 0
\end{align*}
\]

The analytic solution of this optimization problem is \((x_1, x_2) = (0.7864, 0.6177)\) with the function value: \(f = 0.0457\).

For the reported analyses in this presentation it is used \(x_1\) as the parameter that carries uncertainty, while \(x_2\) remain deterministic.
The Rosenbrock function constrained to a unit circle can be posed in the RDO Formulation as follows:

\[
\begin{align*}
\text{minimize} & \quad F(x) = \mu_f(x) + \sigma_f(x) \\
\text{subject to} & \quad G(x) = \mu_g + 2\sigma_g \leq 0
\end{align*}
\]

The factor 2 in the constraint is called **Robustness Constraint**.

To solve this problem, the methods briefly introduced earlier were applied.

Monte Carlo Method was only used for post-optimality analysis of the others selected methods.
RDO Results

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Method</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\mu_f$</th>
<th>$\sigma_f$</th>
<th>$\epsilon \mu_f$</th>
<th>$\epsilon \sigma_f$</th>
<th>Fcalls</th>
<th>Dcalls</th>
<th>Gcalls</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>MM</td>
<td>0.7859</td>
<td>0.6163</td>
<td>0.0462</td>
<td>0.0002</td>
<td>-0.0033</td>
<td>-0.9969</td>
<td>91</td>
<td>91</td>
<td>91</td>
</tr>
<tr>
<td>0.001</td>
<td>SP</td>
<td>0.7858</td>
<td>0.6164</td>
<td>0.0461</td>
<td>0.0002</td>
<td>0.0000</td>
<td>-0.0001</td>
<td>67</td>
<td>0</td>
<td>201</td>
</tr>
<tr>
<td>0.005</td>
<td>MM</td>
<td>0.7831</td>
<td>0.6119</td>
<td>0.0510</td>
<td>0.0053</td>
<td>-0.0741</td>
<td>-0.9996</td>
<td>90</td>
<td>90</td>
<td>90</td>
</tr>
<tr>
<td>0.005</td>
<td>SP</td>
<td>0.7797</td>
<td>0.6065</td>
<td>0.0510</td>
<td>0.0054</td>
<td>0.0000</td>
<td>0.0004</td>
<td>66</td>
<td>0</td>
<td>198</td>
</tr>
<tr>
<td>0.010</td>
<td>MM</td>
<td>0.7797</td>
<td>0.6065</td>
<td>0.0636</td>
<td>0.0210</td>
<td>-0.2336</td>
<td>0.9998</td>
<td>85</td>
<td>85</td>
<td>85</td>
</tr>
<tr>
<td>0.010</td>
<td>SP</td>
<td>0.7795</td>
<td>0.6067</td>
<td>0.0635</td>
<td>0.0210</td>
<td>0.0000</td>
<td>-0.0009</td>
<td>64</td>
<td>0</td>
<td>192</td>
</tr>
</tbody>
</table>

- One can observe that by decrease the covariance of $x_1$, the optimum attained is closest to the global minimum.

- MM method proves to be inaccurate even for low covariances, while the SP is very accurate at the predicting the desired statistical measures, comparing favourably with the very heavy (computational time speaking) Monte Carlo method.

- SP performs even less function evaluations than the MM in return for a considerably higher accuracy, although it presents more constraint function calls.
### RDO Results

![Graph showing the Rosenbrock function with 1 constraint and minimum evolution results with SP and MM methods.]

The table below presents the results for different values of $k \sigma_g$ with the Method being SP for all cases.

<table>
<thead>
<tr>
<th>$k \sigma_g$</th>
<th>Method</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\mu_f$</th>
<th>$\sigma_f$</th>
<th>$\epsilon_{\mu_f}$</th>
<th>$\epsilon_{\sigma_f}$</th>
<th>fcalls</th>
<th>dfcalls</th>
<th>gcalls</th>
<th>dgcalls</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SP</td>
<td>0.7848</td>
<td>0.6148</td>
<td>0.0503</td>
<td>0.0054</td>
<td>0.0000</td>
<td>0.0002</td>
<td>67</td>
<td>0</td>
<td>201</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>SP</td>
<td>0.7830</td>
<td>0.6121</td>
<td>0.0510</td>
<td>0.0054</td>
<td>0.0000</td>
<td>0.0004</td>
<td>66</td>
<td>0</td>
<td>198</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>SP</td>
<td>0.7813</td>
<td>0.6093</td>
<td>0.0517</td>
<td>0.0053</td>
<td>0.0000</td>
<td>-0.0001</td>
<td>63</td>
<td>0</td>
<td>189</td>
<td>0</td>
</tr>
</tbody>
</table>
The Rosenbrock function constrained to a unit circle can be posed in the RBDO Formulation as follows:

