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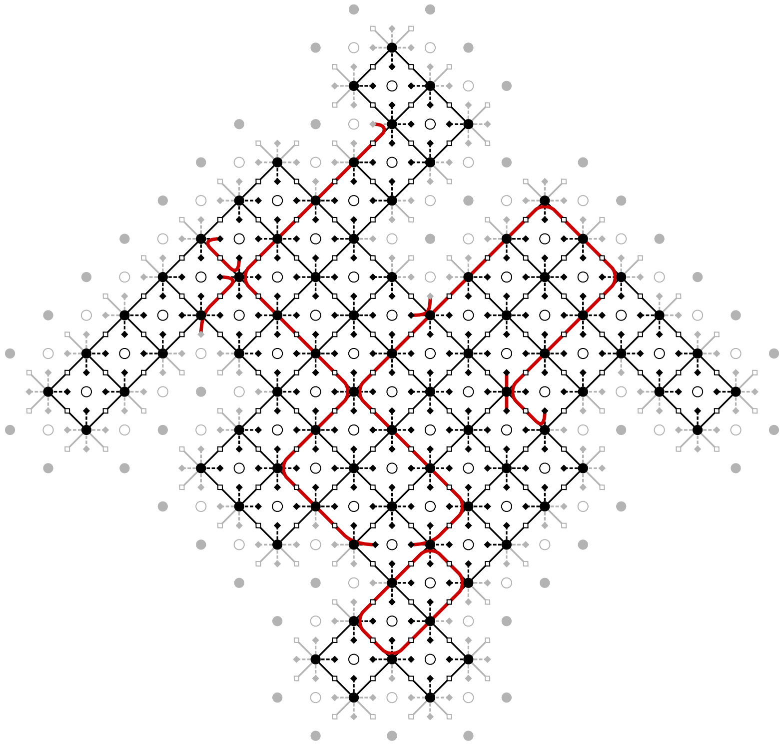

Conformal limit of the planar Ising model with disorder lines

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Thesis to obtain the Master of Science Degree in

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Para a minha mãe.

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Resumo

Nesta dissertação vamos definir o modelo de Ising e explorar alguns arguments combinatórios clássicos, incluíndo uma simetria importante entre baixas e altas temperaturas e conhecida como dualidade. Linhas de desordem são introduzidas no modelo como objetos duais de variáveis de *spin*. O cerne deste trabalho é o estudo do limite de escala do modelo na presença de linhas de desordem e na temperatura crítica $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$. Introduzimos uma definição nova de obserável de spinor que é generalizada ao modelo com linhas de desordem. Usando técnicas recentes relativas a spinores s-holomorfos, provamos formalmente a convergência destes spinores no limite de escala. Como consequência, provamos um resultado que descreve a correlação de variáveis de *spin* em múltiplos pontos e com linhas de desordem em domínios planares e simplesmente conexos.

Palavras-chave: modelo de Ising, linhas de desordem, dualidade de Kramers-Wannier, obserável de spinor, limite de escala

Abstract

In this dissertation we will define the Ising model and explore some standard combinatorial arguments, including an important symmetry between low and high temperatures known as duality. Disorder lines are introduced in the model as dual objects of spin variables. The bulk of this work is dedicated to studying the scaling limit of the model under disorder lines on square lattices at its critical temperature $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$. We provide a new definition of the spinor observables, generalized to the setting with disorder lines. Using recently developed techniques regarding s-holomorphic spinors, we formally prove the convergence of these spinors to the scaling limit. This allows us to derive a result concerning the multi-point spin correlations with disorder lines on simply connected planar domains.

Keywords: Ising model, disorder lines, Kramers-Wannier duality, spinor observable, scaling limit

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Nomenclature

Mathematical constants

π	Pi
e	Euler's constant
i	Imaginary unit
β_c	$\frac{1}{2}\ln\left(\sqrt{2}+1\right)$, critical temperature
α_c	$\exp(-2\beta_c) = \sqrt{2}-1$
λ	$\exp\left(i\frac{\pi}{4}\right)$

Common sets and set theory

$\lvert A\rvert$	Cardinality of set A
$A \oplus B$	Exclusive OR of sets A and B
A^B	Set $\{a_b : a \in A, b \in B\}$
$\emptyset$	Empty set
$\mathbb{Z}$	Set of integer numbers
$\mathbb{Z}^+$	Set of positive integer numbers
$\mathbb{Z}_0^+$	Set of non-negative integer numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{R}_0^+$	Set of non-negative real numbers
$\mathbb{C}$	Set of complex numbers

Ising model

$\mathcal{V}$	Set of vertices
$\mathcal{E}$	Set of edges
$\mathcal{F}$	Set of faces
(vu)	Edge linking vertices v and u
$\mathcal{G}$	Graph $(\mathcal{V}, \mathcal{E})$
$\dagger$	Dual element or set of dual elements

$\mathcal{E}^\dagger$	Set of dual edges
$\mathcal{G}^\dagger$	Dual graph of $\mathcal{G}$
β	Inverse temperature
σ	Spin configuration
H	Hamiltonian
H^Γ	Hamiltonian under disorder lines Γ
${}^\Theta H$	Hamiltonian under order lines Θ
$\mathcal{Z}$	Partition function
$\mathcal{Z}_\beta$	Partition function with inverse temperature β
$\mathcal{Z}_{(J_e)}$	Partition function with coupling constants (J_e)
$\mathcal{Z}_{\Omega_\delta}$	Partition function on the Ising model defined on $\mathcal{G}^\dagger_{\Omega_\delta}$
$\mathcal{Z}^+$	Partition function under + boundary conditions
$\mathcal{Z}^\dagger$	Partition function of dual model
$\mathcal{Z}^\Gamma$	Partition function under disorder lines Γ
${}^\Theta\mathcal{Z}$	Partition function under order lines Θ
$\mathbb{P}$	Probability
$\mathbb{P}^\Gamma$	Probability under disorder lines Γ
${}^\Theta\mathbb{P}$	Probability under order lines Θ
$\mathbb{E}$	Expected value
$\mathbb{E}_{(J_e)}$	Expected value with coupling constants (J_e)
$\mathbb{E}_\mathcal{G}$	Expected value on the Ising model defined on $\mathcal{G}$
$\mathbb{E}_{\Omega_\delta}$	Expected value on the Ising model defined on $\mathcal{G}^\dagger_{\Omega_\delta}$
$\mathbb{E}^+$	Expected value under + boundary conditions
$\mathbb{E}^\Gamma$	Expected value under disorder lines Γ
${}^\Theta\mathbb{E}$	Expected value under order lines Θ
σ_v	Spin or order variable
μ_a	Disorder variable
ψ_c	Fermion variable

Discretizations: $\mathbb{C}$

$\mathbb{C}_\delta$	Square grid of mesh size δ, as a subset of $\mathbb{C}$
$\mathcal{V}_{\mathbb{C}_\delta}$	Set of vertices of $\mathbb{C}_\delta$
$\mathcal{E}_{\mathbb{C}_\delta}$	Set of edges of $\mathbb{C}_\delta$
$\mathcal{F}_{\mathbb{C}_\delta}$	Set of faces of $\mathbb{C}_\delta$
$\mathcal{C}_{\mathbb{C}_\delta}$	Set of corners of $\mathbb{C}_\delta$
$\mathcal{C}^1_{\mathbb{C}_\delta}$	Set of eastern corners of $\mathbb{C}_\delta$
$\mathcal{C}^i_{\mathbb{C}_\delta}$	Set of western corners of $\mathbb{C}_\delta$
$\mathcal{C}^\lambda_{\mathbb{C}_\delta}$	Set of northern corners of $\mathbb{C}_\delta$
$\mathcal{C}^{\bar{\lambda}}_{\mathbb{C}_\delta}$	Set of southward corners of $\mathbb{C}_\delta$

Discretizations: generic set

Ω	Smooth subset of $\mathbb{C}$
δ	Mesh size of discretization
Ω_δ	Discrete domain
$\mathcal{V}_{\Omega_\delta}$	Set of vertices of Ω_δ
$\mathcal{E}_{\Omega_\delta}$	Set of edges of Ω_δ
$\mathcal{F}_{\Omega_\delta}$	Set of faces of Ω_δ
$\mathcal{C}_{\Omega_\delta}$	Set of corners of Ω_δ
$\mathcal{C}^1_{\Omega_\delta}$	Set of eastern corners of Ω_δ
$\mathcal{C}^i_{\Omega_\delta}$	Set of western corners of Ω_δ
$\mathcal{C}^\lambda_{\Omega_\delta}$	Set of northern corners of Ω_δ
$\mathcal{C}^{\bar{\lambda}}_{\Omega_\delta}$	Set of southward corners of Ω_δ
$v(c)$	Vertex adjacent to the corner c
$f(c)$	Face containing the corner c
$x \sim y$	The sites (vertices, edges, faces or corners) x and y are adjacent or incident to each other
(xy)	Edge/half-edge linking the sites x and y
Int	Interior of a set of sites
∂	Boundary of a set of sites

$\mathcal{G}_{\Omega_\delta}$ Graph $(\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$

$\mathcal{G}^\dagger_{\Omega_\delta}$ Dual graph of $\mathcal{G}_{\Omega_\delta}$

Geometrical curves

wind Winding of a curve

$\Gamma^1 \cdot \Gamma^2$ Intersection number of the two curves/collections of curves Γ^1 and Γ^2

Combinatorial paths

ε Empty path

$\langle \alpha, \beta \rangle$ Concatenation of paths α and β

Γ Subset of edges

$\Gamma^{[v_1,\ldots,v_{2m}]}$ Subset of edges where the vertices with odd degree are $v_1, \ldots, v_{2m}$

$\mathscr{C}_{\Omega_\delta}$ Set of collections of loops in $\mathcal{G}_{\Omega_\delta}$

$\mathscr{C}_{\Omega_\delta}(x_1, \ldots, x_{2m})$ Set of contours in $\mathcal{G}_{\Omega_\delta}$ where the sites with odd degree are $x_1, \ldots, x_{2m}$

$\mathrm{p}(Q)$ Special path of a smoothing Q

η_c Square root associated to the corner c

$\mathrm{sign}(s)$ Sign of a permutation s

$\tau(Q)$ Sign of a smoothing Q

Double covers

$[\Omega; b_1, \ldots, b_n]$ Canonical double cover of $\Omega \setminus \{b_1, \ldots, b_n\}$ branching around each of $b_1, \ldots, b_n$

$[\Omega; b_1, \ldots, b_n; b'_1, \ldots, b'_m]$ Double cover $[\Omega; b_1, \ldots, b_n, b'_1, \ldots, b'_m]$

$\widetilde{z}$ Point of double cover whose projection is z

Discrete complex analysis

$\Re$ Real part

$\Im$ Imaginary part

$\mathrm{Proj}_{l(c)}$ Projection onto the line $\eta_c\mathbb{R}$

Δ_δ Discrete Laplacian operator

$\widetilde{\Delta}_\delta$ Modified discrete Laplacian operator

$\partial_\delta, \overline{\partial}_\delta$ Discrete Wirtinger derivatives

hm^L_A Discrete harmonic measure function

$\mathbb{X}_\delta$ Left-slit discrete plane

$\mathbb{Y}_\delta$ Right-slit discrete plane

Spinors

$\mathbf{a}$ Shorthand for $a_1, \ldots, a_n$

$\mathbf{c}$ Shorthand for $c_1, \ldots, c_{2m}$

$\mathbf{u}$ Shorthand for $u_1, \ldots, u_{2m}$

$\mathbf{v}$ Shorthand for $v_1, \ldots, v_{2m}$

$a_1^{\rightarrow}$ $a_1 + \frac{\delta}{2}$

$\phi_{\mathbf{a}}^{\mathbf{c}}$ Complex phase

$\text{sheet}_{\mathbf{a},\mathbf{c}}$ Sheet number

$F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ Discrete spinor observable

$f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ Continuous spinor observable

$F^{\Gamma}_{[\mathbb{C}_\delta;a]}$ Discrete full-plane spinor observable

$f^{\Gamma}_{[\mathbb{C};a]}$ Continuous full-plane spinor observable

$G^{\Gamma}_{[\mathbb{C}_\delta;a]}$ Discrete primitive of $F^{\Gamma}_{[\mathbb{C}_\delta;a]}$

$g^{\Gamma}_{[\mathbb{C};a]}$ Continuous primitive of $f^{\Gamma}_{[\mathbb{C}_\delta;a]}$

$\vartheta(\delta)$ Normalizing factor

$\mathcal{A}^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ Coefficient of $2\sqrt{z-a_1}$ in the expansion of $f^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ around a_1

$\mathcal{A}^{\Gamma}_{\Omega}(\mathbf{a};\mathbf{u})$ Same as $\mathcal{A}^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$

1 Introduction

The Ising model is a mathematical model used in Statistical Physics. The model is defined on a graph and defines random variables associated to the vertices of a graph, which can take one of two values $\{\pm 1\}$. These represent the orientation of dipoles, and the main characteristic is that each dipole can interact with their neighbours: configurations where more neighbouring dipoles agree occur with higher probability.

The model was invented by Wilhelm Lenz in 1920 and first studied by Ernst Ising, who solved its 1-dimensional version (that is, when the graph is $\mathbb{Z}$) in his thesis. Since then, it has been widely studied to this day, for being both a simplified model of reality as well as one of the simplest statistical models to feature a phase transition. Originally conceptualized to be a model for the behaviour of ferromagnetism at an atomic level, where the dipoles represent the spin of electrons and the graph is given by the structure of the material, it has since then found usages for modelling gases, brain activity and even melt ponds on sea ice.

1.1 The model

Given a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, to each vertex $v \in \mathcal{V}$ we associate a variable $\sigma_v \in \{\pm 1\}$, referred to as the *spin* of the vertex. A *spin configuration* is an assignment of spins $\sigma = (\sigma_v)_{v \in \mathcal{V}} \in \{\pm 1\}^{\mathcal{V}}$ to every vertex. The *Hamiltonian* function is defined as

$$\begin{aligned} H\colon\ \{\pm 1\}^{\mathcal{V}} &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto -\sum_{\substack{e \in \mathcal{E} \\ e=(vu)}} \sigma_v \sigma_u \end{aligned} \tag{1}$$

which can be seen as the sum of "contribution" from all edges: each edge $e = (vu) \in \mathcal{E}$ contributes with -1 if $\sigma_v = \sigma_u$ or $+1$ if $\sigma_v = -\sigma_u$. The model is defined by the probability distribution on $\{\pm 1\}^{\mathcal{V}}$ where each configuration σ is proportional to the *weight* of a configuration, which equals $\exp\big(-\beta H(\sigma)\big)$ where $\beta > 0$ is a fixed constant — the Gibbs measure. Such a probability is thus given by

$$\mathbb{P}(\sigma) = \frac{1}{\mathcal{Z}_\beta} \exp\big(-\beta H(\sigma)\big) \tag{2}$$

where

$$\mathcal{Z}_\beta := \sum_{\sigma \in \{\pm 1\}^{\mathcal{V}}} \exp\big(-\beta H(\sigma)\big)$$

is the *partition function* of the model. Another expression for the probability can be obtained by expanding the Hamiltonian:

$$\mathbb{P}(\sigma) = \frac{1}{\mathcal{Z}_\beta} \prod_{\substack{e \in \mathcal{E} \\ e=(vu)}} \exp(\beta \sigma_v \sigma_u).$$

Remark 1.1. In a physical context β is the inverse temperature of the system (assuming the units are such that the Boltzmann constant k_B equals 1) and the Hamiltonian of a configuration is interpreted as its energy.

Some observations If more neighbouring spins agree, the Hamiltonian of a configuration will be lower and its probability will be higher. In addition, flipping all the signs of any configuration yields a second configuration with the same energy. This fact implies the signs $+$ and $-$ are interchangeable, and in particular leads to a simple proof of $\mathbb{E}[\sigma_v] = 0$ for any $v \in \mathcal{V}$.

Fox a fixed graph, $|H(\sigma)| \leq \mathcal{E}$. Because the Hamiltonian is bounded, taking $\beta \longrightarrow 0$ in (2) makes $\exp\big(-\beta H(\sigma)\big) \longrightarrow 1$, therefore all configurations will occur with the same probability. On the other hand,

$$\mathbb{P}(\sigma) \propto \exp\big(-\beta H(\sigma)\big) \propto \Big(-\beta\big(H(\sigma) - \min H\big)\Big) \xrightarrow{\beta \to +\infty} \begin{cases} 1, & H(\sigma) = \min H \\ 0, & H(\sigma) > \min H \end{cases}$$

implying only the configurations where H attains its minimum have probability greater than 0, and have the same probability. For a connected graph, H attains its minimum when either all $\sigma_v = +1$ or all $\sigma_v = -1$; when the graph is disconnected, the spins need only to be aligned in connected components.

Remark 1.2. In ferromagnetism, in high temperature conditions ($\beta \to 0$) the dipoles should behave uniformly random, whereas in a low temperature setting ($\beta \to +\infty$) they should be predominately aligned, which is indeed the case.

1.2 Generalizations

Many changes have been proposed to the model throughout the years. The most common generalization is to introduce interaction constants $(J_e)_{e \in \mathcal{E}}$, allowing for some connections to be stronger than others. The energy function becomes

$$H_{(J_e)}(\sigma) := -\sum_{\substack{e \in \mathcal{E} \\ e=(vu)}} J_e \sigma_v \sigma_u$$

with the probability measure defined the same way:

$$\mathbb{P}_{(J_e)} := \frac{1}{\mathcal{Z}_{(J_e),\beta}} \exp\big(-\beta H_{(J_e)}(\sigma)\big) = \frac{1}{\mathcal{Z}_{(J_e),\beta}} \prod_{\substack{e \in \mathcal{E} \\ e=(vu)}} \exp(\beta J_e \sigma_v \sigma_u) \tag{3}$$

where the partition function is now given by

$$\mathcal{Z}_{(J_e),\beta} := \sum_{\sigma \in \{\pm 1\}^{\mathcal{V}}} \exp\big(-\beta H_{(J_e)}(\sigma)\big) = \sum_{\sigma \in \{\pm 1\}^{\mathcal{V}}} \prod_{\substack{e \in \mathcal{E} \\ e=(vu)}} \exp(\beta J_e \sigma_v \sigma_u).$$

Some combinations of interaction constants have particular importance. An Ising model with *disorder insertions* is a model where $J_e \in \{\pm 1\}$. Informally speaking, an edge with a disorder insertion behaves opposite from normal, making configurations where the neighbouring spins have opposite signs more likely.

One can further generalize by adding an external magnetic field. The Hamiltonian would become

$$-\mu \sum_{v\in\mathcal{V}} h_v\sigma_v - \sum_{\substack{e\in\mathcal{E}\\ e=(vu)}} J_e\sigma_v\sigma_u$$

where μ is the magnetic moment and h_v represents the interaction of the external magnetic moment with the site v. In addition, any of these constants can be taken to be complex, wielding a complex measure over $\{\pm 1\}^{\mathcal{V}}$.

Another possible generalization is the Potts model, which corresponds to an Ising model where spins are allowed to be in more than 2 states [Pot52, Wu82].

1.3 Continuous model and Statistical Field Theory

Statistical Field Theory studies physical phenomena in systems with a very large number (possibly infinitely many) of degrees of freedom. Different phenomena are described under different mathematical models, but a common feature across all of them is the usage of fields to parametrize the freedom of the system. In this context, a *field* is simply a function taking a value at each point of the domain, and the different possibilities for a field encapsulate the degrees of freedom. For the Ising model, this field is the so-called *spin field*, which assigns a ± 1 spin to every site. This idea is useful for defining and studying continuous versions of discrete models, which is the goal of this work.

The first question one encounters is how to formally define a model with infinitely many degrees of freedom, for instance one with a random variable associated to every point of some domain $\Omega \subset \mathbb{R}^n$ (see [Mus10] for a possible approach). A very informal approach is described in [HVK13]. A statistical field theory corresponds to a random field ϕ defined on Ω with an associated measure $\mathbb{P}(\phi) \propto \exp\big(-\mathcal{S}[\phi]\big)$, where $\mathcal{S}$ is a functional of ϕ called the *action*. Other fields $\mathcal{O}_k$ are then defined, which are functions of ϕ in an infinitesimal neighbourhood of some insertion point z_k and are thus called *local fields*. Quantities of interest would be given by *correlations* of these fields and are denoted by $\big\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\big\rangle$. The correlations are given by the functional integrals

$$\big\langle \phi_1 \cdots \phi_n \big\rangle = \frac{\int \mathcal{O}_1(z_1)[\phi]\cdots\mathcal{O}_n(z_n)[\phi]\exp(-\mathcal{S}[\phi])\ \mathcal{D}\phi}{\int \exp(-\mathcal{S}[\phi])\ \mathcal{D}\phi}.$$

For the Ising model, the spin variables σ_j are examples of local fields.

Remark 1.3. A *conformal field theory* arises when a statistical field theory is invariant under conformal transformations of Ω. At its core, it is a symmetry of the action $\mathcal{S}$.