\[
\begin{align*}
\text{minimize} & \quad F(x) = f(\mu_x) \\
\text{subject to} & \quad g^f c(x) = \leq 0
\end{align*}
\]

Two different methods of FORM were implemented:

- RIA;
- PMA.
One can observe that by decrease the covariance of $x_1$, the optimum attained is closest to the global minimum.

The objective function is defined using the mean values of the design variables as input, becoming virtually the same as if using the first order Method of Moments (RDO).

The errors ($\epsilon_\beta$) are obtained by comparison of the real reliability with the target.

The accuracy of both formulations in terms of the reliability index is acceptable, especially considering that FORM uses a first order approximation to the real failure hypersurface.

From the table above one can notice that the PMA approach requires less constraint function calls than RIA, although presents a little more functions calls.
## RBDO Results

### Analytic Problem - Rosenbrock Function with 1 Constraint

#### RBDO approach

<table>
<thead>
<tr>
<th>$k_{\beta_{reqd}}$</th>
<th>Method</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$f$</th>
<th>$\epsilon_{\beta}$</th>
<th>fcalls</th>
<th>gcalls</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>RIA</td>
<td>0.7847</td>
<td>0.6149</td>
<td>0.0464</td>
<td>0.0689</td>
<td>60</td>
<td>683</td>
</tr>
<tr>
<td>2</td>
<td>RIA</td>
<td>0.7829</td>
<td>0.6122</td>
<td>0.0472</td>
<td>0.0165</td>
<td>60</td>
<td>685</td>
</tr>
<tr>
<td>3</td>
<td>RIA</td>
<td>0.7811</td>
<td>0.6094</td>
<td>0.0480</td>
<td>0.0263</td>
<td>58</td>
<td>679</td>
</tr>
</tbody>
</table>
The Rosenbrock function constrained to a unit circle can be posed in the $R^2BDO$ Formulation as follows:

$$\minimize_{x=(x_1,x_2)} F(x) = \mu_f(x) + \sigma_f(x)$$

subject to $g^r c(x) \leq 0$

It was used the Sigma Point method to compute the mean and standard deviation of the objective function.

For the constraints it was selected the FORM with PMA.
As expected the computational time has increased significantly since this hybrid method uses the parts of RDO and RBDO that require more function calls such as the computation of the mean and standard deviation and the MPP problem in the reliability constraint.
R²BDO Results

Analytic Problem - Rosenbrock Function with 1 Constraint

R²BDO Approach

F. Afonso, L. Amândio, A. Marta and A. Suleman
Robust and Reliability Based Design Optimization
May 22, 2015 50 / 69
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F. Afonso, L. Amândio, A. Marta and A. Suleman
Robust and Reliability Based Design Optimization
May 22, 2015
The scope of this test case is to evaluate not only the SORA and RDS methods not used in the previous problem, but also to assess if all the methods still remain accurate if more than one constraint exists.

The problem was introduced by Shan and Wang [25] and is posed as:

\[
\begin{align*}
\text{minimize} & \quad f(\mu_1, \mu_2) = \mu_1 + \mu_2 \\
\text{subject to} & \quad P(g_i(X) \geq 0) \geq R_i, \; i = 1, 2, 3 \\
& \quad g_1(X) = X_1^2 X_2/20 - 1 \\
& \quad g_2(X) = (X_1 + X_2 - 5)^2/30 + (X_1 - X_2 - 12)^2/120 - 1 \\
& \quad g_3(X) = 80/(X_1^2 + 8X_2 + 5) - 1 \\
& \quad 0 \leq \mu_j \leq 10, \; j = 1, 2 \\
& \quad \sigma_1 = \sigma_2 = 0.3\beta_i = 3i, \; i = 1, 2, 3
\end{align*}
\]

where \( \mu_1, \mu_2, \sigma_1 \) and \( \sigma_2 \) are the mean values and standard deviations of 2 random variables \( (X_1 \text{ and } X_2) \), and \( R_i \) is the required reliability (which is the same for every constraint).