The problem with the formulation above is that it is difficult to formalize for infinitely many degrees of freedom. The approach followed in this work considers a continuous model on Ω defined as a *scaling limit of discrete models*. One considers instead a family of models defined on appropriate discretizations Ω_δ of Ω for each $\delta > 0$ (Figure 1), which converge in some sense to the original domain as $\delta \longrightarrow 0$. For example, given $a, b \in \Omega$, the expected value of a product of spin variables $\sigma_a\sigma_b$ for the continuous Ising

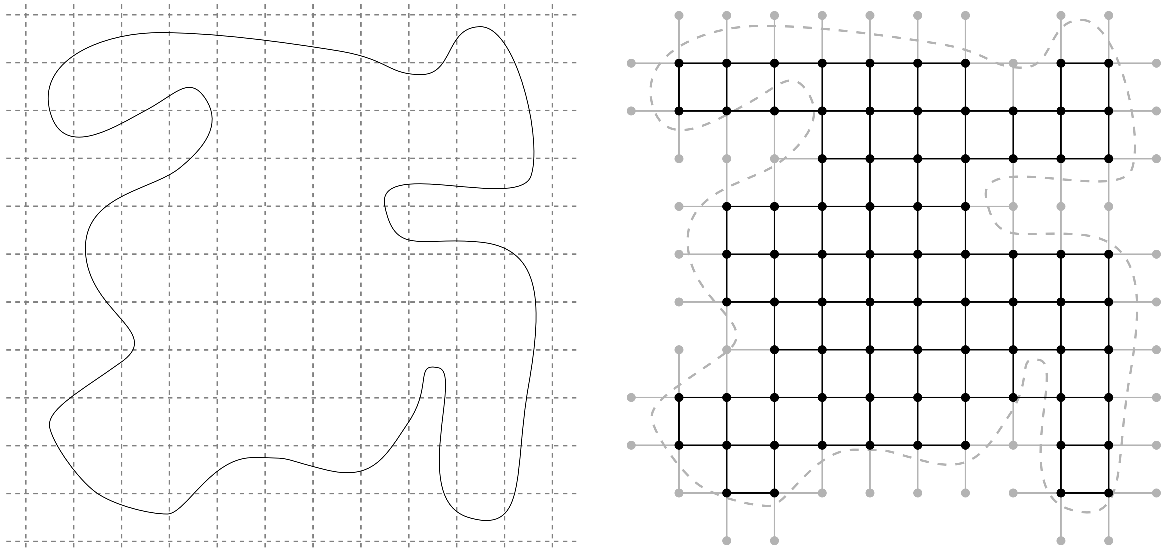

Figure 1: A domain Ω with a square grid (on the left) and an example of a discretization Ω_δ (on the right). Boundary sites are often considered for the model, and coloured grey.

model would be defined as:

$$\mathbb{E}_\Omega[\sigma_a\sigma_b] := \lim_{\delta\to 0} \mathbb{E}_{\Omega_\delta}[\sigma_a\sigma_b]$$

and note that there is an abuse of notation here: the sites on the right-hand side may not be a and b but instead appropriate approximations of these points on Ω_δ.

For each discrete model, the previous ideas are simple to define: the fields ϕ_δ and the actions $\mathcal{S}_\delta$ can be accurately defined, local fields $(\mathcal{O}_k)_\delta$ are formalized as random variables independent of ϕ_δ on all but a finite number of neighbours of the insertion point, and the correlations become expected values of random variables. On the downside, one has to prove that such limits exist and do not depend on the discretizations. These are usually done using some lattice, most commonly the square lattice [CHI15, Dub11, Smi10a, CI13], but other works have considered discretizations on other types of graphs [CS12].

Remark 1.4. It is very common for these models to be defined using *external parameters*: as an example, the Ising model has β and coupling constants J_e can be introduced. For discrete models, probability distribution functions and field correlations are smooth functions of these parameters. However, when passing to the scaling limit, it is possible that the limit versions of these become discontinuous. This implies the limit theory having different properties under different regions of the external parameter space. These regions are called *phases* of the model, and crossing such regions corresponds to *phase transitions*. Since a statistical field theory is usually based on a few, universal physical principals, these transitions have often universal properties and behaviours. Such phase transitions are often the most important objects of study in a statistical field theory [GJ87]. Some examples are the superfluidity transition in quantum fluids, the superconductivity transition of metallic materials at low temperatures and the para-to-ferro magnetic transition of magnetic materials, which can be seen in the Ising model at dimensions 2 and greater [Ons44].

A major problem of this approach is that properties of the scaling limit may not happen in the lattice models. The most relevant is the conformal invariance property, which physical arguments suggest holds for the scaling limit of multiple 2D models at their continuous phase transitions, even though there is no rigorous proof for most cases [Smi06].

The 2D Ising model is a cornerstone of statistical and conformal field theories for being both one of the first and most fundamental examples. It is also one of the few cases where formal proofs of the conformal invariance of scaling limits have been given [Smi06]. The main technique used in these proofs is to study the properties of *spinor observables*: first introduced in [Smi06] and further explored in other papers [Smi10b, CS12, CHI15, CI13, CCK17], they are functions defined at the lattice level which can be proven to converge to a continuous counterpart, usually defined by boundary value problem. They provide a means of accurately stating and proving scaling limit results.

1.4 Main results

The first part of this work is dedicated to a presentation of common combinatorial arguments used in the Ising model — namely, high and low-temperature expansions and domain wall configurations —, followed by an introduction to Kramers-Wannier duality. First described in 1941 [KW41a, KW41b], this phenomenon relates the Ising model on a planar graph with another defined on its dual graph, where a spin is assigned to every face; furthermore, it exposes a link between high and low-temperature models, and models with boundary conditions — that is, where the spins at the boundary of the graph at fixed as + — and with no boundary conditions. It also motivates the usage of disorder insertions as being the dual objects of spin variables [KC71]. It was an important stepping stone for the computation of the exact solution of the 2D Ising model by Onsager in 1944 [Ons44], one of the landmarks in Theoretical Physics [BK95].

The exposition ends with Theorem 2.36, a well-known result in literature which can be roughly stated as follows:

Theorem 1.5 (Theorem 2.36)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an adequate subgraph of a square lattice with faces $\mathcal{F}$ and let $\partial\mathcal{F}$ be its boundary faces. Consider an Ising model on $\mathcal{G}$ with parameter β, together with another Ising model on $\mathcal{G}^\dagger = (\mathcal{F} \cup \partial\mathcal{F}, \mathcal{E}^\dagger)$ — where $\mathcal{E}^\dagger$ is the set of dual edges of $\mathcal{E}$ — with parameter $\beta^\dagger$ and + boundary conditions, achieved by fixing all the spins of $\partial\mathcal{F}$ as being +.*

Take any $\Theta \subseteq \mathcal{E}$ and let $v_1, \ldots, v_{2m} \in \mathcal{V}$ be the vertices that are endpoints of an odd number of elements of Θ. Likewise, take any $\Gamma \subseteq \mathcal{E}^\dagger$ and let $a_1, \ldots, a_{2n} \in \mathcal{F} \cup \partial\mathcal{F}$ be the faces that are endpoints of an odd number of elements of Γ. If $\tanh\beta = \exp(-2\beta^\dagger)$, then

$$\mathbb{E}_{\mathcal{G}}\left[\prod_{k=1}^{2m} \sigma_{v_k} \left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma}\right] = (-1)^{|\Theta \cap \Gamma^\dagger|} \cdot \mathbb{E}^{+}_{\mathcal{G}^\dagger}\left[\prod_{j=1}^{2n} \sigma_{a_j} \left(\prod_{k=1}^{2m} \mu_{v_k}\right)_{\Theta}\right].$$

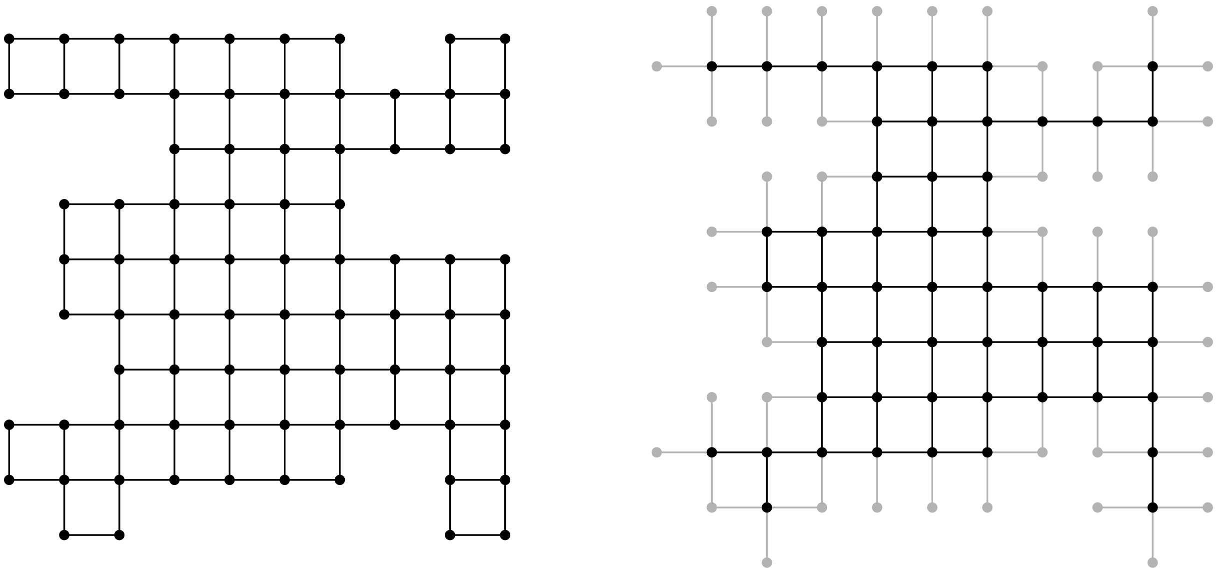

Figure 2: Examples of graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (on the left) and $\mathcal{G}^\dagger = (\mathcal{F} \cup \partial\mathcal{F}, \mathcal{E}^\dagger)$ (on the right) for which Theorem 1.5 holds. The faces $\partial\mathcal{F}$ and dual edges linked to a boundary faces are coloured grey.

where the above random variables are given by

$$\left(\prod_{j=1}^{2n} \mu_{a_j}\right)_\Gamma = \prod_{\substack{e\in\Gamma^\dagger \\ e=(vu)}} \exp(-2\beta\sigma_v\sigma_u) \qquad \left(\prod_{k=1}^{2m} \mu_{v_k}\right)_\Theta = \prod_{\substack{e^\dagger\in\Theta \\ e^\dagger=(vu)}} \exp(-2\beta^\dagger\sigma_v\sigma_u).$$

The random variables mentioned in the result are called disorder variables and they "encode" the effect of disorder insertions in the Ising model.

The bulk of this work is dedicated to a new generalization of the aforementioned spinor observables for the Ising model with disorder insertions at the critical point $\beta = \frac{1}{2}\ln(\sqrt{2}+1)$. We define the objects $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ at the lattice level, describe its properties, define their continuous counterpart $f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}$ and prove convergence. These functions are defined on double covers, branching around the spin sites and endpoints of disorder insertions. The main result of the work is Theorem 7.12, a rather technical result requiring tools from discrete complex analysis to handle functions that are discrete holomorphic in some way.

Theorem 1.6 (Theorem 7.12)**.** *Given a bounded, simply connected domain $\Omega \subset \mathbb{C}$, let $\boldsymbol{a}, \boldsymbol{u} \in \Omega$ be a collection of adequate points and let $\Gamma \subseteq \Omega$ be a collection of paths linking $\boldsymbol{u}$. Let Ω_δ be a family of discretizations of Ω by the square grids $(1+i)\delta\mathbb{Z}^2$. Then, under general conditions, for any $\varepsilon > 0$,*

$$\frac{1}{\vartheta(\delta)} F^\Gamma_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]} \xrightarrow{\delta\to 0} f^\Gamma_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}$$

uniformly on compact sets of distance at least ε from the branching points.

As an example of how spinor observables can be used to find conformal invariance results, the following result is proven.

Theorem 1.7 (Theorem 7.19)**.** *Under the same conditions of Theorem 1.6, define $\mathcal{A}^\Gamma_\Omega(\boldsymbol{a};\boldsymbol{u})$ as the* *following coefficient in the expansion of* $f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}$ *near the first branching point* a_1*:*

$$f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}(z) = \frac{1}{\sqrt{z-a_1}} + 2\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\sqrt{z-a_1} + O\big(|z-a_1|^{3/2}\big).$$

This coefficient verifies the conformal covariance rule

$$\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u}) = \varphi'(a_1)\cdot\mathcal{A}^{\varphi(\Gamma)}_{\Omega'}\big(\varphi(\boldsymbol{a});\varphi(\boldsymbol{u})\big) + \frac{1}{8}\frac{\varphi''(a_1)}{\varphi'(a_1)}$$

for any conformal mapping $\varphi:\Omega\to\Omega'$*. In addition,*

$$\lim_{\delta\to 0}\frac{1}{2\delta}\left(\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]}-1\right) = \Re\Big(\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\Big)$$
$$\lim_{\delta\to 0}\frac{1}{2\delta}\left(\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2i\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]}-1\right) = -\Im\Big(\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\Big)$$

with the respective values computed on Ising models at the critical temperature, with $+$ *boundary conditions and with disorder lines* Γ *on graphs defined on* Ω_δ*.*

This result is a generalization of Theorem 1.5 from [CHI15], where this result is proven in the absence of disorder insertions: no $\mathbf{u}$ are considered, or equivalently $\Gamma=\emptyset$. In this case, the statement holds for all possible values of $\mathbf{a}$, allowing for a clean integration which is not possible otherwise — for instance, one cannot integrate on points where $\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]=0$. Furthermore, the authors are able to explicitly compute $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ for the domain $\Omega=\mathbb{H}$, which together with the conformal covariance rule leads to a full proof of the conformal invariance of multi-point spin correlations (see Theorem 1.2 in [CHI15]).

1.5 Key steps and organization of the work

Theorem 1.5 is proven in Part I using Ising models with adequate coupling constants. This requires generalizing a number of results to this setting, but the arguments used are fundamentally identical. The notation is introduced in Section 2.1, and is consistent with future sections. In Sections 2.2 and 2.3 we describe the high and low-temperature expansions, which are linked in Section 2.4. In Section 2.5 we introduce disorder insertions as dual objects of spin variables, and Section 2.6 is dedicated to the proof of Theorem 2.36.

We then proceed to the exploration of the spinor observables, which comprises Parts II and III. The former is dedicated to the study of their variables, whereas the latter handles the convergence problem with an approach heavily inspired by [CHI15]. Section 3 establishes the notation used in the sequel, organized according to the setting where it is used. Some definitions from Section 3.4 require proofs of well-definedness, which are relegated to the end of the section. In addition, we prove Lemmas 3.2 and 3.3 which will be useful for arguments at the lattice level.

In Section 4.1 we give an informal intuition to the spinor observables by defining them using fermionic random variables. In Section 4.2 we establish a combinatorial expression for correlations through Proposition 4.8, a result from [CCK17]. The spinor observables are formally defined in Section 4.3. In Section

4.4 we create a connection to the ideas from Section 4.1, as well as prove Proposition 4.17 which is fundamental in extracting information from the spinors to prove Theorem 7.19. In Section 4.5 we determine the properties necessary to prove the convergence result, namely a discrete version of homomorphism called s-holomorphism.

Section 5 marks the start of the convergence proof. In Section 5.1 we describe why a direct proof is not possible and we must instead prove $\int (F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]})^2 \to \int (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$. The remainder of this section is dedicated to the construction of technical tools necessary to handle these kinds of functions. Section 5.2 starts the study of discrete holomorphic functions and discrete primitives. In Section 5.3 the notion of s-holomorphism, a property stronger than the usual discrete holomorphism, is introduced. In Section 5.4 the object $\int F^2$ is accurately defined for lattices, and Section 5.5 describes the boundary modification trick for simplification of future arguments. In Section 5.6 we establish further properties used in the convergence proofs.

In Section 6 we define two auxiliary functions for the convergence proof, which are used to describe the behaviour of the spinor observables near the branching points. The results necessary for the convergence proof are described in Section 6.1, some auxiliary facts related to the discrete harmonic measure are proven in Section 6.2 and the study of these two functions is don in Sections 6.3 and 6.4.

Section 7 is where the convergence proof is done. The continuous spinors observables are defined in Section 7.1 and the primitive of their squares are described in Section 7.2. In Sections 7.3 and 7.4 we prove Theorems 2.36 and 7.12, respectively.

PART I

ORDER-DISORDER DUALITY

2 Order-disorder duality

In this section we present some combinatorial arguments commonly used to study the Ising model, explore the Kramers-Wannier duality and introduce disorder lines as dual objects of order lines. We emphasize the approaches used to derive the results, which can be adapted to other setups.

2.1 Setup

The results of this section regard the standard Ising model, with coupling constants introduced whenever specified. Some of these are done in a generic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, while others require a more elaborate setup to be expressed. We will thus enunciate them in accordance to Parts II and III, introducing the notation necessary for now.

We consider a discrete domain Ω_δ which is the union of faces of a square grid with mesh size $\delta > 0$. Such faces are called *interior faces* and are denoted by $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$. Given such a domain, the set of *interior vertices* is the set $\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ of vertices of the grid that are corners to any face from $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$ and the set of *interior edges* is the set $\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$ of edges that are adjacent to any face of $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$.

Additionally, we define the sets of *boundary faces*, *vertices* and *edges* as being the respective elements adjacent/incident to their interior counterparts that do not belong to those sets, and are denoted by $\partial\mathcal{F}_{\Omega_\delta}$, $\partial\mathcal{V}_{\Omega_\delta}$ and $\partial\mathcal{E}_{\Omega_\delta}$. The sets of *faces*, *vertices* and *edges* are the union of the respective interior and boundary elements: $\mathcal{F}_{\Omega_\delta} := \mathrm{Int}\,\mathcal{F}_{\Omega_\delta} \cup \partial\mathcal{F}_{\Omega_\delta}$, $\mathcal{V}_{\Omega_\delta} := \mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{V}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta} := \mathrm{Int}\,\mathcal{E}_{\Omega_\delta} \cup \partial\mathcal{E}_{\Omega_\delta}$. Figure 1 shows an example of such a discretization with boundary elements coloured grey.

The domain Ω_δ is any polygonal domain resulting from the union of square grid faces, and to simplify arguments we will assume that Ω_δ is simply connected and any edges connecting vertices of $\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ belong to $\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$. We will study the Ising model defined on various graphs formed by these vertices, edges and faces; for the two next subsections we will always focus on the graph $(\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$. The parameter $\beta > 0$ is considered fixed, and no coupling constants are considered unless specified otherwise.

2.2 High-temperature expansion

Take a generic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and let us find another way of expressing the partition function $\mathcal{Z}_\beta$. By separating the exponential in its even and odd parts, the dependence of $\exp(\beta\sigma_v\sigma_u)$ on $\sigma_v\sigma_u$ (which can only take the values ± 1) can be conveniently rewritten as

$$\exp(\beta\sigma_v\sigma_u) = \cosh(\beta\sigma_v\sigma_u) + \sinh(\beta\sigma_v\sigma_u) = \cosh\beta + \sigma_v\sigma_u \sinh\beta$$

therefore

$$\begin{aligned}
\mathcal{Z}_\beta &= \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \exp\big(\beta\sigma_u\sigma_v\big)\\
&= \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \big(\cosh\beta + \sigma_v\sigma_u \sinh\beta\big)\\
&= (\cosh\beta)^{|\mathcal{E}|} \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \big(1 + \sigma_v\sigma_u \tanh\beta\big).
\end{aligned}$$

Let us expand the inner product of factors $\big(1+\sigma_v\sigma_u\tanh\beta\big)$. Every term of the expanded sum can be computed by picking either 1 or $\sigma_v\sigma_u\tanh\beta$ for each $e=(vu)\in\mathcal{E}$ and then multiplying all of the chosen factors together. Therefore, there is a bijection between subsets of $\mathcal{E}$ and terms of the sum: to each $E\subseteq\mathcal{E}$ we associate the term $\prod_{e\in E,\, e=(vu)}\sigma_v\sigma_u\tanh\beta$. This leads to

$$\begin{aligned}
(\cosh\beta)^{|\mathcal{E}|} \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \big(1 + \sigma_v\sigma_u \tanh\beta\big) &=\\
&= (\cosh\beta)^{|\mathcal{E}|} \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \left(\sum_{E\subseteq\mathcal{E}} \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u\tanh\beta \right)\\
&= (\cosh\beta)^{|\mathcal{E}|} \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \left(\sum_{E\subseteq\mathcal{E}} (\tanh\beta)^{|E|} \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u \right).
\end{aligned}$$

At this point, we have written the weight of each configuration as a sum of monomers of σ_v. This is interesting by itself because it allows connections between the Ising model and other probabilistic models that assign weights to products of sign variables in other contexts, see [Dub11] for an example.

The key step is to use the outer sum to cancel some of the products of monomers. To do that, we swap the sums:

$$\begin{aligned}
(\cosh\beta)^{|\mathcal{E}|} \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \left(\sum_{E\subseteq\mathcal{E}} (\tanh\beta)^{|E|} \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u \right) &=\\
&= (\cosh\beta)^{|\mathcal{E}|} \sum_{E\subseteq\mathcal{E}} \left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} (\tanh\beta)^{|E|} \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u \right)\\
&= (\cosh\beta)^{|\mathcal{E}|} \sum_{E\subseteq\mathcal{E}} (\tanh\beta)^{|E|} \left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u \right) \qquad (4)
\end{aligned}$$

Let us compute the inner sum for a fixed $E\subseteq\mathcal{E}$. Think of the product as $\sigma_{v_1}^{\alpha_{v_1}}\sigma_{v_2}^{\alpha_{v_2}}\cdots\sigma_{v_n}^{\alpha_{v_n}}$, where $\{v_1,\dots,v_n\}=\mathcal{V}$ and $\alpha_{v_k}\in\mathbb{Z}_0^+$ is the exponent of σ_{v_k}, which is the number of edges of E that have v_k as an endpoint. Since $\sigma_{v_k}\in\{\pm1\}$, if some α_k is odd then factoring $\sigma_{v_k}^{\alpha_{v_k}}$ out yields two terms which cancel each other. On the other hand, if all α_k are even then all $\sigma_{v_k}^{\alpha_{v_k}}$ are equal to 1 and the sum becomes $\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} 1 = 2^{|\mathcal{V}|}$. Hence, the inner sum only survives when every vertex is the endpoint of an even number of edges of E, in which case it always equals $2^{|\mathcal{V}|}$. Plugging this, we get

$$(\cosh\beta)^{|\mathcal{E}|}\sum_{E\subseteq\mathcal{E}}(\tanh\beta)^{|E|}\left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}}\prod_{\substack{e\in E\\ e=(vu)}}\sigma_v\sigma_u\right)=$$

$$=(\cosh\beta)^{|\mathcal{E}|}\sum_{\substack{E\subseteq\mathcal{E}\\ \text{All } v\in\mathcal{V}\text{ have even degree in } E}}(\tanh\beta)^{|E|}\cdot 2^{|\mathcal{V}|}=$$

$$=2^{|\mathcal{V}|}(\cosh\beta)^{|\mathcal{E}|}\sum_{\substack{E\subseteq\mathcal{E}\\ \text{All } v\in\mathcal{V}\text{ have even degree in } E}}(\tanh\beta)^{|E|}$$

where the degree of a vertex v in $E\subset\mathcal{E}$ is the cardinality $|\{e\in E:\ v \text{ is an endpoint of } e\}|$.

We claim that E verifies this property if and only if it can be written as a collection of edge-disjoint loops in $\mathcal{G}$:

1. If E is a collection of edge-disjoint loops then it easily follows that all vertices have even degree in E.

2. If all vertices have even degree in E, then we describe a procedure to decompose it into edge-disjoint loops. Pick a vertex v_1 incident to at least one edge of E and build a path π by passing though vertices $v_1, v_2, \ldots, v_n$ such that $(v_i v_{i+1})\in E$ and no edge is used twice, which stops when it is not possible to continue. Due to the stopping condition, all edges of E incident to v_n must be used in π. If $v_n\neq v_1$ then the number of such edges is $2|\{i : v_i = v_n\}|-1$, implying v_n has odd degree in E, which is impossible. Therefore, $v_n = v_1$ and π is a loop. Repeating the procedure recursively for $E\setminus\pi$ until there are no more edges left, we find a valid decomposition.

Proposition 2.1. *For the Ising model on any $\mathcal{G}=(\mathcal{V},\mathcal{E})$,*

$$\mathcal{Z}_\beta = 2^{|\mathcal{V}|}(\cosh\beta)^{|\mathcal{E}|}\sum_{\substack{E\subseteq\mathcal{E}\\ E\text{ collection of loops}}}(\tanh\beta)^{|E|}.$$

This is the *high-temperature expansion* of the Ising model, so called because it was historically used to study the model at high temperatures, corresponding to the case of small β[1]. This expansion allows bijections between other statistical models which impose a probability measure on the power set of edges of $\mathcal{G}$. The most common example of this is the dual Ising model, which we will later see (Proposition 2.12).

The same strategy can be used to expand a variety of similar sums.

Example 2.2. Let us compute the expected value of the product of two spins at two fixed vertices $v_1, v_2\in\mathcal{V}$ according to the probability measure (2). That sum would be

$$\mathbb{E}[\sigma_{a_1}\sigma_{a_2}] = \frac{1}{\mathcal{Z}_\beta}\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}}\sigma_{v_1}\sigma_{v_2}\left(\prod_{\substack{e\in\mathcal{E}\\ e=(vu)}}\exp(\beta\sigma_v\sigma_u)\right)$$

[1]Note that no assumptions on β were made.

and we have already discussed how to deal with $\mathcal{Z}_\beta$. Using the previous ideas for this sum, we eventually obtain

$$\mathbb{E}[\sigma_{v_1}\sigma_{v_2}] = \frac{1}{\mathcal{Z}_\beta}(\cosh\beta)^{|\mathcal{E}|}\sum_{E\subseteq\mathcal{E}}(\tanh\beta)^{|E|}\left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}}\sigma_{v_1}\sigma_{v_2}\prod_{\substack{e\in E\\ e=(vu)}}\sigma_v\sigma_u\right)$$

and, as before, the inner sum equals $2^{|\mathcal{V}|}$ if all σ_v appear an even number of times inside the sum and 0 otherwise. Seeing as we already have a $\sigma_{v_1}\sigma_{v_2}$ factor, E must be such that both v_1 and v_2 have odd degree in E while all the other vertices have even degree E. This is equivalent to stating that E can be decomposed into a collection of edge-disjoint loops of $\mathcal{E}$ and a path between v_1 and v_2: the ($\Leftarrow$) implication is straightforward and for the ($\Rightarrow$) one can find the decomposition by proceeding in similar fashion, with the added detail that starting at v_1 implies ending at v_2. Putting everything together, one arrives at

$$\mathbb{E}[\sigma_{v_1}\sigma_{v_2}] = \frac{\sum_{\substack{E\subseteq\mathcal{E}\\ E\text{ loops + path } v_1\leftrightarrow v_2}}(\tanh\beta)^{|E|}}{\sum_{\substack{E\subseteq\mathcal{E}\\ E\text{ loops}}}(\tanh\beta)^{|E|}} \tag{5}$$

and as corollary we get a formal proof of $\mathbb{E}[\sigma_{v_1}\sigma_{v_2}] > 0$, since $\beta > 0 \Rightarrow \tanh\beta > 0$.

Example 2.3. Computing $\mathbb{E}[\sigma_{v_1}]$ in a similar fashion yields

$$\mathbb{E}[\sigma_{v_1}] = \frac{1}{\mathcal{Z}_\beta}(\cosh\beta)^{|\mathcal{E}|}\sum_{E\subseteq\mathcal{E}}(\tanh\beta)^{|E|}\left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}}\sigma_{v_1}\prod_{\substack{e\in E\\ e=(vu)}}\sigma_v\sigma_u\right) \tag{6}$$

but now the inner sum only survives if v_1 has odd degree in E and all other vertices have even degree, which is impossible because the sum of the degrees of all vertices must be equal to $2|E|$. Therefore $\mathbb{E}[\sigma_{a_1}] = 0$, as was seen earlier. This argument can be generalized to prove $\mathbb{E}[\sigma_{v_1}\cdots\sigma_{v_{2m+1}}] = 0$.

Example 2.4. Take a non-empty $P \subseteq \mathcal{V}$ and consider the model with all the spins of P fixed as $+$. The partition function of the model is a sum over all configurations $\sigma \in \{\pm1\}^{\mathcal{V}\setminus P} \times \{+1\}^P$. The expansion remains largely the same as the usual $\mathcal{Z}_\beta$ up to (4), yielding

$$(\cosh\beta)^{|\mathcal{E}|}\sum_{E\subseteq\mathcal{E}}(\tanh\beta)^{|E|}\left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}\setminus P}\times\{+1\}^P}\prod_{\substack{e\in E\\ e=(vu)}}\sigma_v\sigma_u\right).$$

Recall that the inner sums would cancel out if some σ_v appeared an odd number of times in the product, because we can factor out the σ_v and get two terms with opposite signs. Since the spins at P are fixed, it is impossible to factor out σ_v when $v \in P$ because it can only take one value. For the sum to survive, we only require that σ_v appears an even number of times for every $v \in \mathcal{V}\setminus P$, in which case it equals $2^{|\mathcal{V}\setminus P|}$. Therefore, the sum survives if and only if $E \subseteq \mathcal{E}$ is such that all vertices of $\mathcal{V}\setminus P$ have even degree in E, which is equivalent to stating that E is a collection of edge-disjoint loops and paths connecting vertices of P.

Example 2.5. As for the expected value of a single spin variable at $v_1 \notin P$ when the spins of P are fixed as $+$, the same argument up to (6) yields

$$\frac{1}{\mathcal{Z}_\beta}(\cosh\beta)^{|\mathcal{E}|}\sum_{E\subseteq\mathcal{E}}(\tanh\beta)^{|E|}\left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}\setminus P}\times\{+1\}^P}\sigma_{v_1}\prod_{\substack{e\in E\\ e=(vu)}}\sigma_v\sigma_u\right)$$

(note that $\mathcal{Z}_\beta$ here is the partition function of the model conditioned to the spins of V being $+$, which was explored in the previous example) and now not all inner sums vanish immediately. For a fixed $E\subseteq\mathcal{E}$, the sum does not vanish if σ_{v_1} appears an odd number of times in the product and σ_v appears an even number of times for every $v\in\mathcal{V}\setminus P$. This is possible because some σ_v with $v\in P$ can appear an odd number of times to compensate for σ_{v_1}, whereas before we had $P=\emptyset$ and no such thing could happen. The inner sum equals $2^{|\mathcal{V}\setminus P|}$ if v_1 has odd degree in E and all other $v\in\mathcal{V}\setminus P$ have even degree in E, which is equivalent to requiring that E is a collection of edge-disjoint loops, paths connecting vertices of P and a path running from v_1 to any $v\in P$. Together with the previous example, we can write this expected value as

$$\frac{\sum_{\substack{E\subseteq\mathcal{E}\\ E\text{ loops + paths }P\leftrightarrow P+\text{ path }v_1\leftrightarrow P}}(\tanh\beta)^{|E|}}{\sum_{\substack{E\subseteq\mathcal{E}\\ E\text{ loops + paths }P\leftrightarrow P}}(\tanh\beta)^{|E|}}$$

and note how it equals 1 if $v_1\in P$, so it can be extended to those cases as well.

Let us now obtain the high-temperature expansion for the partition function $\mathcal{Z}_{(J_e),\beta}$ of the Ising model with coupling constants.

Proposition 2.6. *For the Ising model on any $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with coupling constants (J_e),*

$$\mathcal{Z}_{(J_e),\beta}=2^{|\mathcal{V}|}\left(\prod_{e\in\mathcal{E}}\cosh\beta J_e\right)\sum_{\substack{E\subseteq\mathcal{E}\\ E\text{ collection of loops}}}\left(\prod_{e\in E}\tanh\beta J_e\right).$$

Proof. The algebraic manipulations and arguments are the same as before with simple adaptations. We limit ourselves to showing the intermediate steps.

$$\begin{aligned}
\mathcal{Z}_{(J_e),\beta} &= \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \exp(\beta J_e \sigma_v\sigma_u) \\
&= \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \big(\cosh\beta J_e + \sigma_v\sigma_u \sinh\beta J_e\big) \\
&= \left(\prod_{e\in\mathcal{E}} \cosh\beta J_e\right) \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \big(1 + \sigma_v\sigma_u \tanh\beta J_e\big) \\
&= \left(\prod_{e\in\mathcal{E}} \cosh\beta J_e\right) \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \left[\sum_{E\subseteq\mathcal{E}} \left(\prod_{e\in E} \tanh\beta J_e\right) \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u\right] \\
&= \left(\prod_{e\in\mathcal{E}} \cosh\beta J_e\right) \sum_{E\subseteq\mathcal{E}} \left(\prod_{e\in E} \tanh\beta J_e\right) \left(\sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in E\\ e=(vu)}} \sigma_v\sigma_u\right) \\
&= \left(\prod_{e\in\mathcal{E}} \cosh\beta J_e\right) \sum_{\substack{E\subseteq\mathcal{E}\\ E \text{ collection of loops}}} \left(\prod_{e\in E} \tanh\beta J_e\right) \cdot 2^{|\mathcal{V}|} \\
&= 2^{|\mathcal{V}|} \left(\prod_{e\in\mathcal{E}} \cosh\beta J_e\right) \sum_{\substack{E\subseteq\mathcal{E}\\ E \text{ collection of loops}}} \left(\prod_{e\in E} \tanh\beta J_e\right).
\end{aligned}$$

□

2.3 Low-temperature expansion

Consider the standard Ising model on $(\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$ but with *boundary conditions*; that is, the vertices of $\partial\mathcal{V}_{\Omega_\delta} \subset \mathcal{V}_{\Omega_\delta}$ are fixed as $+$. Another way of representing a configuration σ is as follows: for every edge $e = (vu) \in \mathcal{E}_{\Omega_\delta}$, one draws the *dual edge* $e^\dagger$, which connects the center of the two faces that share e as a boundary component, if $\sigma_v \neq \sigma_u$ (Figure 3). The result is a subset $E^\dagger$ of the set of *dual edges*, which will be denoted as $\mathcal{E}^\dagger_{\Omega_\delta}$. This representation is called the *domain wall* configuration, because the drawn edges separate regions where the spin variables have opposite signs.

Remark 2.7. Just like with $\mathcal{E}^\dagger_{\Omega_\delta}$, we will employ the notation $E^\dagger$ to denote $\{e^\dagger : e \in E\}$ for any generic set of edges $E \subseteq \mathcal{E}_{\Omega_\delta}$. In addition, we will write $\partial\mathcal{E}^\dagger_{\Omega_\delta} \equiv (\partial\mathcal{E}_{\Omega_\delta})^\dagger$ and $\mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta} \equiv (\mathrm{Int}\,\mathcal{E}_{\Omega_\delta})^\dagger$ for simplicity. Finally, note that the dual of a dual edge is itself and therefore the same is true for any set of edges.

Not all subsets of $\mathcal{E}^\dagger_{\Omega_\delta}$ yield valid spin configurations. Let $E^\dagger \subseteq \mathcal{E}^\dagger_{\Omega_\delta}$ be a valid domain walls configuration and say we start going around a loop on the graph $(\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$. Suppose we know the spin at the starting site is $+$ and we mark the remaining spins on the vertices as we progress. By definition of the domain wall configuration, neighbouring spins are opposite if and only if a dual edge from $E^\dagger$ is crossed. For the domain walls configuration to be consistent, one has to mark the spin at the starting site as being $+$, implying an even number of dual edges in $E^\dagger$ were crossed. Note how the argument still holds if the starting spin was $-$. We thus conclude that any loop in $(\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$ must cross an even number of elements of $E^\dagger$. In particular, using loops that go around a single face, every $f \in \mathcal{F}_{\Omega_\delta}$ is an endpoint of

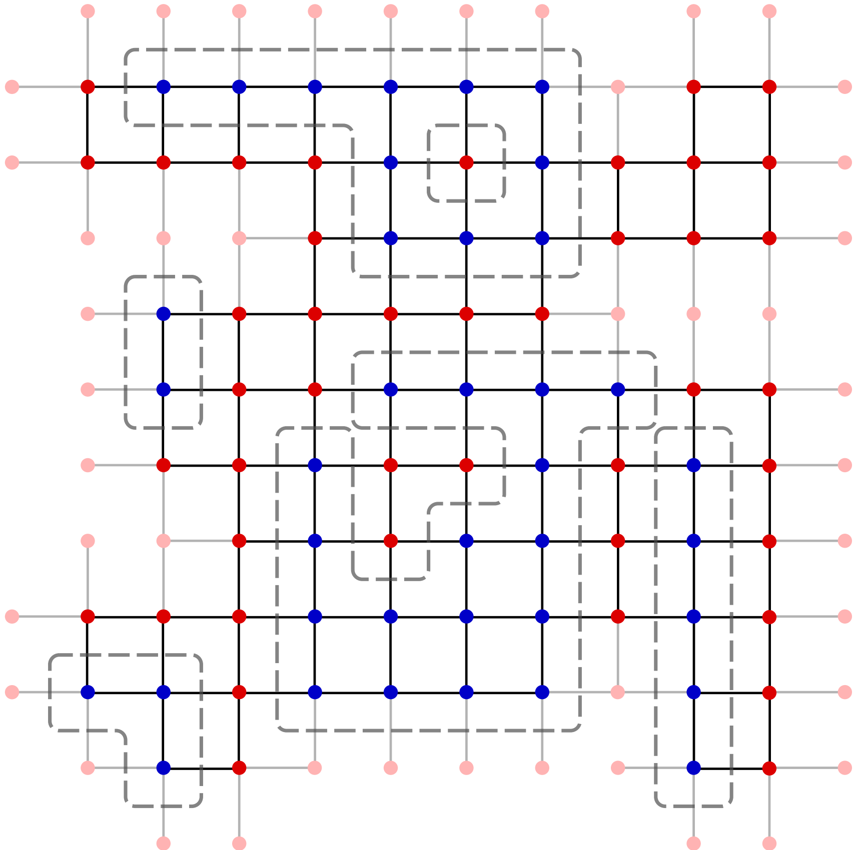

Figure 3: Domain wall representation for a spin configuration (+ in red, − in blue). The dual edges are drawn in a way that highlights a possible decomposition into collections of loops.

an even number of dual edges of $E^\dagger$, hence $E^\dagger$ must be a collection of loops.

Proving that $E^\dagger \subseteq \mathcal{E}^\dagger_{\Omega_\delta}$ being a collection of loops is a sufficient condition for $E^\dagger$ to be a valid domain wall configuration is simpler. If $E^\dagger = \emptyset$, then σ is the configuration where all spins are positive. Each time a loop l is added to $E^\dagger$, taking a valid configuration for $E^\dagger$ and flipping the spins of the vertices inside l yields a valid configuration for $E^\dagger \cup l$ (Figure 4). Note that this operation is always possible because there will never be a boundary vertex inside one of these loops: the discretized domain Ω_δ is assumed to be simply connected, so there are no "holes" inside, and the fixed spins occur at $\partial\mathcal{V}_{\Omega_\delta}$, which is adequately positioned at the boundary — in fact, there are not even edges between vertices of $\partial\mathcal{V}_{\Omega_\delta}$.

Let us write the partition function using a combinatorial sum over domain wall configurations, which we denote by $\mathcal{Z}^+_\beta$. Multiplying the partition function by $\exp\big(-|\mathcal{E}_{\Omega_\delta}|\beta\big)$ — a factor $\exp(-\beta)$ for each edge in $\mathcal{E}_{\Omega_\delta}$ — yields

$$\begin{aligned}\exp\big(-|\mathcal{E}_{\Omega_\delta}|\beta\big)\cdot\mathcal{Z}^+_\beta &= \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}\times\{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}}\prod_{\substack{e\in\mathcal{E}_{\Omega_\delta}\\ e=(vu)}}\exp\big(-\beta(1-\sigma_v\sigma_u)\big)\\ &= \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}\times\{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}}\prod_{\substack{e\in\mathcal{E}_{\Omega_\delta}\\ e=(vu)}}\exp\big(-2\beta\mathbf{1}(\sigma_v\neq\sigma_u)\big)\end{aligned}$$

where $\mathbf{1}(\sigma_v \neq \sigma_u)$ equals 1 if $\sigma_v \neq \sigma_u$, otherwise it equals 0. For a configuration $\sigma \in \{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}} \times \{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}$, the edges contributing to the corresponding product are the ones whose endpoints have oppo-

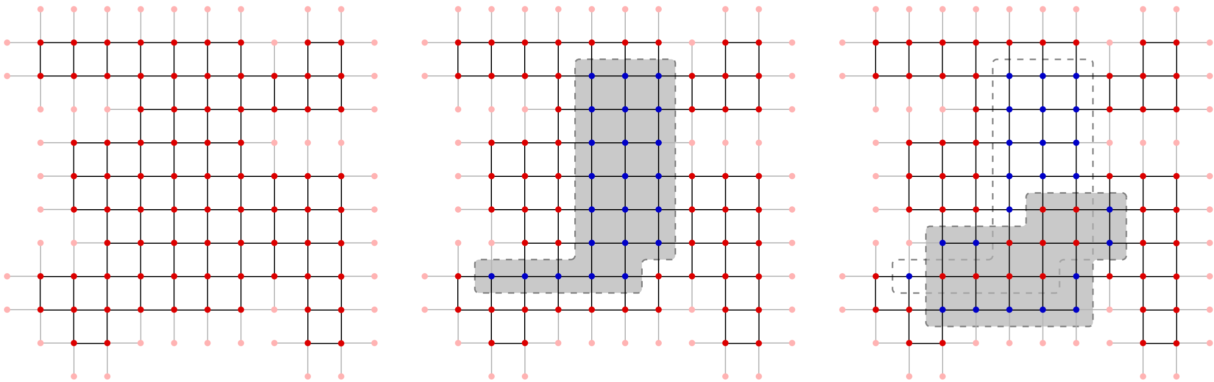

Figure 4: Iterative process for building a domain wall configuration

site spins, which are the ones that have a dual edge drawn in a domain wall configuration. Therefore, if $E^\dagger \subseteq \mathcal{E}^\dagger_{\Omega_\delta}$ is the domain wall configuration of σ,

$$\prod_{\substack{e\in\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \exp\big(-2\beta\mathbf{1}(\sigma_v\neq\sigma_u)\big) = \exp\big(-2\beta|E^\dagger|\big)$$

and summing over all configurations σ yields

$$\sum_{\sigma\in\{\pm1\}^{\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}}\times\{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}} \prod_{\substack{e\in\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \exp\big(-2\beta\mathbf{1}(\sigma_v\neq\sigma_u)\big) = \sum_{\substack{E^\dagger\subseteq\mathcal{E}^\dagger_{\Omega_\delta}\\ E^\dagger \text{ collection of loops}}} \exp\big(-2\beta|E^\dagger|\big).$$

Proposition 2.8. *For the Ising model on* $(\mathcal{V}_{\Omega_\delta},\mathcal{E}_{\Omega_\delta})$,

$$\mathcal{Z}^+_\beta = \exp\big(|\mathcal{E}_{\Omega_\delta}|\beta\big) \sum_{\substack{E^\dagger\subseteq\mathcal{E}^\dagger_{\Omega_\delta}\\ E^\dagger \text{ collection of loops}}} \exp\big(-2\beta|E^\dagger|\big)$$

As the reader might have guessed, this is the *low-temperature expansion* of the Ising model, whose name derives from the fact it was originally used to study the model at low temperatures. Nowadays, much of the literature regarding the Ising model considers configurations in a domain wall format by default [CS12, CHI15], a trend this work will follow.