A target reliability index \( (\beta_{reqd} = 3) \) and a standard deviation of the random variables \( (\sigma = 0.3) \) was chosen.
This results of this problem were computed by L. Amândio [28] [29]

<table>
<thead>
<tr>
<th>Method</th>
<th>RIA</th>
<th>PMA</th>
<th>PMA_alt</th>
<th>SORA</th>
<th>SORA_alt</th>
<th>RDS</th>
<th>MM</th>
<th>SP</th>
<th>$R^2$BDO</th>
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<tr>
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<tr>
<td>$\mu_2$</td>
<td>3.2866</td>
<td>3.2866</td>
<td>3.2866</td>
<td>3.2866</td>
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<td>3.2800</td>
<td>3.4442</td>
<td>3.4164</td>
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<tr>
<td>$\varepsilon_{\beta_1}$ [%]</td>
<td>$</td>
<td>\varepsilon</td>
<td>&lt; 2$</td>
<td>$</td>
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<td>$\varepsilon_{\beta_3}$ [%]</td>
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<tr>
<td>Function Calls</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>#Obj. func. eval</td>
<td>18</td>
<td>18</td>
<td>21</td>
<td>51</td>
<td>51</td>
<td>18</td>
<td>15*</td>
<td>75**</td>
<td>105**</td>
</tr>
<tr>
<td>#Const. func. eval</td>
<td>1137</td>
<td>1956</td>
<td>688</td>
<td>367</td>
<td>234</td>
<td>54*</td>
<td>45*</td>
<td>225**</td>
<td>688</td>
</tr>
<tr>
<td>#Total. func. eval</td>
<td>1155</td>
<td>1974</td>
<td>709</td>
<td>418</td>
<td>285</td>
<td>72*</td>
<td>60*</td>
<td>300**</td>
<td>709**</td>
</tr>
</tbody>
</table>

* the number of function evaluations required to determine partial derivatives, were not accounted for
** the number of necessary function calls within the main function evaluations, were taken into account
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The original problem was first introduced by Sellar et al [30].

In this presented test case it was reduced the number of global design variables by erasing the $x_2$ variable from discipline 1.

This analytic problem was chosen due to its simplicity and despite its low dimensionality, it presents features of larger Multidisciplinary Design Optimization (MDO) problems, which allows each of the MDO architecture implementations to be verified before passing to more complex problems.

This problem in the standard formulation is given by:

\[ \begin{align*}
\text{minimize} & \quad f(z_1, x_1, x_2) = x_1^2 + x_2 + y_1 + e^{-y_2} \\
\text{subject to} & \quad 1 - \frac{y_1}{3.16} \leq 0 \\
& \quad \frac{y_2}{24} - 1 \leq 0 \\
& \quad -10 \leq z_1 \leq 10 \\
& \quad 0 \leq x_1, x_2 \leq 10 \\
\end{align*} \]

Discipline 1 \quad y_1(z_1, x_1, y_2) = z_1^2 + x_1 - 0.2y_2

Discipline 2 \quad y_2(z_1, x_2, y_1) = \sqrt{y_1} + z_1 + x_2

The introduction of the uncertainties in the design optimization was performed using the Multi-Discipline Feasible (MDF) architecture, due to it simplicity and because the MDF solve a single optimization problem, which makes easier the introduction of uncertainties in the MDO architectures study.