Just like with the high-temperature expansion, it is interesting to consider different settings and adaptations of the low-temperature expansion.

Example 2.9. A common case of study is to take *wired boundary conditions* in which the spins are fixed as + along an arc of the boundary and as − along the complementary arc. The valid domain wall configurations are collections of dual loops together with a dual path between the faces separating the oppositely wired arcs.

Example 2.10. If a setup has no boundary, the allowed domain wall configurations are still the collections loops. Some examples include infinite graphs like $\mathbb{Z}^2$ (which require a more delicate definition of the model), but also finite graphs embedded on a torus.

The argument applies with little changes when coupling constants are introduced.

Proposition 2.11. *For the Ising model on $(\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$ with coupling constants (J_e),*

$$\mathcal{Z}_{(J_e),\beta} = \left(\prod_{e\in\mathcal{E}_{\Omega_\delta}} \exp\big(\beta J_e\big)\right) \sum_{\substack{E^\dagger\subseteq\mathcal{E}^\dagger_{\Omega_\delta} \\ E^\dagger \text{ collection of loops}}} \left(\prod_{e^\dagger\in E^\dagger} \exp\big(-2\beta J_e\big)\right)$$

Proof. For each configuration σ, its domain walls representation is done in the same way. As before, the collections $E^\dagger \subseteq \mathcal{E}^\dagger_{\Omega_\delta}$ corresponding to valid representations are collections of edge-disjoint loops. Multiplying the usual partition function by $\prod_{e\in\mathcal{E}_{\Omega_\delta}} \exp\big(-\beta J_e\big)$ yields

$$\left(\prod_{e\in\mathcal{E}_{\Omega_\delta}} \cdot \exp\big(-\beta J_e\big)\right)\mathcal{Z}_{(J_e),\beta} = \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}\times\{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}} \prod_{\substack{e\in\mathcal{E}_{\Omega_\delta} \\ e=(vu)}} \exp\big(-2\beta J_e\mathbf{1}(\sigma_v\neq\sigma_u)\big)$$

and if $E^\dagger \subseteq \mathcal{E}^\dagger_{\Omega_\delta}$ is the domain wall representation of the configuration $\sigma \in \{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}} \times \{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}$ then

$$\prod_{\substack{e\in\mathcal{E}_{\Omega_\delta} \\ e=(vu)}} \exp\big(-2\beta J_e\mathbf{1}(\sigma_v\neq\sigma_u)\big) = \prod_{e^\dagger\in E^\dagger} \exp\big(-2\beta J_e\big),$$

therefore

$$\sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}\times\{+1\}^{\partial\mathcal{V}_{\Omega_\delta}}} \prod_{\substack{e\in\mathcal{E}_{\Omega_\delta} \\ e=(vu)}} \exp\big(-2\beta J_e\mathbf{1}(\sigma_v\neq\sigma_u)\big) = \sum_{\substack{E^\dagger\subseteq\mathcal{E}^\dagger_{\Omega_\delta} \\ E^\dagger \text{ collection of loops}}} \left(\prod_{e^\dagger\in E^\dagger} \exp\big(-2\beta J_e\big)\right).$$

□

2.4 Kramers-Wannier Duality

Taken together, Propositions 2.1 with $\mathcal{G} = (\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$ and 2.8 give two different expressions for $\mathcal{Z}_\beta$ with a remarkably similar structure: up to a multiplicative constant, both are sums over subsets of either $\mathcal{E}_{\Omega_\delta}$ or $\mathcal{E}^\dagger_{\Omega_\delta}$ of some function of β taken to the power of the cardinality of the subset, and there is an obvious bijection between $\mathcal{E}_{\Omega_\delta}$ and $\mathcal{E}^\dagger_{\Omega_\delta}$. However, Proposition 2.1 has a sum over subsets of $\mathcal{E}_{\Omega_\delta}$ that are collections of loops whereas the sum of Proposition 2.1 is over subsets of $\mathcal{E}^\dagger_{\Omega_\delta}$ that are collections of loops, and if $E \subseteq \mathcal{E}_{\Omega_\delta}$ is a collection of loops there is no guarantee that $E^\dagger$ is also a collection of loops.

We can indeed make those sums match. The trick is to consider two Ising models: one on $\mathcal{G}_\delta := (\operatorname{Int}\mathcal{V}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}_{\Omega_\delta})$ and another on $\mathcal{G}^\dagger_\delta := (\operatorname{Int}\mathcal{F}^\dagger_{\Omega_\delta} \cup \partial\mathcal{F}^\dagger_{\Omega_\delta}, \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta}) = (\mathcal{F}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta})$, keeping free boundary conditions on the former while setting + boundary conditions on the latter — namely, on the spins of $\partial\mathcal{F}^\dagger_{\Omega_\delta}$. Let $\beta, \beta^\dagger > 0$ be the respective parameters of these two models. The high-temperature expansion on $\mathcal{G}_\delta$ is given by Proposition 2.1, and yields

$$\mathcal{Z}_\beta = 2^{|\operatorname{Int}\mathcal{V}_{\Omega_\delta}|}(\cosh\beta)^{|\operatorname{Int}\mathcal{E}_{\Omega_\delta}|} \sum_{\substack{E\subseteq\operatorname{Int}\mathcal{E}_{\Omega_\delta} \\ E \text{ collection of loops}}} (\tanh\beta)^{|E|}.$$

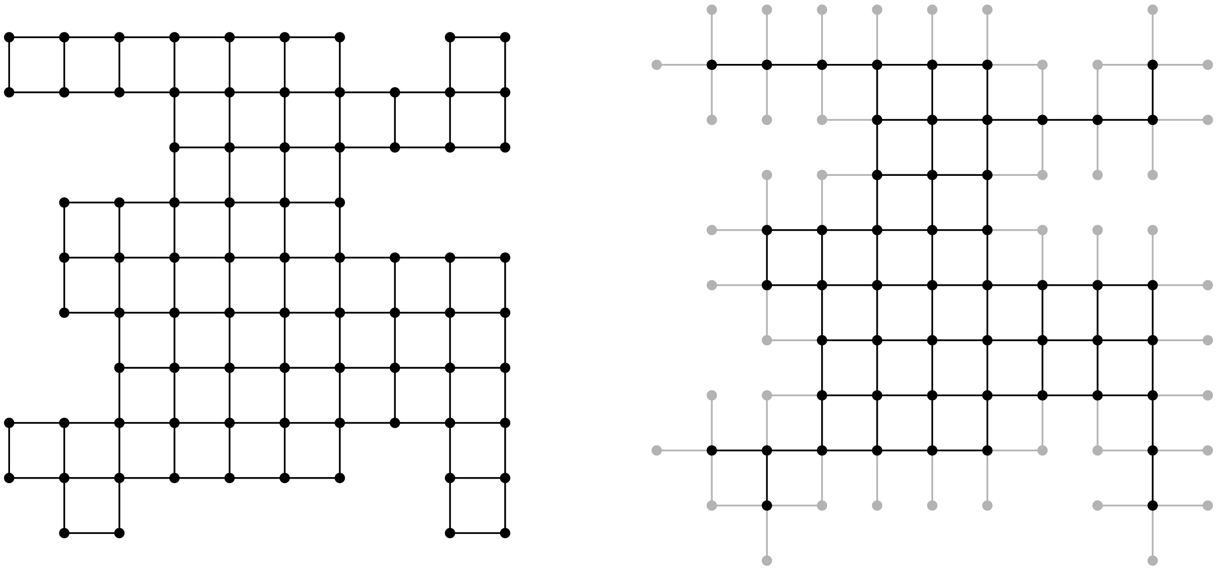

Figure 5: The graph $\mathcal{G}_\delta = (\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}_{\Omega_\delta})$ on the left, and the graph $\mathcal{G}_\delta^\dagger = (\mathcal{F}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta})$ on the right. The faces $\partial\mathcal{F}_{\Omega_\delta}$ and dual edges linking an interior face with a boundary one are coloured grey.

Conversely, Proposition 2.8 as it is stated cannot be directly applied to $\mathcal{G}_\delta^\dagger$, but the arguments apply the same way. If $\mathcal{Z}^{\dagger,+}_{\beta^\dagger}$ is the partition function of the model on $\mathcal{G}_\delta^\dagger$, then the low-temperature expansion on $\mathcal{G}_\delta^\dagger$ gives

$$\begin{aligned}\mathcal{Z}^{\dagger,+}_{\beta^\dagger} &= \exp\big(|\mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}|\beta^\dagger\big) \sum_{\substack{E\subseteq(\mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta})^\dagger \\ E \text{ collection of loops}}} \exp\big(-2\beta^\dagger|E|\big) \\ &= \exp\big(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger\big) \sum_{\substack{E\subseteq\mathrm{Int}\,\mathcal{E}_{\Omega_\delta} \\ E \text{ collection of loops}}} \exp\big(-2\beta^\dagger|E|\big)\end{aligned}$$

where we have used the fact that the dual of a dual set is the original set and $|\mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}| = |\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|$. If we now pick $\beta, \beta^\dagger$ so that $\tanh\beta = \exp(-2\beta^\dagger)$, then the sums match perfectly.

Proposition 2.12. *Consider an Ising model on $\mathcal{G}_\delta = (\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}_{\Omega_\delta})$ with parameter β and no boundary conditions, together with another Ising model on $\mathcal{G}_\delta^\dagger = (\mathcal{F}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta})$ with parameter $\beta^\dagger$ and $+$ boundary conditions. If $\tanh\beta = \exp(-2\beta^\dagger)$, then*

$$\mathcal{Z}_\beta = 2^{|\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}|}(\cosh\beta)^{|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|}\exp\big(-|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta\big)\cdot\mathcal{Z}^{\dagger,+}_{\beta^\dagger}.$$

This is the *Kramers-Wannier duality.* At its core, it is a symmetry relating a low-temperature Ising model with a high-tempertaure Ising model: notice how the relation

$$\tanh\beta = \exp(-2\beta^\dagger) \Leftrightarrow \exp(2\beta + 2\beta^\dagger) = \exp(2\beta) + \exp(2\beta^\dagger) + 1$$

is symmetric on β and $\beta^\dagger$, and that increasing one implies decreasing the other.

Remark 2.13. Although further technical details are needed for a complete proof, one can get an intuition as to how to find such a critical point. Assuming there is a single phase transition, it would occur when the model is self-dual: that is,

$$\tanh\beta = \exp(-2\beta) \Leftrightarrow \beta = \frac{1}{2}\ln(\sqrt{2}+1)$$

which is indeed the case.

Remark 2.14. With the introduction of the dual graph $\mathcal{G}_\delta^\dagger$ we clarify that the words "vertex", "edge" and "face" are used in reference to the original graph $\mathcal{G}_\delta$ — that is, for elements of $\mathcal{V}_{\Omega_\delta}$, $\mathcal{E}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta}$ respectively —, unless explicitly stated otherwise. In addition, we will drop the + superscript from the partition function of the dual model since it will always be considered with + boundary conditions.

The same duality occurs when coupling constants are considered.

Proposition 2.15. *Consider an Ising model on $\mathcal{G}_\delta = (\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}_{\Omega_\delta})$ with parameter β, coupling constants (J_e) and no boundary conditions, together with another Ising model on $\mathcal{G}_\delta^\dagger = (\mathcal{F}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}^\dagger)$ with parameter $\beta^\dagger$, coupling constants $(J_{e^\dagger})$ and + boundary conditions. If $\tanh(\beta J_e) = \exp(-2\beta^\dagger J_{e^\dagger})$ for all $e \in \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$, then*

$$\mathcal{Z}_{(J_e),\beta} = 2^{|\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}|}\left(\prod_{e\in\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}} \cosh\left(\beta J_e\right)\right)\left(\prod_{e\in\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}} \exp\left(-\beta^\dagger J_{e^\dagger}\right)\right)\cdot \mathcal{Z}^\dagger_{(J_{e^\dagger}),\beta^\dagger}.$$

Proof. Taking the high-temperature expansion of the $\mathcal{G}_\delta$ model together with the low-temperature expansion of the $\mathcal{G}_\delta^\dagger$ model, the equality follows from Proposition 2.6 and (the corresponding statement for $\mathcal{G}_\delta^\dagger$ of) Proposition 2.11. □

2.5 Order and disorder variables, order and disorder lines

As showcased before, when using Kramers-Wannier duality many settings of the model are exchanged. The graph is replaced by the dual graph, no boundary conditions are replaced by + boundary conditions and $\tanh(\beta)$ is replaced by $\exp(-2\beta)$. One might wonder how to translate the computations of correlations from one model to the other, especially since a high-temperature expansion computation for such a correlation is already worked out in (5).

Consider the two models on $\mathcal{G}_\delta$ and $\mathcal{G}_\delta^\dagger$, pick $v_1, v_2 \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ and let us try to compute the expected value $\mathbb{E}_{\mathcal{G}_\delta}[\sigma_{v_1}\sigma_{v_2}]$ of the product of spin variables σ_{v_1} and σ_{v_2} in $\mathcal{G}_\delta$ using the dual model. The high-temperature expansion yields (5) (with $\mathcal{E} = \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$), and note how the denominator is expressed using the $\mathcal{G}_\delta^\dagger$ model according to Proposition 2.12. Only the sum

$$\sum_{\substack{E\subseteq\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1\leftrightarrow v_2}} (\tanh\beta^\dagger)^{|E|}$$

remains to be worked out. The low-temperature expansion on $\mathcal{G}_\delta^\dagger$ would give a sum over collections of loops, but we require a sum over collections of loops coupled with a path between v_1 and v_2.

One can get such a sum by tweaking the $\mathcal{G}_\delta^\dagger$ model. Fix a path $\theta \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta}$ connecting v_1 to v_2 and consider the Ising model on $\mathcal{G}_\delta^\dagger$. According to (1), the Hamiltonian can be seen as follows: for every edge $e^\dagger = (uv) \in \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta}$, one adds up -1 if $\sigma_u = \sigma_v$ and $+1$ if $\sigma_u = -\sigma_v$. Our tweak consists on making the dual edges crossed by θ (that is, such that $e \in \theta$) behave the opposite way: if $e^\dagger = (uv)$ is crossed by γ, then the contribution of $e^\dagger$ to the Hamiltonian is changed to be $+1$ if $\sigma_v = \sigma_u$ and -1 if $\sigma_v = -\sigma_u$. This would change the definition of the Hamiltonian to be

$$-\sum_{\substack{e^\dagger \in \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta} \setminus \theta^\dagger \\ e^\dagger = (vu)}} \sigma_v \sigma_u + \sum_{\substack{e^\dagger \in \theta^\dagger \\ e^\dagger = (vu)}} \sigma_v \sigma_u.$$

and the probability of a configuration is still given by (2). Informally speaking, spin variables connected by an edge crossed by θ behave as if that neighbouring spin was the opposite of its real value.

Following the low-temperature expansion argument, the valid domain wall configurations are still collections of loops $E \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta}$. In the usual setting, the edges in $E^\dagger$ are the ones that contribute with a $+1$ to the Hamiltonian. In this case, the edges crossed by θ are behaving opposite from normal, hence it is the edges from $(E \oplus \theta)^\dagger$ that are contributing with a $+1$ to the modified Hamiltonian. The desired sum will appear because the collection of subsets

$$\{E \oplus \gamma : E \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta} \text{ collection of loops}\}$$

are precisely the subsets of $\operatorname{Int}\mathcal{E}_{\Omega_\delta}$ formed by loops of $\operatorname{Int}\mathcal{E}_{\Omega_\delta}$ together with a path from v_1 to v_2: one way to check this is

$$\begin{aligned}
&\{E \oplus \gamma : E \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta} \text{ collection of loops}\} = \\
&\qquad = \{E \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta} :\ v_1 \text{ and } v_2 \text{ have odd degree in } E \text{ and every other vertex has even degree in } E\} \\
&\qquad = \{E : E \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta} \text{ collection of loops + path linking } v_1 \text{ and } v_2\}.
\end{aligned}$$

The computations can be carried out using the ideas from Proposition 2.11. This modification corresponds to considering a model on $\mathcal{G}_\delta^\dagger$ with coupling constants given by

$$J_{e^\dagger} = \begin{cases} -1, & e \in \theta \\ 1, & e \notin \theta \end{cases}, \tag{7}$$

therefore

$$\begin{aligned}
\mathcal{Z}^{\dagger}_{(J_{e^\dagger}),\beta^\dagger} &= \exp\Big(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger - 2|\theta|\beta^\dagger\Big) \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ collections of loops}}} \Big(\prod_{e\in E} \exp\big(-2\beta^\dagger J_{e^\dagger}\big)\Big) && (8)\\
&= \exp\Big(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger - 2|\theta|\beta^\dagger\Big) \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} \Big(\prod_{e\in E\oplus\theta} \exp\big(-2\beta^\dagger J_{e^\dagger}\big)\Big) && (9)\\
&= \exp\Big(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger - 2|\theta|\beta^\dagger\Big) \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} \Big(\prod_{e\notin E, e\in\theta} \exp\big(2\beta^\dagger\big)\Big)\Big(\prod_{e\in E, e\notin\theta} \exp\big(-2\beta^\dagger\big)\Big)\\
&= \exp\Big(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger\Big) \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} \Big(\prod_{e\in E, e\in\theta} \exp\big(-2\beta^\dagger\big)\Big)\Big(\prod_{e\in E, e\notin\theta} \exp\big(-2\beta^\dagger\big)\Big)\\
&= \exp\Big(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger\Big) \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} \exp\big(-2\beta^\dagger|E|\big).
\end{aligned}$$

where (8) is the corresponding statement from Proposition 2.11 applied to $\mathcal{G}_\delta^\dagger$ and (9) uses the bijection

$$\begin{aligned}
\{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta} \text{ collections of loops}\} &\longrightarrow \{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta} \text{ loops + path } v_1 \leftrightarrow v_2\}\\
E &\longmapsto E\oplus\theta.
\end{aligned}$$

The complete answer to our question is the following result from [KC71]:

Proposition 2.16. *Take $v_1, v_2 \in \mathcal{V}_{\Omega_\delta}$ and θ a path in $\mathcal{G}_\delta$ linking v_1 and v_2. Consider an Ising model on $\mathcal{G}_\delta$ with parameter β and no boundary conditions, together with two models on $\mathcal{G}_\delta^\dagger$ with parameter $\beta^\dagger$ and + boundary conditions. Make one of them have no coupling constants whilst setting the other with coupling constants $(J_{e^\dagger})$ given by (7). If $\tanh\beta = \exp(-2\beta^\dagger)$, then*

$$\mathbb{E}_{\mathcal{G}_\delta}[\sigma_{v_1}\sigma_{v_2}] = \frac{\mathcal{Z}^{\dagger}_{(J_{e^\dagger}),\beta^\dagger}}{\mathcal{Z}^{\dagger}_{\beta^\dagger}}$$

Proof. The high-temperature expansion (5) yields

$$\mathbb{E}_{\mathcal{G}_\delta}[\sigma_{v_1}\sigma_{v_2}] = \frac{\sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} (\tanh\beta)^{|E|}}{\sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops}}} (\tanh\beta)^{|E|}}$$

and the expansion done before shows that

$$\sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} (\tanh\beta)^{|E|} = \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E \text{ loops + path } v_1 \leftrightarrow v_2}} \exp\big(-2\beta^\dagger|E|\big) = \exp\Big(-|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger\Big)\cdot \mathcal{Z}^{\dagger}_{(J_{e^\dagger}),\beta^\dagger}$$

whereas the denominator is computed using Proposition 2.12:

$$\sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E\text{ loops}}} (\tanh\beta)^{|E|} = \sum_{\substack{E\subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}\\ E\text{ loops}}} \exp\big(-2\beta^\dagger|E|\big) = \exp\Big(-|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta^\dagger\Big)\cdot \mathcal{Z}^\dagger_{\beta^\dagger}.$$

□

This result reveals a dual connection between spin variables and lines along which the spins behave opposite from normal. The former are order (or spin) variables, whereas the latter are formally known as disorder lines.

Definition 2.17. Given an Ising model on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an *order variable* is a random variable of the form σ_v with $v \in \mathcal{V}$.

Definition 2.18. Given an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with faces $\mathcal{F}$, a *disorder line* is a path $\gamma \subseteq \mathcal{E}^\dagger$ connecting two elements from $\mathcal{F}$. We say the model has a disorder line γ if the Hamiltonian is defined as

$$H^\gamma(\sigma) := -\sum_{\substack{e\in\mathcal{E}\setminus\gamma^\dagger\\ e=(vu)}} \sigma_v\sigma_u + \sum_{\substack{e\in\gamma^\dagger\\ e=(vu)}} \sigma_v\sigma_u$$

for any configuration σ. The partition function, probability and expected values of the model are written as $\mathcal{Z}^\gamma_\beta$, $\mathbb{P}^\gamma$ and $\mathbb{E}^\gamma$, respectively.

Remark 2.19. The change of notation from θ to γ is to underline that γ belongs to the dual graph — in fact, in this section we will always take $\gamma \subseteq \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}$ — but the θ previously considered belonged to $\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$ because it was a path in the dual graph of $\mathcal{G}^\dagger_\delta$.

When handling disorder lines, a recurring theme is that many properties are essentially dependent of the endpoints only. As an example, we have the following:

Proposition 2.20. *Let $\gamma_1, \gamma_2 \subseteq \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}$ be two paths connecting the same faces. Let $\mathcal{Z}^{\gamma_1}_\beta$ and $\mathcal{Z}^{\gamma_2}_\beta$ be the partition functions of two Ising models on $\mathcal{G}_\delta$ with parameter β and disorder line γ_1 and γ_2, respectively. Then,*

$$\mathcal{Z}^{\gamma_1}_\beta = \mathcal{Z}^{\gamma_2}_\beta.$$

Proof. Take the loop l obtained from concatenating γ_1 to γ_2 and assume for simplicity that l does not go through any face more than once; in particular, γ_1 and γ_2 are edge-disjoint. Given a configuration σ for the first model, let $\widetilde{\sigma}$ be the configuration for the second model obtained from σ after flipping all the spins of the vertices inside l. Note how this operation establishes a 1-to-1 correspondence between configurations and $\sigma_v\sigma_u = -\widetilde{\sigma}_v\widetilde{\sigma}_u$ if and only if $(vu) \in l^\dagger = (\gamma_1^\dagger \cup \gamma_2^\dagger)$. Thus,

$$\begin{aligned}
\mathcal{Z}_\beta^{\gamma_1} &= \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \Bigg(\prod_{\substack{e\in \operatorname{Int}\mathcal{E}_{\Omega_\delta}\setminus\gamma_1^\dagger \\ e=(vu)}} \exp(\beta\sigma_v\sigma_u)\Bigg)\Bigg(\prod_{\substack{e\in\gamma_1^\dagger \\ e=(vu)}} \exp(-\beta\sigma_v\sigma_u)\Bigg) \\
&= \sum_{\widetilde{\sigma}\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \Bigg[\Bigg(\prod_{\substack{e\in \operatorname{Int}\mathcal{E}_{\Omega_\delta}\setminus(\gamma_1^\dagger\cup\gamma_2^\dagger) \\ e=(vu)}} \exp(\beta\widetilde{\sigma}_v\widetilde{\sigma}_u)\Bigg)\Bigg(\prod_{\substack{e\in\gamma_2^\dagger \\ e=(vu)}} \exp(-\beta\widetilde{\sigma}_v\widetilde{\sigma}_u)\Bigg)\Bigg]\Bigg(\prod_{\substack{e\in\gamma_1^\dagger \\ e=(vu)}} \exp(\beta\widetilde{\sigma}_v\widetilde{\sigma}_u)\Bigg) \\
&= \sum_{\widetilde{\sigma}\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \Bigg(\prod_{\substack{e\in \operatorname{Int}\mathcal{E}_{\Omega_\delta}\setminus\gamma_2^\dagger \\ e=(vu)}} \exp(\beta\widetilde{\sigma}_v\widetilde{\sigma}_u)\Bigg)\Bigg(\prod_{\substack{e\in\gamma_2^\dagger \\ e=(vu)}} \exp(-\beta\widetilde{\sigma}_v\widetilde{\sigma}_u)\Bigg) \\
&= \mathcal{Z}_\beta^{\gamma_2}.
\end{aligned}$$

For the general case one should take $l = \gamma_1 \oplus \gamma_2$, and if l goes through any face more than once — making the decision of which vertices are "inside l" ambiguous —, then divide it into edge-disjoint cycles and, for each one, flip all spins inside it. This will make the contribution of the edges in $\gamma_1 \oplus \gamma_2$ flip[2], which yields the desired result. □

It is possible to consider a model with multiple disorder lines, which may share edges or endpoints. Seeing as a disorder line makes the separated spins behave opposite from normal, two disorder lines in the same place should have no effect (which is the dual fact of how two order variables associated to the same vertex cancel each other out). That means adding a new disorder line should correspond to taking the $\oplus$ operation with the already existing disorder lines. It is thus often assumed that the set of disorder lines has no overlapping edges so as to simplify arguments. Here, we make the choice of defining disorder lines as a generic subset of edges of the dual graph rather than a collection of paths in the dual graph, and write said subset of edges as paths whenever needed. Note how this definition is fundamentally equal to Definition 2.18.

Definition 2.21. Given an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with faces $\mathcal{F}$, a *set of disorder lines* is a set $\Gamma \subseteq \mathcal{E}^\dagger$, and *adding a disorder line* γ means replacing Γ with $\Gamma \oplus \gamma$. We say the model has a set of disorder lines Γ if the Hamiltonian is defined as

$$H^\Gamma(\sigma) := -\sum_{\substack{e\in\mathcal{E}\setminus\Gamma^\dagger \\ e=(vu)}} \sigma_v\sigma_u + \sum_{\substack{e\in\Gamma^\dagger \\ e=(vu)}} \sigma_v\sigma_u$$

for any configuration σ. The partition function, probability and expected values of the model are written as $\mathcal{Z}_\beta^\Gamma$, $\mathbb{P}^\Gamma$ and $\mathbb{E}^\Gamma$, respectively.

When it is useful to think of Γ as a collection of edge-disjoint disorder lines, an important fact is that the endpoints of such lines must be the faces with odd degree in Γ and therefore they must exist in even number. To emphasize them, we write $\Gamma \equiv \Gamma[a_1, \ldots, a_{2n}]$ when referring to a set of disorder lines Γ where the vertices with odd degree are $a_1, \ldots, a_{2n}$.

[2]This can also be seen as flipping the contributions from γ_1 and then γ_2. The edges in $\gamma_1 \cap \gamma_2$ behave as if nothing happened.

Remark 2.22. When employing the notation $\Gamma[a_1, \ldots, a_{2n}]$ (and more generally, when speaking about disorder lines), we allow for some of the endpoints a_k to repeat. The faces with odd degree in Γ are the ones that appear an odd number of times on the list $a_1, \ldots, a_{2n}$ (notice how the list must still have even length). We will make an abuse of language when referring to $a_1, \ldots, a_{2n}$ as the vertices with odd degree in $\Gamma[a_1, \ldots, a_{2n}]$, even if some a_k repeat. This "cancellation" will be a recurring pattern in the sequel, and will be left implicit.

There is another way of expressing the effect of disorder lines in an Ising model, as noted in [KC71]. For a set Γ of disorder lines and any configuration σ we have

$$\begin{aligned} \mathbb{P}^{\Gamma}(\sigma) &= \frac{1}{\mathcal{Z}_{\beta}^{\Gamma}} \Bigg(\prod_{\substack{e \in \operatorname{Int} \mathcal{E}_{\Omega_\delta} \setminus \Gamma^\dagger \\ e=(vu)}} \exp(\beta \sigma_v \sigma_u) \Bigg) \Bigg(\prod_{\substack{e \in \Gamma^\dagger \\ e=(vu)}} \exp(-\beta \sigma_v \sigma_u) \Bigg) \\ &= \frac{1}{\mathcal{Z}_{\beta}^{\Gamma}} \Bigg(\prod_{\substack{e \in \operatorname{Int} \mathcal{E}_{\Omega_\delta} \\ e=(vu)}} \exp(\beta \sigma_v \sigma_u) \Bigg) \Bigg(\prod_{\substack{e \in \Gamma^\dagger \\ e=(vu)}} \exp(-2\beta \sigma_v \sigma_u) \Bigg) \end{aligned}$$

and the left factor is $\exp\big(- \beta H(\sigma) \big)$ where H is the standard Hamiltonian. Therefore, considering disorder lines Γ can be coded in the standard model as inserting the right factor as a random variable in the computations. For future usage, these variables are defined in the general case with coupling constants.

Definition 2.23. Given an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with faces $\mathcal{F}$ with coupling constants (J_e), a *disorder variable* is a random variable of the form

$$(\mu_a \mu_b)_{\gamma, (J_e)} := \prod_{\substack{e \in \gamma^\dagger \\ e=(vu)}} \exp(-2\beta J_e \sigma_v \sigma_u)$$

where $\gamma \subseteq \mathcal{E}^\dagger$ is a disorder line connecting $a, b \in \mathcal{F}$. If $\Gamma \equiv \Gamma[a_1, \ldots, a_{2n}] \subseteq \mathcal{E}$ is a set of disorder lines, we use the shorthand

$$\Bigg(\prod_{j=1}^{2n} \mu_{a_j} \Bigg)_{\Gamma, (J_e)} := \prod_{j=1}^{n} (\mu_{a_{2j-1}} \mu_{a_{2j}})_{\gamma_j, (J_e)} = \prod_{\substack{e \in \Gamma^\dagger \\ e=(vu)}} \exp(-2\beta J_e \sigma_v \sigma_u)$$

where $\gamma_1, \ldots, \gamma_n$ is a decomposition of Γ into edge-disjoint paths such that γ_k runs from a_{2j-1} to a_{2j}. The coupling constants will often be left implicit from the notation.

Let us formally prove how disorder variables encode the effect of disorder lines.

Proposition 2.24. *Consider an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with parameter β, coupling constants (J_e) and probability measure given by (3). Let $\Gamma \equiv \Gamma[a_1, \ldots, a_{2n}] \subseteq \mathcal{E}^\dagger$ be a set of disorder* *lines. Then,*

$$\mathbb{P}_{\left(\widetilde{J}_e\right)}(\sigma) = \frac{\left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)}}{\mathbb{E}_{(J_e)}\left[\left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)}\right]} \mathbb{P}_{(J_e)}(\sigma) \tag{10}$$

where $\mathbb{P}_{\left(\widetilde{J}_e\right)}$ *is the probability of the Ising model on* $\mathcal{G}$ *with parameter* β *and coupling constants*

$$\widetilde{J}_e = \begin{cases} -J_e, & e \in \Gamma^\dagger \\ J_e, & e \notin \Gamma^\dagger \end{cases}.$$

Proof. Let $\widetilde{\mathbb{P}}(\sigma)$ be the function given by the right-hand side of (10). We have $\sum_{\sigma\in\{\pm 1\}^{\mathrm{Int}\,\mathcal{V}}} \widetilde{\mathbb{P}}(\sigma) = 1$, hence $\widetilde{\mathbb{P}}$ is a probability measure over $\{\pm 1\}^{\mathrm{Int}\,\mathcal{V}}$. In addition,

$$\begin{aligned}
\widetilde{\mathbb{P}}(\sigma) &\propto \left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)} \mathbb{P}_{(J_e)}(\sigma) \\
&\propto \left(\prod_{\substack{e\in\Gamma^\dagger \\ e=(vu)}} \exp(-2\beta J_e \sigma_v \sigma_u)\right)\left(\prod_{\substack{e\in\mathcal{E} \\ e=(vu)}} \exp(\beta J_e \sigma_v \sigma_u)\right) \\
&= \left(\prod_{\substack{e\in\mathcal{E}\setminus\Gamma^\dagger \\ e=(vu)}} \exp(\beta \sigma_v \sigma_u)\right)\left(\prod_{\substack{e\in\Gamma^\dagger \\ e=(vu)}} \exp(-\beta \sigma_v \sigma_u)\right) \\
&= \exp\Big(- \beta H_{\left(\widetilde{J}_e\right)}(\sigma)\Big).
\end{aligned} \tag{11}$$

This means $\widetilde{\mathbb{P}}$ is a probability measure such that $\widetilde{\mathbb{P}} \propto \exp\Big(- \beta H_{\left(\widetilde{J}_e\right)}\Big)$, implying $\widetilde{\mathbb{P}} = \mathbb{P}_{\left(\widetilde{J}_e\right)}$. ☐

Proposition 2.25. *In the conditions of Proposition 2.24,*

$$\mathbb{E}_{(J_e)}\left[\mathbb{X} \cdot \left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)}\right] = \frac{\mathcal{Z}_{(\widetilde{J}_e),\beta}}{\mathcal{Z}_{(J_e),\beta}} \cdot \mathbb{E}_{\left(\widetilde{J}_e\right)}[\mathbb{X}]$$

where $\mathbb{X} \equiv \mathbb{X}(\sigma)$ *is any random variable depending on the spin variables.*

Proof. We essentially want to compute the ratio between the normalizing constants of $\mathbb{P}_{(J_e)}$ and $\widetilde{\mathbb{P}}$, which can be done by collecting the multiplicative constants in the computation (11) and knowing

$$\widetilde{\mathbb{P}}(\sigma) = \mathbb{P}_{\left(\widetilde{J}_e\right)}(\sigma) = \frac{1}{\mathcal{Z}_{(\widetilde{J}_e),\beta}} \exp\Big(- \beta H_{\left(\widetilde{J}_e\right)}(\sigma)\Big).$$

Equivalently, the intermediate steps of (11) show that

$$\left(\prod_{\substack{e\in\Gamma^\dagger \\ e=(vu)}} \exp(-2\beta J_e \sigma_v \sigma_u)\right)\left(\prod_{\substack{e\in\mathcal{E} \\ e=(vu)}} \exp(\beta J_e \sigma_v \sigma_u)\right) = \exp\Big(- \beta H_{\left(\widetilde{J}_e\right)}(\sigma)\Big)$$

for any configuration $\sigma \in \{\pm 1\}^{\mathcal{V}}$, which can be rewritten as

$$\left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)} \cdot \mathcal{Z}_{(J_e),\beta}\mathbb{P}_{(J_e)}(\sigma) = \mathcal{Z}_{(\widetilde{J}_e),\beta}\mathbb{P}_{\left(\widetilde{J}_e\right)}(\sigma)$$

implying

$$\mathbb{X}(\sigma)\left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)} \cdot \mathbb{P}_{(J_e)}(\sigma) = \frac{\mathcal{Z}_{(\widetilde{J}_e),\beta}}{\mathcal{Z}_{(J_e),\beta}}\mathbb{X}(\sigma) \cdot \mathbb{P}_{\left(\widetilde{J}_e\right)}(\sigma)$$

and now summing both sides over all $\sigma \in \{\pm 1\}^{\mathcal{V}}$ finishes the proof. □

Corollary 2.26. *In the conditions of the Proposition 2.24,*

$$\mathbb{E}_{(J_e)}\left[\left(\prod_{j=1}^{2n} \mu_{a_j}\right)_{\Gamma,(J_e)}\right] = \frac{\mathcal{Z}_{(\widetilde{J}_e),\beta}}{\mathcal{Z}_{(J_e),\beta}}$$

This corollary allows us to rewrite Proposition 2.16 in a more symmetric way, using order and disorder variables.

Proposition 2.27. *Consider an Ising model on $\mathcal{G}_\delta = (\operatorname{Int}\mathcal{V}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}_{\Omega_\delta})$ with parameter β and no boundary conditions, together with another Ising model on $\mathcal{G}_\delta^\dagger = (\mathcal{F}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}_{\Omega_\delta}^\dagger)$ with parameter $\beta^\dagger$ and + boundary conditions. Let $\theta \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta}$ be a path connecting $v_1, v_2 \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$. If $\tanh\beta = \exp(-2\beta^\dagger)$, then*

$$\mathbb{E}_{\mathcal{G}_\delta}[\sigma_{v_1}\sigma_{v_2}] = \mathbb{E}_{\mathcal{G}_\delta^\dagger}[(\mu_{v_1}\mu_{v_2})_\theta].$$

An immediate conclusion of this result is that the expected value $\mathbb{E}_{\mathcal{G}_\delta^\dagger}[(\mu_{v_1}\mu_{v_2})_\theta]$ is independent of the path θ as long as the endpoints are the same. This is why the notation for disorder variables highlights the endpoints of the path, and in fact many authors choose to drop the subscript entirely. We will leave it, and leave a full description of how tweaking disorder lines may change the expected value for Corollary 2.37.

One might wonder whether there is some way of defining "order lines", which would express order variables with in terms of a modification of the Hamiltonian. That is indeed the case, although these objects appear seldom in the modern literature. Nonetheless, it is nice to complete the quartet of order/disorder variables/lines.

As before, an algebraic manipulation should make it clear how to define the modified Hamiltonian. The new trick here is to find a pair of non-zero symmetric complex numbers that can be written as an exponential of those same numbers: that is, $\exp(\pm z) = \pm z$, possibly with a multiplicative constant applied on the right-hand side. This was done in [KC71] with $\exp\left(\pm \frac{i\pi}{2}\right) = \pm i$. Let us illustrate this strategy by computing $\mathbb{E}_{\mathcal{G}_\delta}[\sigma_{v_1}\sigma_{v_2}]$, where we make use of the path $\theta \subseteq \mathcal{E}_{\Omega_\delta}$ running between two sites

$v_1, v_2 \in \mathcal{V}_{\Omega_\delta}$:

$$
\begin{aligned}
\mathbb{E}_{\mathcal{G}_\delta}[\sigma_{v_1}\sigma_{v_2}] &= \frac{1}{\mathcal{Z}_\beta} \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \left(\sigma_{v_1}\sigma_{v_2} \prod_{\substack{e\in\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \exp(\beta\sigma_v\sigma_u)\right) \\
&= \frac{1}{\mathcal{Z}_\beta} \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \left(\left[\prod_{\substack{e\in\theta\\ e=(vu)}} \sigma_v\sigma_u\right] \prod_{\substack{e\in\operatorname{Int}\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \exp(\beta\sigma_v\sigma_u)\right) \qquad (12) \\
&= \frac{1}{\mathcal{Z}_\beta}(-i)^{|\theta|} \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \left(\left[\prod_{\substack{e\in\theta\\ e=(vu)}} i\sigma_v\sigma_u\right] \prod_{\substack{e\in\operatorname{Int}\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \exp(\beta\sigma_v\sigma_u)\right) \\
&= \frac{1}{\mathcal{Z}_\beta}(-i)^{|\theta|} \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \left(\left[\prod_{\substack{e\in\theta\\ e=(vu)}} \exp\left(\frac{i\pi}{2}\sigma_v\sigma_u\right)\right]\left[\prod_{\substack{e\in\operatorname{Int}\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \exp(\beta\sigma_v\sigma_u)\right]\right) \\
&= \frac{1}{\mathcal{Z}_\beta}(-i)^{|\theta|} \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}} \exp\left(-\beta\left[-\sum_{\substack{e\in\operatorname{Int}\mathcal{E}_{\Omega_\delta}\\ e=(vu)}} \sigma_v\sigma_u - \sum_{\substack{e\in\theta\\ e=(vu)}} \frac{i\pi}{2\beta}\sigma_v\sigma_u\right]\right)
\end{aligned}
$$

where (12) holds because spins of vertices other than the endpoints of θ appear twice in the product $\prod_{(uv)\in\delta}\sigma_v\sigma_u$. Seeing as the term inside the final sum should be $\exp\big(-\beta H(\sigma)\big)$, we have a motivation as to how to define the modified Hamiltonian, which can be easily generalized for multiple order lines.