The local design variables, $(x_1, x_2)$, remain deterministic, while $z_1$ carries now uncertainty.
RDO Formulation

\[
\begin{align*}
\text{minimize} & \quad F(x, r) = W_f \mu_f + W_g \sigma_f \\
\text{subject to} & \quad G(x) = K_{\mu_g} \mu_g K_{\sigma_g} \sigma_g = 1 \\
& \quad -10 \leq \mu_{z_1} \leq 10 \\
& \quad 0 \leq x_1, x_2 \leq 10 \\
\text{with} & \quad f = x_1^2 + x_2 + y_1(x_1, y_2, \mu_{z_1}) + e^{-y_2(x_2, y_1, \mu_{z_1})} \\
& \quad g = \begin{cases} 
1 - \frac{y_1(x_1, y_2, \mu_{z_1})}{3.16} & \leq 0 \\
\frac{y_2(x_2, y_1, \mu_{z_1})}{24} - 1 & \leq 0 
\end{cases} \\
& \quad \mu_{z_1} = z_1 \\
& \quad \sigma_{z_1} = \text{c.o.v.}_{z_1} \mu_{z_1} \\
& \quad W_f = 1 \\
& \quad W_g = 1 \\
& \quad K_{\mu_g} = 1 \\
& \quad K_{\sigma_g} = 2 \\
\text{Discipline 1} & \quad y_1(x_1, y_2, \mu_{z_1}) = \mu_{z_1}^2 + x_1 - 0.2 y_2 \\
\text{Discipline 2} & \quad y_2(x_2, y_1, \mu_{z_1}) = \sqrt{y_1} + \mu_{z_1} + x_2
\end{align*}
\]
### RDO Results

<table>
<thead>
<tr>
<th>Optimum</th>
<th>$c.o.v.\ z_1 = 0.001$</th>
<th>$c.o.v.\ z_1 = 0.005$</th>
<th>$c.o.v.\ z_1 = 0.01$</th>
<th>$c.o.v.\ z_1 = 0.015$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_f$</td>
<td>3.1834</td>
<td>3.1973</td>
<td>3.2543</td>
<td>3.3282</td>
</tr>
<tr>
<td>$\sigma_f$</td>
<td>0</td>
<td>0.0070</td>
<td>0.0353</td>
<td>0.0720</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0.0000</td>
<td>0.0118</td>
<td>0.0274</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$z_1$</td>
<td>1.9776</td>
<td>1.9816</td>
<td>1.9945</td>
<td>2.0109</td>
</tr>
</tbody>
</table>

- One can note that when the random parameter $z_1$ covariance increases the objective function becomes farther, as it was expected.
- With $c.o.v.\ z_1 = 0.005$, mean and standard deviation errors under 1% in a 0.5960 second were achieved, which is just a little more than using the MDF (0.5914 second) without uncertainties.
RBDO Formulation

\[
\begin{align*}
\text{minimize} & \quad f(x, r) = x_1^2 + x_2 + y_1(x_1, y_2, \mu_{z_1}) + e^{-y_2(x_2, y_1, \mu_{z_1})} \\
\text{subject to} & \quad g_i^r c(x, r) = \begin{cases} 
\beta_{\text{reqd}} - \beta \leq 0 & \text{if RIA} \\
g^*(r(u)) \leq 0 & \text{if PMA}
\end{cases} \\
& \quad -10 \leq \mu_{z_1} \leq 10 \\
& \quad 0 \leq x_1, x_2 \leq 10 \\
\text{with} & \quad \mu_{z_1} = z_1 \\
& \quad \sigma_{z_1} = c.o.v.z_1 \mu_{z_1} \\
\text{Discipline 1} & \quad y_1(x_1, y_2, \mu_{z_1}) = \mu_{z_1}^2 + x_1 - 0.2y_2 \\
\text{Discipline 2} & \quad y_2(x_2, y_1, \mu_{z_1}) = \sqrt{y_1} + \mu_{z_1} + x_2
\end{align*}
\]
RBDO Formulation

- **RIA Formulation:**
  \[
  \begin{align*}
  &\text{minimize} & \left( u^T u \right)^{\frac{1}{2}} \\
  &\text{subject to} & g(r(u)) = 0 \\
  &\text{with} & g(r(u)) = \begin{cases} 1 - \frac{y_1}{3.16} & \\
  \frac{y_2}{24} & \end{cases}
  \end{align*}
  \]

- **PMA Formulation:**
  \[
  \begin{align*}
  &\text{minimize} & -g(u) \\
  &\text{subject to} & \left( u^T u \right)^{\frac{1}{2}} - \beta_{\text{reqd}} \\
  &\text{with} & -g(u) = -[(3.16 - y_1) + (y_2 - 24)]
  \end{align*}
  \]

- The r-space to u-space transformation is using a Normal Distribution.
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