Definition 2.28. Given an Ising model on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an *order line* is a path $\theta \subseteq \mathcal{E}$, a *set of order lines* is a set $\Theta \subseteq \mathcal{E}$ and *adding an order line* θ means replacing Θ with $\Theta \oplus \theta$. We say the model has a set of order lines Θ if the Hamiltonian is defined as

$$
{}^{\Theta}H(\sigma) := -\sum_{\substack{e\in\mathcal{E}\\ e=(vu)}} \sigma_v\sigma_u - \sum_{\substack{e\in\Theta\\ e=(vu)}} \frac{i\pi}{2\beta}\sigma_v\sigma_u
$$

for any configuration σ. The partition function, probability and expected values of the model are written as ${}^{\Theta}\mathcal{Z}_\beta$, ${}^{\Theta}\mathbb{P}$ and ${}^{\Theta}\mathbb{E}$, respectively.

Remark 2.29. Like with disorder lines, we want to simplify arguments and define order lines as subsets of $\mathcal{E}$ rather than collections of paths of $\mathcal{E}$. However, it is not obvious at first glance why two overlapping order lines would cancel each other out. If $e = (vu) \in \mathcal{E}$ were to contribute to the Hamiltonian with the effect of two order lines, then ${}^{\Theta}H(\sigma)$ would have an added $-\frac{i\pi}{\beta}\sigma_v\sigma_u$ term, which yields an $\exp(i\pi\sigma_v\sigma_u) = -1$ factor to $\exp {}^{\Theta}H(\sigma)$. Since this is common to all σ, the partition function will also have this extra -1 factor, effectively cancelling the effect of e on ${}^{\Theta}\mathbb{P}$.

Remark 2.30. On a related topic, note that every edge of an order line adds an $e^{\frac{i\pi}{2}\sigma_v\sigma_u} = \pm i$ factor to $\exp {}^{\Theta}H(\sigma)$, hence $\exp {}^{\Theta}H(\sigma) \in i^{|\Theta|}\mathbb{R}$ for all σ. The partition function will remove this common multiplicative constant $i^{|\Theta|}$[3], therefore the measure remains real, albeit not necessarily positive for every configuration.

Similar to disorder lines, we write $\Theta \equiv \Theta[v_1, \dots, v_{2m}]$ when referring to a collection of order lines

[3] A consequence of this is the added factor in Corollary 2.33.

where the vertices with odd degree are $v_1, \dots, v_{2m}$, possibly with repetition (Remark 2.22).

Let us state the relation between order variables and order lines.

Proposition 2.31. *Consider an Ising model on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with parameter β, coupling constants (J_e) and probability measure given by (3). Let $\Theta \equiv \Theta[v_1, \dots, v_{2m}] \subseteq \mathcal{E}$ be a set of order lines. Then,*

$$\mathbb{P}_{\left(\widetilde{J}_e\right)}(\sigma) = \frac{\prod_{k=1}^{2m} \sigma_{v_k}}{\mathbb{E}_{(J_e)}\left[\prod_{k=1}^{2m} \sigma_{v_k}\right]} \mathbb{P}_{(J_e)}(\sigma) \tag{13}$$

where $\mathbb{P}_{\left(\widetilde{J}_e\right)}$ is the probability of the Ising model on $\mathcal{G}$ with parameter β and coupling constants

$$\widetilde{J}_e = \begin{cases} J_e + \frac{i\pi}{2\beta}, & e \in \Theta \\ J_e, & e \notin \Theta \end{cases}.$$

Proof. Let $\widetilde{\mathbb{P}}(\sigma)$ be the function given by the right-hand side of (13). We have $\sum_{\sigma \in \{\pm 1\}^{\mathcal{V}}} \widetilde{\mathbb{P}}(\sigma) = 1$, hence it $\widetilde{P}$ a normalized measure over $\{\pm 1\}^{\mathcal{V}}$. In addition,

$$\begin{aligned}
\widetilde{\mathbb{P}}(\sigma) &\propto \left(\prod_{k=1}^{2m} \sigma_{v_k}\right) \mathbb{P}_{(J_e)}(\sigma) \\
&\propto \left(\prod_{\substack{e \in \Theta \\ e=(vu)}} i\sigma_v \sigma_u\right)\left(\prod_{\substack{e \in \mathcal{E} \\ e=(vu)}} \exp(\beta J_e \sigma_v \sigma_u)\right) \\
&= \left(\prod_{\substack{e \in \Theta \\ e=(vu)}} \exp\left(\frac{i\pi}{2}\sigma_v \sigma_u\right)\right)\left(\prod_{\substack{e \in \mathcal{E} \\ e=(vu)}} \exp(-\beta \sigma_v \sigma_u)\right) \\
&= \exp\Big(-\beta H_{\left(\widetilde{J}_e\right)}(\sigma)\Big).
\end{aligned} \tag{14}$$

This means $\widetilde{\mathbb{P}}$ is a probability measure such that $\widetilde{\mathbb{P}} \propto \exp\Big(-\beta H_{\left(\widetilde{J}_e\right)}\Big)$, implying $\widetilde{\mathbb{P}} = \mathbb{P}_{\left(\widetilde{J}_e\right)}$. □

Proposition 2.32. *In the conditions of Proposition 2.31,*

$$\mathbb{E}_{(J_e)}\left[\mathbb{X} \cdot \prod_{k=1}^{2m} \sigma_{v_k}\right] = (-i)^{|\Theta|} \frac{\mathcal{Z}_{(\widetilde{J}_e),\beta}}{\mathcal{Z}_{(J_e),\beta}} \cdot \mathbb{E}_{\left(\widetilde{J}_e\right)}[\mathbb{X}]$$

where $\mathbb{X} \equiv \mathbb{X}(\sigma)$ is any random variable depending on the spin variables.

Proof. We want to compute the ratio between the normalizing constants of $\mathbb{P}_{(J_e)}$ and $\widetilde{\mathbb{P}}$, which can be done by collecting the multiplicative constants in the computation (14) and knowing

$$\widetilde{\mathbb{P}}(\sigma) = \mathbb{P}_{\left(\widetilde{J}_e\right)}(\sigma) = \frac{1}{\mathcal{Z}_{(\widetilde{J}_e),\beta}} \exp\Big(-\beta H_{\left(\widetilde{J}_e\right)}(\sigma)\Big).$$

Equivalently, the intermediate steps of (14) show that

$$\Bigg(\prod_{\substack{e\in\Theta\\ e=(vu)}} i\sigma_v\sigma_u\Bigg)\Bigg(\prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \exp(\beta J_e\sigma_v\sigma_u)\Bigg) = \exp\Big(-\beta H_{\big(\widetilde{J}_e\big)}(\sigma)\Big)$$

for any configuration $\sigma \in \{\pm1\}^{\mathcal{V}}$, which can be rewritten as

$$i^{|\Theta|}\left(\prod_{k=1}^{2m}\sigma_{v_k}\right)\cdot \mathcal{Z}_{(J_e),\beta}\mathbb{P}_{(J_e)}(\sigma) = \mathcal{Z}_{(\widetilde{J}_e),\beta}\mathbb{P}_{\big(\widetilde{J}_e\big)}(\sigma)$$

implying

$$\mathbb{X}(\sigma)\left(\prod_{k=1}^{2m}\sigma_{v_k}\right)\cdot \mathbb{P}_{(J_e)}(\sigma) = (-i)^{|\Theta|}\frac{\mathcal{Z}_{(\widetilde{J}_e),\beta}}{\mathcal{Z}_{(J_e),\beta}}\mathbb{X}(\sigma)\cdot \mathbb{P}_{\big(\widetilde{J}_e\big)}(\sigma)$$

and now summing both sides over all $\sigma \in \{\pm1\}^{\mathcal{V}}$ finishes the proof. □

Corollary 2.33. *In the conditions of Proposition 2.31,*

$$\mathbb{E}_{(J_e)}\left[\prod_{k=1}^{2m}\sigma_{v_k}\right] = (-i)^{|\Theta|}\frac{\mathcal{Z}_{(\widetilde{J}_e),\beta}}{\mathcal{Z}_{(J_e),\beta}}$$

Finally, we write Proposition 2.16 using order and disorder lines.

Proposition 2.34. *Let θ be a path in $\mathcal{G}_\delta$. Consider two Ising models on $\mathcal{G}_\delta$ with parameter β and no boundary conditions, one being the standard one and the other having θ as an order line. In addition, consider two Ising models on $\mathcal{G}_\delta^\dagger$ with parameter $\beta^\dagger$ and no boundary conditions, one being the standard one and the other having θ as a disorder line. If $\tanh\beta = \exp(-2\beta)$, then*

$$(-i)^{|\theta|}\frac{{}^{\theta}\mathcal{Z}_\beta}{\mathcal{Z}_\beta} = \frac{\mathcal{Z}^{\dagger,\theta}_{\beta^\dagger}}{\mathcal{Z}^{\dagger}_{\beta^\dagger}}$$

Remark 2.35. Propositions 2.24 and 2.31 can be generalized to models where some of the spins are fixed. One should take $\mathbb{P}_{(J_e)}$ and $\mathbb{P}_{\big(\widetilde{J}_e\big)}$ as being the conditional probability measures, together with $\mathbb{E}_{(J_e)}$ being the expected conditional value conditioned in the same way. The arguments do not change.

2.6 Duality of order/disorder variables

The ideas from Proposition 2.16 can be generalized to any number of order and disorder variables. We follow the same strategy: start by writing the expected values using partition functions of Ising models with modified coupling constants (which can be seen as order and disorder lines) and then apply Proposition 2.15 to pass to the dual model. We state the result in its most symmetrical and common form, using order and disorder variables, making this a direct generalization of Proposition 2.27.

Theorem 2.36. *Consider an Ising model on $\mathcal{G}_\delta = (\operatorname{Int}\mathcal{V}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}_{\Omega_\delta})$ with parameter β and no boundary conditions, together with another Ising model on $\mathcal{G}_\delta^\dagger = (\mathcal{F}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta})$ with parameter $\beta^\dagger$ and + boundary conditions. Let $\Theta \equiv \Theta[v_1,\dots,v_{2m}] \subseteq \operatorname{Int}\mathcal{E}_{\Omega_\delta}$ and $\Gamma \equiv \Gamma[a_1,\dots,a_{2n}] \subseteq \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta}$. If $\tanh\beta = \exp(-2\beta^\dagger)$,*

then
$$\mathbb{E}_{\mathcal{G}_\delta}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_\Gamma\right] = (-1)^{|\Theta\cap\Gamma^\dagger|}\cdot\mathbb{E}_{\mathcal{G}_\delta^\dagger}\left[\prod_{j=1}^{2n}\sigma_{a_j}\left(\prod_{k=1}^{2m}\mu_{v_k}\right)_\Theta\right].$$

Proof. The model on $\mathcal{G}_\delta$ is equivalent to one with coupling constants $J_e = 1$. Let $\mathbb{P}$ be the probability measure, let $\mathcal{Z}$ be the partition function and let $\mathbb{E}$ be the expected value. In addition, consider two other models on $\mathcal{G}_\delta$ with parameter β, no boundary conditions and coupling constants

$$\widetilde{J}_e = \begin{cases} -J_e, & e\in\Gamma^\dagger \\ J_e, & e\notin\Gamma^\dagger \end{cases} \quad \text{and} \quad \widetilde{\widetilde{J}}_{e^\dagger} = \begin{cases} \widetilde{J}_e + \frac{i\pi}{2\beta^\dagger}, & e\in\Theta \\ \widetilde{J}_e, & e\notin\Theta \end{cases}$$

and define $\widetilde{\mathbb{P}}, \widetilde{\mathcal{Z}}, \widetilde{\mathbb{E}}$ and $\widetilde{\widetilde{\mathbb{P}}}$, $\widetilde{\widetilde{\mathcal{Z}}}$, $\widetilde{\widetilde{\mathbb{E}}}$ in an analogous way. Then,

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_\Gamma\right] &= \sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}}\left(\prod_{k=1}^{2m}\sigma_{v_k}\right)\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Theta,(J_e)}\mathbb{P}(\sigma) \\
&= \mathbb{E}\left[\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_\Gamma\right]\cdot\sum_{\sigma\in\{\pm1\}^{\operatorname{Int}\mathcal{V}_{\Omega_\delta}}}\left(\prod_{k=1}^{2m}\sigma_{v_k}\right)\widetilde{\mathbb{P}}(\sigma) && (15)\\
&= \frac{\widetilde{\mathcal{Z}}}{\mathcal{Z}}\cdot\widetilde{\mathbb{E}}\left[\prod_{k=1}^{2m}\sigma_{v_k}\right] && (16)\\
&= \frac{\widetilde{\mathcal{Z}}}{\mathcal{Z}}\cdot(-i)^{|\Theta|}\frac{\widetilde{\widetilde{\mathcal{Z}}}}{\widetilde{\mathcal{Z}}} && (17)\\
&= (-i)^{|\Theta|}\cdot\frac{\widetilde{\widetilde{\mathcal{Z}}}}{\mathcal{Z}} && (18)
\end{aligned}$$

using Proposition 2.24, Corollary 2.26 and Corollary 2.33 on (15), (16) and (17), respectively.

Now take the dual graph and consider three separate models with parameter $\beta^\dagger$, + boundary conditions and coupling constants $(J_{e^\dagger})$, $(\widetilde{J}_{e^\dagger})$ and $\left(\widetilde{\widetilde{J}}_{e^\dagger}\right)$, respectively. These are set so that the condition from Proposition 2.15 holds for the respective pairs:

$$\tanh\left(\beta J_e\right) = \exp\left(-2\beta^\dagger J_{e^\dagger}\right) \quad , \quad \tanh\left(\beta\widetilde{J}_e\right) = \exp\left(-2\beta^\dagger\widetilde{J}_{e^\dagger}\right) \quad , \quad \tanh\left(\beta\widetilde{\widetilde{J}}_e\right) = \exp\left(-2\beta^\dagger\widetilde{\widetilde{J}}_{e^\dagger}\right)$$

(note that these conditions do not completely define the coupling constants). Setting $J_e = 1$ makes the first model matches the one in the statement, and define $\mathbb{P}^\dagger, \mathcal{Z}^\dagger, \mathbb{E}^\dagger$, $\widetilde{\mathbb{P}}^\dagger, \widetilde{\mathcal{Z}}^\dagger, \widetilde{\mathbb{E}}^\dagger$ and $\widetilde{\widetilde{\mathbb{P}}}^\dagger, \widetilde{\widetilde{\mathcal{Z}}}^\dagger, \widetilde{\widetilde{\mathbb{E}}}^\dagger$ similarly. Applying Proposition 2.15 to both sides of the fraction in (18) yields

$$\frac{\widetilde{\widetilde{\mathcal{Z}}}}{\mathcal{Z}} = \frac{\left[\prod_{e\in\operatorname{Int}\mathcal{E}_{\Omega_\delta}}\cosh\left(\beta\widetilde{\widetilde{J}}_e\right)\right]\left[\prod_{e^\dagger\in\operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta}}\exp\left(-\beta^\dagger\widetilde{\widetilde{J}}_{e^\dagger}\right)\right]}{\left[\prod_{e\in\operatorname{Int}\mathcal{E}_{\Omega_\delta}}\cosh\left(\beta J_e\right)\right]\left[\prod_{e^\dagger\in\operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta}}\exp\left(-\beta^\dagger J_{e^\dagger}\right)\right]}\cdot\frac{\widetilde{\widetilde{\mathcal{Z}}}^\dagger}{\mathcal{Z}^\dagger}. \tag{19}$$

Let us study the conditions imposed on the coupling constants $\widetilde{J}_{e^\dagger}$ and $\widetilde{\widetilde{J}}_{e^\dagger}$ and define them unambiguously:

1. Regarding $\widetilde{J}_{e^\dagger}$, if $e^\dagger \notin \Gamma$ then setting $\widetilde{J}_{e^\dagger} = J_{e^\dagger}$ is possible. For the case $e^\dagger \in \Gamma$,

$$\exp\Big(-2\beta^\dagger \widetilde{J}_{e^\dagger}\Big) = \tanh\Big(\beta \widetilde{J}_e\Big) = \tanh\Big(-\beta J_e\Big) = -\tanh\Big(\beta J_e\Big) = -\exp\Big(-2\beta^\dagger J_{e^\dagger}\Big)$$

therefore the condition on $\widetilde{J}_{e^\dagger}$ is equivalent to stating that replacing $J_{e^\dagger} \to \widetilde{J}_e$ makes the exponential $\exp\Big(-2\beta J_{e^\dagger}\Big)$ flip its sign. Hence, we take $\widetilde{J}_{e^\dagger} = J_{e^\dagger} + \frac{i\pi}{2\beta^\dagger}$. Note how this model is the standard one with order lines Γ.

2. For the third model, if $e^\dagger \notin \Theta^\dagger$ we set $\widetilde{\widetilde{J}}_{e^\dagger} = \widetilde{J}_{e^\dagger}$, whereas if $e^\dagger \in \Theta^\dagger$ we have

$$\exp\Big(-2\beta^\dagger \widetilde{\widetilde{J}}_{e^\dagger}\Big) = \tanh\Big(\beta \widetilde{\widetilde{J}}_e\Big) = \tanh\Big(\beta \widetilde{J}_e + \frac{i\pi}{2}\Big) = \tanh\Big(\beta \widetilde{J}_e\Big)^{-1} = \exp\Big(-2\beta^\dagger \widetilde{J}_{e^\dagger}\Big)^{-1}$$

so changing $\widetilde{J}_e \to \widetilde{\widetilde{J}}_e$ must invert the exponential $\exp\Big(-2\beta^\dagger \widetilde{J}_{e^\dagger}\Big)$. Taking $\widetilde{\widetilde{J}}_{e^\dagger} = -\widetilde{J}_{e^\dagger}$ does this. Informally speaking, this is the model with $(\widetilde{J}_{e^\dagger})$ with disorder lines Θ applied on top of it.

We expand 19 by retracing the steps done previously but in opposite order. Recall that the results can be applied in the presence of boundary conditions all the same (Remark 2.35). Notice how the fact the disorder variables depend on the coupling constants is relevant here:

$$\begin{aligned}
\frac{\widetilde{\widetilde{\mathcal{Z}}}^\dagger}{\mathcal{Z}^\dagger} &= \frac{\widetilde{\mathcal{Z}}^\dagger}{\mathcal{Z}} \cdot \frac{\widetilde{\widetilde{\mathcal{Z}}}^\dagger}{\widetilde{\mathcal{Z}}^\dagger} \\
&= i^{|\Gamma|}\mathbb{E}\left[\prod_{j=1}^{2n} \sigma_{a_j}\right] \cdot \widetilde{\mathbb{E}}\left[\left(\prod_{k=1}^{2m} \mu_{v_k}\right)_{\Theta,\left(\widetilde{J}_{e^\dagger}\right)}\right] \\
&= i^{|\Gamma|}\mathbb{E}\left[\prod_{j=1}^{2n} \sigma_{a_j}\right] \cdot \sum_{\sigma \in \{\pm 1\}^{\mathrm{Int}\,\mathcal{F}\Omega_\delta} \times \{+1\}^{\partial \mathcal{F}\Omega_\delta}} \left(\prod_{k=1}^{2m} \mu_{v_k}\right)_{\Theta,\left(\widetilde{J}_{e^\dagger}\right)} \cdot \widetilde{\mathbb{P}}(\sigma) \\
&= i^{|\Gamma|} \cdot \sum_{\sigma \in \{\pm 1\}^{\mathrm{Int}\,\mathcal{F}\Omega_\delta} \times \{+1\}^{\partial \mathcal{F}\Omega_\delta}} \left(\prod_{j=1}^{2n} \sigma_{a_j}\right)\left(\prod_{k=1}^{2m} \mu_{v_k}\right)_{\Theta,\left(\widetilde{J}_{e^\dagger}\right)} \cdot \mathbb{P}(\sigma)
\end{aligned}$$

and this is the expected value we want with the nuance of the coupling constants in the disorder variables. Changing those leads to

$$\begin{aligned}
\left(\prod_{k=1}^{2m} \mu_{v_k}\right)_{\Theta,\left(\widetilde{J}_{e^\dagger}\right)} &= \prod_{\substack{e^\dagger \in \Theta^\dagger \\ e^\dagger = (vu)}} \exp\big(-2\beta \widetilde{J}_{e^\dagger} \sigma_v \sigma_u\big) \\
&= \prod_{\substack{e^\dagger \in \Theta^\dagger \\ e^\dagger = (vu)}} \exp\big(-2\beta J_{e^\dagger} \sigma_v \sigma_u\big) \prod_{\substack{e^\dagger \in \Theta^\dagger \cap \Gamma \\ e = (vu)}} \exp\big(-i\pi \sigma_v \sigma_u\big) \\
&= \left(\prod_{k=1}^{2m} \mu_{v_k}\right)_{\Theta,(J_{e^\dagger})} \cdot (-1)^{|\Theta^\dagger \cap \Gamma|}.
\end{aligned}$$

Collecting all multiplicative constants, we arrive at

$$\mathbb{E}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Gamma}\right] =$$

$$= (-i)^{|\Theta|} i^{|\Gamma|} (-1)^{|\Theta^\dagger\cap\Gamma|} \frac{\left[\prod_{e\in \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}} \cosh\left(\beta\widetilde{\widetilde{J}}_e\right)\right]\left[\prod_{e^\dagger\in \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}} \exp\left(-\beta^\dagger\widetilde{\widetilde{J}}_{e^\dagger}\right)\right]}{\left[\prod_{e\in \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}} \cosh\left(\beta J_e\right)\right]\left[\prod_{e^\dagger\in \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}} \exp\left(-\beta^\dagger J_{e^\dagger}\right)\right]} \cdot \mathbb{E}^\dagger\left[\prod_{j=1}^{2n}\sigma_{a_j}\left(\prod_{k=1}^{2m}\mu_{v_k}\right)_{\Theta}\right]$$

and we are left with simplifying the right-hand side. Recalling the definitions of $\widetilde{\widetilde{J}}_e$ and $\widetilde{J}_e$,

$$\cosh\left(\beta\widetilde{\widetilde{J}}_e\right) = \begin{cases}\cosh\left(\beta\widetilde{J}_e\right), & e\notin\Theta\\ i\sinh\left(\beta\widetilde{J}_e\right), & e\in\Theta\end{cases} = \begin{cases}\cosh\beta, & e\notin\Theta, e\notin\Gamma^\dagger\\ \cosh\beta, & e\notin\Theta, e\in\Gamma^\dagger\\ i\sinh\beta, & e\in\Theta, e\notin\Gamma^\dagger\\ -i\sinh\beta, & e\in\Theta, e\in\Gamma^\dagger\end{cases}$$

(at the end we use $J_e = 1$) and as for the coupling constants of the dual model

$$\exp\left(-\beta^\dagger\widetilde{\widetilde{J}}_{e^\dagger}\right) = \begin{cases}\exp\left(-\beta^\dagger\widetilde{J}_{e^\dagger}\right), & e^\dagger\notin\Theta^\dagger\\ \exp\left(-\beta^\dagger\widetilde{J}_{e^\dagger}\right)^{-1}, & e^\dagger\in\Theta^\dagger\end{cases} = \begin{cases}\exp(-\beta^\dagger), & e^\dagger\notin\Theta^\dagger, e^\dagger\notin\Gamma\\ -i\exp(-\beta^\dagger), & e^\dagger\notin\Theta^\dagger, e^\dagger\in\Gamma\\ \exp(-\beta^\dagger)^{-1}, & e^\dagger\in\Theta^\dagger, e^\dagger\notin\Gamma\\ i\exp(-\beta^\dagger)^{-1}, & e^\dagger\in\Theta^\dagger, e^\dagger\in\Gamma\end{cases}$$

which yields

$$(-i)^{|\Theta|} i^{|\Gamma|} (-1)^{|\Theta^\dagger\cap\Gamma|} \frac{\left[\prod_{e\in \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}} \cosh\left(\beta\widetilde{\widetilde{J}}_e\right)\right]\left[\prod_{e^\dagger\in \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}} \exp\left(-\beta^\dagger\widetilde{\widetilde{J}}_{e^\dagger}\right)\right]}{\left[\prod_{e\in \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}} \cosh\left(\beta J_e\right)\right]\left[\prod_{e^\dagger\in \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}} \exp\left(-\beta^\dagger J_{e^\dagger}\right)\right]} =$$

$$= (-i)^{|\Theta|} i^{|\Gamma|} (-1)^{|\Theta^\dagger\cap\Gamma|} \cdot \left[(-1)^{|\Theta^\dagger\cap\Gamma|} i^{|\Theta|} \prod_{e\in\Theta}\tanh\beta\right] \cdot \left[(-1)^{|\Gamma|-|\Theta^\dagger\cap\Gamma|} i^{|\Gamma|} \prod_{e^\dagger\in\Theta^\dagger}\exp(-2\beta^\dagger)^{-1}\right]$$

$$= (-1)^{|\Theta^\dagger\cap\Gamma|} \cdot \prod_{e\in\Theta}\frac{\tanh\beta}{\exp(-2\beta^\dagger)}$$

$$= (-1)^{|\Theta^\dagger\cap\Gamma|}$$

where we use $\tanh\beta = \exp(-2\beta^\dagger)$ in the last step. □

This is a rather strong result, and a couple of consequences follow easily from it. First, one can avoid studying order variables in one of the boundary settings (free or +) by introducing disorder variables. Second, we can describe how changing disorder lines affects these expected values without the hassle of generalizing the argument of Proposition 2.20. Modifying Γ while keeping the endpoints constant does not change the expected value of the right-hand side in Theorem 2.36, so only the sign factor can change.

Corollary 2.37. *For the Ising model on $\mathcal{G}_\delta$ with free boundary conditions, the expected value*

$$\mathbb{E}_{\mathcal{G}_\delta}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Gamma_1}\right]$$

is independent of the choice of Γ up to a sign.

In addition, let $\Gamma_1 \equiv \Gamma_1[a_1, \ldots, a_{2n}]$ and $\Gamma_2 \equiv \Gamma_2[a_1, \ldots, a_{2n}]$ be two sets of disorder lines. Pick a set $\Theta \equiv \Theta[v_1, \ldots, v_{2m}]$ of order lines. Then

$$\mathbb{E}_{\mathcal{G}_\delta}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Gamma_1}\right] = (-1)^{|\Theta^\dagger\cap(\Gamma_1\oplus\Gamma_2)|}\cdot\mathbb{E}_{\mathcal{G}_\delta}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Gamma_2}\right].$$

Proof. Considering the dual model on $\mathcal{G}_\delta^\dagger$, we have

$$\begin{aligned}\mathbb{E}_{\mathcal{G}_\delta}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Gamma_1}\right] &= (-1)^{|\Theta\cap\Gamma_1^\dagger|}\cdot\mathbb{E}_{\mathcal{G}_\delta^\dagger}\left[\prod_{j=1}^{2n}\sigma_{a_j}\left(\prod_{k=1}^{2m}\mu_{v_k}\right)_{\Theta}\right]\\ &= (-1)^{|\Theta\cap\Gamma_1^\dagger|+|\Theta\cap\Gamma_2^\dagger|}\cdot\mathbb{E}_{\mathcal{G}_\delta}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_{\Gamma_2}\right].\end{aligned}$$

and as for the exponent of the sign

$$\begin{aligned}|\Theta\cap\Gamma_1^\dagger| + |\Theta\cap\Gamma_2^\dagger| &= |\Theta^\dagger\cap\Gamma_1| + |\Theta^\dagger\cap\Gamma_2|\\ &= |\Theta^\dagger\cap(\Gamma_1\setminus\Gamma_2)| + |\Theta^\dagger\cap(\Gamma_2\setminus\Gamma_1)| + 2|\Theta^\dagger\cap(\Gamma_1\cap\Gamma_2)|\\ &= |\Theta^\dagger\cap(\Gamma_1\setminus\Gamma_2)| + |\Theta^\dagger\cap(\Gamma_2\setminus\Gamma_1)| \mod 2\\ &= |\Theta^\dagger\cap(\Gamma_1\oplus\Gamma_2)|\end{aligned}$$

□

The number $|\Theta^\dagger\cap(\Gamma_1\oplus\Gamma_2)|$ can be understood in another way: it equals the number of edges of Θ that cross (dual) edges of $\Gamma_1\oplus\Gamma_2$. A consequence of Corollary 2.37 is that the expected value is independent of the choice of Γ as long as any changes to it occur "far away" from the spin sites.

PART II

LATTICE MODEL DESCRIPTION

3 Notation and definitions

We define the notation needed for Parts II and III, some of which was already introduced in Part I. Most of it refers to discretization, which in done in a square lattice of mesh size $\sqrt{2}\delta$ and rotated by 45°

3.1 Sites and graph notation

For $\delta > 0$, we consider the square grid $\mathbb{C}_\delta = \delta(1+i)(\mathbb{Z}+i\mathbb{Z})$. On it,

1. A *vertex* is an element $v \in \mathbb{C}_\delta$.
2. An *edge* is a set of the form $e \equiv e_{m,n,d} := \{\delta(1+i)(m+in) + \delta(1+i)d \cdot a : a \in [0,1]\}$ (where $m, n \in \mathbb{Z}$ and $d \in \{1, i\}$), which is identified with its midpoint.
3. A *face* is a set of the form $f \equiv f_{m,n} := \{\delta(1+i)(a+ib) : (a,b) \in [m, m+1] \times [n, n+1]\}$ (where $m, n \in \mathbb{Z}$), which is identified with its center.

Note that the distance between two adjacent faces is $\sqrt{2}\delta$, whereas the distance between a vertex and an incident face is δ.

A *discrete domain* is an union of faces $\Omega_\delta \subset \mathbb{C}$. Given such a domain,

1. Denote by $\operatorname{Int}\mathcal{V}_{\Omega_\delta}$ the set of vertices adjacent to faces of Ω_δ, by $\operatorname{Int}\mathcal{E}_{\Omega_\delta}$ the set of edges incident to faces of Ω_δ and by $\operatorname{Int}\mathcal{F}_{\Omega_\delta}$ the set of faces contained in Ω_δ.
2. Denote by $\partial\mathcal{V}_{\Omega_\delta}$, $\partial\mathcal{E}_{\Omega_\delta}$ and $\partial\mathcal{F}_{\Omega_\delta}$ the set of *boundary vertices*, *edges* and *faces*: the vertices, edges and faces that are adjacent to their counterparts $\mathcal{V}_{\Omega_\delta}$, $\mathcal{E}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta}$, respectively, but do not belong to those sets.
3. Denote by $\mathcal{V}_{\Omega_\delta} := \operatorname{Int}\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{V}_{\Omega_\delta}$, $\mathcal{E}_{\Omega_\delta} := \operatorname{Int}\mathcal{E}_{\Omega_\delta} \cup \partial\mathcal{E}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta} := \operatorname{Int}\mathcal{F}_{\Omega_\delta} \cup \partial\mathcal{F}_{\Omega_\delta}$.

From now on, we assume that Ω_δ is simply connected and all edges joining vertices from $\operatorname{Int}\mathcal{V}_{\Omega_\delta}$ are in $\operatorname{Int}\mathcal{E}_{\Omega_\delta}$. These conditions are not restrictive and will simplify future arguments.

The † symbol represents dual elements. The *dual edge* $e^\dagger$ of an edge e connects the center of the two faces that have e as a boundary component. In addition, for any $E \subseteq \mathcal{E}_{\Omega_\delta}$ we write $E^\dagger := \{e^\dagger : e \in E\}$. We will make an abuse of notation by writing $(\operatorname{Int}\mathcal{E}_{\Omega_\delta})^\dagger$ as $\operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta}$.

Using these definitions, the graph $\mathcal{G}_{\Omega_\delta} := (\mathcal{V}_{\Omega_\delta}, \mathcal{E}_{\Omega_\delta})$ will be the main focus, whereas the graph $\mathcal{G}^\dagger_{\Omega_\delta} := (\mathcal{F}_{\Omega_\delta}, \operatorname{Int}\mathcal{E}^\dagger_{\Omega_\delta})$ is where the Ising model is defined. The terms 'vertex', 'edge' and 'face' will be used when referring to elements of $\mathcal{V}_{\Omega_\delta}$, $\mathcal{E}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta}$ respectively, even when focusing on $\mathcal{G}^\dagger_{\Omega_\delta}$ or other graphs where, for instance, their vertices may not be in $\mathcal{V}_{\Omega_\delta}$.

Some extra sites will be needed in the future. For each vertex v, consider four neighbour *corners*: the points $v \pm \frac{\delta}{2}$ and $v \pm i\frac{\delta}{2}$. Given a corner c, we denote by $v(c)$ the vertex adjacent to it and $f(c)$ the face where it is included. In addition, we collect all corners according to their relative position by setting

$$\mathcal{C}^1_{\Omega_\delta} := \left\{v + \frac{\delta}{2} : v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}\right\} \qquad \mathcal{C}^i_{\Omega_\delta} := \left\{v - \frac{\delta}{2} : v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}\right\}$$
$$\mathcal{C}^{\bar\lambda}_{\Omega_\delta} := \left\{v + i\frac{\delta}{2} : v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}\right\} \qquad \mathcal{C}^{\lambda}_{\Omega_\delta} := \left\{v - i\frac{\delta}{2} : v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}\right\}$$

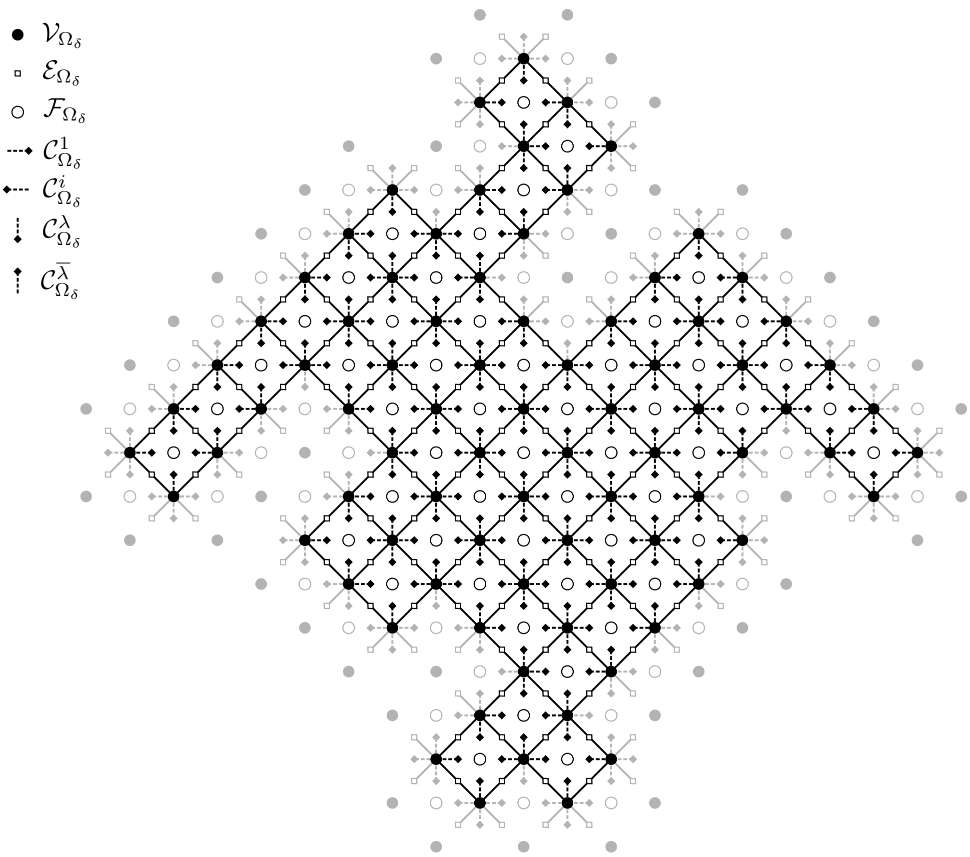

Figure 6: Example of sites for a discretization Ω_δ, with boundary elements coloured grey.

and associating to each corner a complex number amongst $1, i, \lambda := e^{i\frac{\pi}{4}}$ and $\overline{\lambda}$ depending on its set, for reasons that will become apparent later. We also denote $\mathcal{C}_{\Omega_\delta} := \mathcal{C}^{\overline{\lambda}}_{\Omega_\delta} \cup \mathcal{C}^1_{\Omega_\delta} \cup \mathcal{C}^\lambda_{\Omega_\delta} \cup \mathcal{C}^i_{\Omega_\delta}$ and partition $\mathcal{C}_{\Omega_\delta} = \mathrm{Int}\mathcal{C}_{\Omega_\delta} \cup \partial\mathcal{C}_{\Omega_\delta}$ as follows: if $f(c) \in \mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$ then $c \in \mathrm{Int}\mathcal{C}_{\Omega_\delta}$, otherwise $c \in \partial\mathcal{C}_{\Omega_\delta}$.

In figures, vertices are represented by $\bullet$, edges are represented by $\square$ in their endpoints, faces are represented by $\circ$ and corners are represented by $\blacklozenge$, as exemplified in Figure 6.

The notation $x \sim y$ will be used when x and y are vertices, edges, faces or corners and they are adjacent or incident to each other. When $x \in \mathcal{V}_{\Omega_\delta}$ and $y \in \mathcal{C}_{\Omega_\delta} \cup \mathcal{E}_{\Omega_\delta} \cup \mathcal{V}_{\Omega_\delta}$, we use (xy) and (yx) to denote the edge (if $y \in \mathcal{V}_{\Omega_\delta}$) or *half-edge* (otherwise) between x and y. Recall that elements of $\mathcal{E}_{\Omega_\delta}$ are identified with their midpoint, therefore for $x \in \mathcal{E}_{\Omega_\delta}$ incident to $y \in \mathcal{V}_{\Omega_\delta}$ the notation (xy) represents the half-edge connecting the midpoint of the edge x to the vertex y; similarly, faces are identified with their center.

3.2 Ising model setup with disorder lines

We consider the Ising model on the graph $\mathcal{G}^\dagger_{\Omega_\delta}$ at the critical temperature $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$ with a collection of disorder lines $\Gamma \subseteq \mathrm{Int}\mathcal{E}_{\Omega_\delta}$ and boundary conditions $+$ imposed on the spins of $\partial\mathcal{F}_{\Omega_\delta}$. The configurations will be studied in a domain walls configuration, defined in Part I. Define $\alpha_c := \exp(-2\beta_c) = \sqrt{2}-1$, a constant that will appear very commonly because of the low-temperature expansion. The partition function of the model is represented by $\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}$ and the expected value of random variables for this model is denoted by $\mathbb{E}^{\Gamma,+}_{\Omega_\delta}$.

Following Part I, recall that we write $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}]$ to highlight the vertices $v_1, \ldots, v_{2m} \in \operatorname{Int} \mathcal{V}_{\Omega_\delta}$ with odd degree on Γ. We will often decompose Γ into edge-disjoint paths, and note that their endpoints must be $v_1, \ldots, v_{2m}$. We will also allow some of these vertices to repeat, for more details see Remark 2.22.

We will often consider pairwise disjoint corners $c_1, \ldots, c_{2m} \in \operatorname{Int} \mathcal{C}_{\Omega_\delta}$ such that c_k is adjacent to v_k and faces $u_1, \ldots, u_{2m} \in \operatorname{Int} \mathcal{F}_{\Omega_\delta}$ defined by $u_k = f(c_k)$ (note that some u_k may be equal). The spin variables will be associated to faces $a_1, \ldots, a_n \in \operatorname{Int} \mathcal{F}_{\Omega_\delta}$, which may repeat or be equal to some u_k.

3.3 Double covers

This work focuses on *spinors*, holomorphic functions defined on a double cover of some domain and with opposite signs on opposite sheets (that is, with a multiplicative monodromy of -1 around the branching points), both in a discrete and continuous setting. We define notation regarding the domains of spinors.

Given $\Omega \subseteq \mathbb{C}$ and $b_1, \ldots, b_n \in \Omega$, we denote by $[\Omega; b_1, \ldots, b_n]$ the canonical double cover of $\Omega \setminus \{b_1, \ldots, b_n\}$ branching around each $b_1, \ldots, b_n$. Some notes:

1. We will often consider two types of branching points. For this reason, the notation

$$[\Omega; b_1, \ldots, b_{n'}; b_{n'+1}, \ldots, b_n]$$

   will be used to separate them, denoting the same double cover as $[\Omega; b_1, \ldots, b_{n'}, b_{n'+1}, \ldots, b_n]$.

2. Double covers branching around the same points are considered equal, even though the branching points may appear in the notation $[\Omega; b_1, \ldots, b_{n'}; b_{n'+1}, \ldots, b_n]$ in a different order.

3. There is a 2-to-1 correspondence between $[\Omega; b_1, \ldots, b_n]$ and $\Omega \setminus \{b_1, \ldots, b_n\}$. The 2 representatives of $z \in \Omega$ in $[\Omega; b_1, \ldots, b_n]$ and Ω are called the *lifts* of z, whereas the representative of $\widetilde{z} \in [\Omega; b_1, \ldots, b_n]$ in Ω is the *projection* or *base point* of $\widetilde{z}$.

4. We often write $\widetilde{z}$ when referring to an element in $[\Omega; b_1, \ldots, b_n]$ whose projection is z.

5. For $z \in \Omega$ and $w \in \mathbb{C}$, when a lift $\widetilde{z}$ of z is already fixed, we will write $\widetilde{z} + w$ for the lift of $z + w$ that is "in the same sheet" as $\widetilde{z}$. This will only be done when such a description is clear, usually with small values of w.

6. In the same vein, for $z, w \in \Omega$ close to each other and lifted to the same sheet, we write $\frac{1}{2}(\widetilde{z} + \widetilde{w})$ for the lift of $\frac{1}{2}(z + w)$ on the same sheet as a and w.

7. We will compare spinors defined on $[\Omega; b_1, \ldots, b_n]$ with ones defined in $[\mathbb{C}; b_k]$ around small enough neighbourhoods of b_k. In these situations, we assume a proper correspondence between the neighbourhoods of these domains.

For discrete domains Ω_δ, the same notation applies. When referring to the lifts of sets of Ω_δ in $[\Omega_\delta; b_1, \ldots, b_n]$, we denote them by $[\cdot; b_1, \ldots, b_n]$. For instance, $[\mathcal{V}_{\Omega_\delta}; b_1, \ldots, b_n]$ are the points in $[\Omega_\delta; b_1, \ldots, b_n]$ whose projection belongs to $\mathcal{V}_{\Omega_\delta}$.

3.4 Paths and Contours

Due to domain wall arguments, we will deal with contours and paths in $\mathcal{G}_{\Omega_\delta}$. A number of intuitive facts will be assumed; the proof of those can be found at the end of this Section. In addition, we state and prove Lemmas 3.2 and 3.3, which will be useful in Section 4.

The term "path" will be used both in its geometrical meaning of "the image of a smooth function from $[0,1]$ to Ω_δ" as well as in the graph theory meaning of "a sequence of edges consecutively joined by a shared endpoint". Arguments involving the former meaning will be presented with less formal details, so as to spare the reader from even more technical details. In addition, graph paths will often be "smoothed" so that they can be seen as geometrical paths (see Figures 7 and 8 for an example).

Geometric paths. Paths in this context consist of smooth oriented curves and may be either open or closed. We will always assume they are in *general position*: all self-intersections and intersections with other curves are transverse double points. For instance, we exclude cases where curves are tangent but do not cross.

Given such a curve γ, the *winding* of γ is the increment of the argument of the velocity vector of γ and is denoted by $\mathrm{wind}(\gamma)$. If γ is closed, then $\mathrm{wind}(\gamma)$ corresponds to the total rotation angle of its velocity vector when going around γ once.

For two curves γ^1 and γ^2, we denote by $\gamma^1 \cdot \gamma^2$ the *intersection number* of γ^1 and γ^2: the number of times γ^1 and γ^2 intersect each other. When $\gamma^1 = \gamma^2$, $\gamma^1 \cdot \gamma^2$ is the number of self-intersections of the curve. For two collections of paths $\Gamma^1 = \bigcup_i \gamma_i^1$ and $\Gamma^2 = \bigcup_j \gamma_j^2$, such number is defined as $\Gamma^1 \cdot \Gamma^2 := \sum_{i,j} \gamma_i^1 \cdot \gamma_j^2$.

Combinatorial paths. For $x, y \in \mathcal{C}_{\Omega_\delta} \cup \mathcal{E}_{\Omega_\delta} \cup \mathcal{V}_{\Omega_\delta}$, a path in $\mathcal{G}_{\Omega_\delta}$ from x to y is a set of edges or semi-edges $\{(xv_1), (v_1v_2), \dots, (v_my)\}$ corresponding to a sequence $x \sim v_1 \sim \cdots \sim v_m \sim y$ with all $v_k \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$. Such a path is a loop if $x = y \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$. For $x, y \in \mathcal{C}_{\Omega_\delta} \cup \mathcal{E}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta}$, a *path in* $\mathcal{G}^\dagger_{\Omega_\delta}$ is represented by a sequence $x \sim f_1 \sim \cdots \sim f_m \sim y$ where all $f_k \in \mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$.

Set

$$\mathscr{C}_{\Omega_\delta} := \{\omega \subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta} : \text{every } v \in \mathcal{V}_{\Omega_\delta} \text{ has even degree on } \omega\}$$

(recall that the degree of a vertex v in $E \subseteq \mathcal{E}_{\Omega_\delta}$ is the number of edges of E incident to v). Note that these are the collections of loops (Proposition 3.4) and paths starting or ending on elements of $\mathcal{C}_{\Omega_\delta}$ or $\mathcal{E}_{\Omega_\delta}$ are forbidden.

Given $v_1, \dots, v_{2m} \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ in even number (not necessarily distinct) and $\Gamma \equiv \Gamma[v_1, \dots, v_{2m}] \subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$, we set the following collection of *contours*:

$$\mathscr{C}_{\Omega_\delta}(v_1, \dots, v_{2m}) := \{P = \omega \oplus \Gamma : \omega \in \mathscr{C}_{\Omega_\delta}\}.$$

This is the collection of $P \subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$ such that every v_k appearing an odd number of times in the list $v_1, \dots, v_{2m}$ has odd degree in P and all other vertices have even degree (Proposition 3.5), in particular implying the definition is independent of the choice of Γ. The low-temperature expansion for the partition

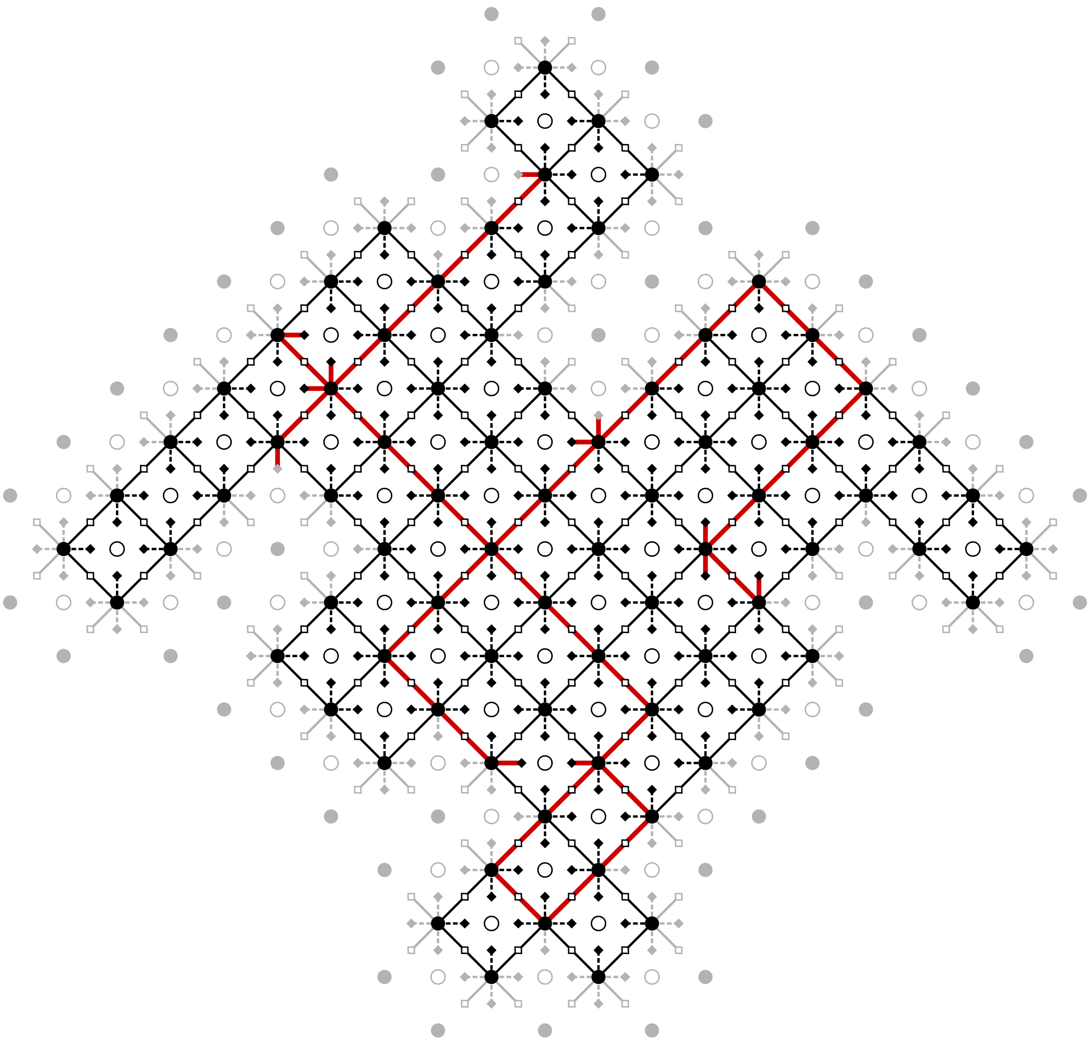

Figure 7: Example of a collection of contours with endpoints on corners.

function yields

$$\mathcal{Z}^{\Gamma,+}_{\Omega_\delta} = \exp(|\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}|\beta_c) \sum_{P \in \mathscr{C}_{\Omega_\delta}(v_1,\ldots,v_{2m})} \alpha_c^{|P|}$$

since $\alpha_c = \exp(-2\beta_c)$, as previously defined.

For $c_1, \ldots, c_{2m} \in \mathcal{C}_{\Omega_\delta}$ in even number and pairwise distinct, set

$$\mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m}) := \{Q = P \cup (v_1c_1) \cup \cdots \cup (v_{2m}c_{2m}) : P \in \mathscr{C}_{\Omega_\delta}(v_1, \ldots, v_{2m})\}$$

where $v_i = v(c_i)$ are the adjacent vertices. For every $Q \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$, there exists a *smoothing* of Q: that is, a decomposition

$$Q = \omega \ \cup \ \gamma_1 \ \cup \ \cdots \ \cup \ \gamma_m$$

in a collection of loops ω and paths γ_k associated to a permutation $s \in S_{2m}$ such that

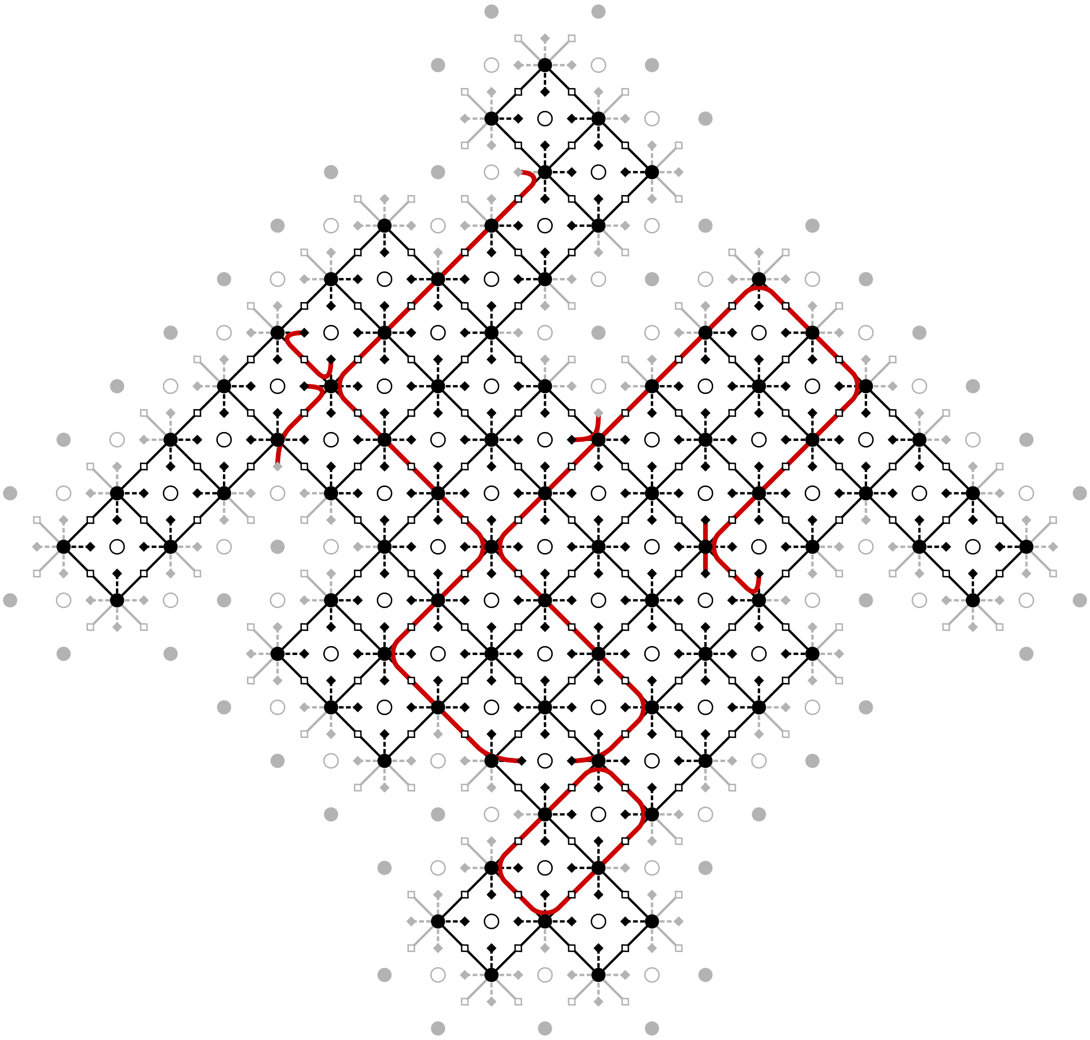

Figure 8: Example of a smoothing for the configuration of Figure 7. In many arguments where smoothings are considered, we assume the paths are as shown (smoothed out and disjoint).

(i) Each path γ_k goes from $c_{s(2k-1)}$ to $c_{s(2k)}$.

(ii) If $u \sim v \sim w$ is part of the loop or path[4], then the edges or half-edges (vu) and (vw) must be consecutive when rotating in v in either clockwise or counter-clockwise order.

The proof of existence is left for Proposition 3.6, and note that such decomposition is in general not unique. Intuitively, a smoothing should be seen as a decomposition of Q in loops and paths that do not intersect each other, a procedure referred to as *solving the crossings*. As the name implies, when using smoothings one considers such paths and loops to be smoothed while keeping them disjoint (Figure 8). The path starting or ending in c_{2m} will be referred to as $\mathrm{p}(Q)$[5].

[4]This also applies to endpoints of loops: if $v \sim u \sim \ldots \sim w \sim v$ is a loop, then u, v, w must satisfy the property.

[5]This notation will be used in situations where the choice of smoothing is not relevant, something that must be proven beforehand.

We extend the $\mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$ definition to the case where the last entry is an edge $e \in \mathcal{E}_{\Omega_\delta}$:

$$\mathscr{C}_\delta(c_1, \ldots, c_{2m-1}, e) := \{Q = P \cup (v_1c_1) \cup \cdots \cup (v_{2m-1}c_{2m-1}) \oplus (ve) : P \in \mathscr{C}(v_1, \ldots, v_{2m-1}, v)\}$$

where $v \in \operatorname{Int} \mathcal{V}_{\Omega_\delta}$ is any endpoint of e — if both endpoints of e are in $\operatorname{Int} \mathcal{V}_{\Omega_\delta}$, it does not matter which one is chosen (Proposition 3.7). The elements of this set also admit smoothings (Proposition 3.8).

Finally, we define the notion of sign of elements of $\mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$, first introduced in [Hon10] and further explored in [CCK17] (from which we borrow notation, but slightly tweak the definition). For each corner $c \in \mathcal{C}_{\Omega_\delta}$ we forever fix a square root of the direction of the segment from $v(c)$ to c and set η_c to be its complex conjugate. Given a smoothing of $Q \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$, the *sign* of Q is defined as

$$\tau(Q) := \operatorname{sign}(s) \cdot \prod_{k=1}^{m} i\eta_{c_{s(2k-1)}} \overline{\eta}_{c_{s(2k)}} \exp\left(-\frac{i}{2} \operatorname{wind}(\gamma_k)\right) \tag{20}$$

where $\operatorname{sign}(s) \in \{\pm 1\}$ is the sign of the permutation s, which can be defined as

$$\operatorname{sign}(s) = (-1)^m$$

where m is the number of transpositions in a decomposition of s, or as

$$\operatorname{sign}(s) = (-1)^{N(s)}, \quad N(s) = \{(i, j) : i < j, s(i) > s(j)\}.$$

Note that $\tau(Q) \in \{\pm 1\}$ (Proposition 3.9). The proof of the unambiguity of $\tau(Q)$ is left for Proposition 3.11. In addition, define the *modified sign* of $Q \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$ as

$$\widetilde{\tau}(Q) := \frac{\tau(Q)}{\overline{\eta}_{c_{2m}}} \tag{21}$$

a notion which can be extended to $Q \in \mathscr{C}_\delta(c_1, \ldots, c_{2m-1}, e)$ for $e \in \mathcal{E}_{\Omega_\delta}$ by setting $\eta_e := 1$.

Remark 3.1. For a corner $c \in \mathcal{C}^1_{\Omega_\delta}$ we have $\eta_c = \pm 1$, whereas if $c \in \mathcal{C}^\lambda_{\Omega_\delta}$ then $\eta_c = \pm e^{i\frac{\pi}{4}} = \pm \lambda$ and so on. This observation motivates the notation used for the sets $\mathcal{C}^{\overline{\lambda}}_{\Omega_\delta}, \mathcal{C}^1_{\Omega_\delta}, \mathcal{C}^\lambda_{\Omega_\delta}$ and $\mathcal{C}^i_{\Omega_\delta}$.

Related results and statement proofs. We now prove some results related to these definitions.

Lemma 3.2. *Let γ be a smooth oriented closed curve in the plane in general position. Then*

$$\exp\left(\frac{i}{2} \operatorname{wind}(\gamma)\right) = (-1)^{\gamma \cdot \gamma + 1}$$

Proof. Decompose γ into a collection of n curves, $\gamma = \bigcup_{k=1}^n \gamma_k$, by solving all the crossings (Figure 9). Note that $\exp\left(\frac{i}{2} \operatorname{wind}(\gamma)\right) = \prod_{k=1}^n \exp\left(\frac{i}{2} \operatorname{wind}(\gamma_k)\right)$ and $\operatorname{wind}(\gamma_k) = \pm 2\pi$, since they are simple closed curves. Hence, $\exp\left(\frac{i}{2} \operatorname{wind}(\gamma)\right) = (-1)^n$.

To determine $n \bmod 2$, note that the number of curves either increases or decreases by 1 each time a crossing is solved. Therefore, it always changes mod 2, and since it starts as 1 we have $n = \gamma \cdot \gamma + 1$

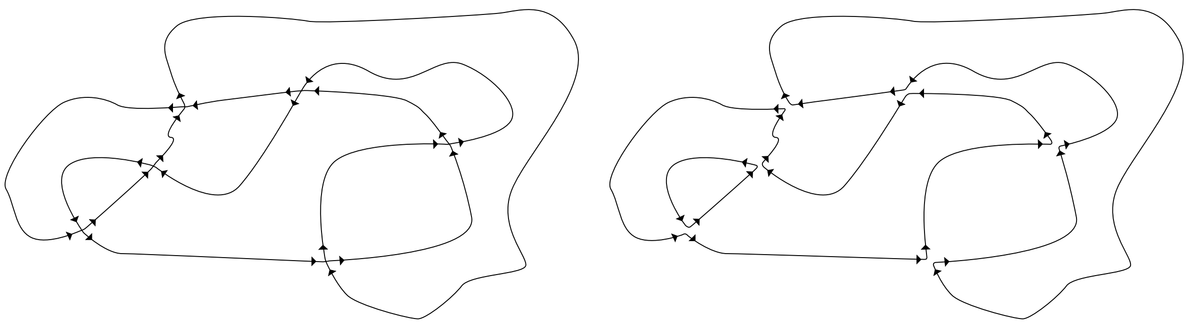

Figure 9: Example of solving the crossings for a curve.

mod 2. □

Lemma 3.3. *Let ω be a piecewise smooth closed curve in Ω_δ in general position and $a_1, \dots, a_n \in \operatorname{Int} \mathcal{F}_{\Omega_\delta}$ such that $a_1, \dots, a_n \notin \gamma$. Let Θ be a collection of edge-disjoint paths in $\mathcal{G}^\dagger_{\Omega_\delta}$ linking $a_1, \dots, a_n$ and possibly $a_{\text{out}} \in \partial\mathcal{F}_{\Omega_\delta}$. Then, ω lifts to a loop in $[\Omega_\delta; a_1, \dots, a_n]$ if and only if $(-1)^{\omega\cdot\Theta} = 1$.*

Proof. To determine if ω lifts to a loop in $[\Omega_\delta; a_1, \dots, a_n]$, we take each branching point a_k, compute the number $\text{loops}_{a_k}(\omega)$ of times ω goes around a_k and then take $\text{loops}_{a_1,\dots,a_n}(\omega) = \sum_k \text{loops}_{a_k}(\omega)$. ω lifts to a loop if and only if $\text{loops}_{a_1,\dots,a_n}(\omega)$ is even, or alternatively $(-1)^{\text{loops}_{a_1,\dots,a_n}(\omega)} = 1$.

Solve the crossings of ω without creating or undoing any intersections of ω with Θ and without changing any of the values $\text{loops}_{a_k}(\omega)$ (Figure 10). Take a partition $\omega = \dot{\bigcup}_j \omega_j$ of ω into disjoint non-intersecting cycles ω_j. We have $\text{loops}_{a_k}(\omega) = \sum_j \text{loops}_{a_k}(\omega_j)$ and thus

$$\text{loops}_{a_1,\dots,a_n}(\omega) = \sum_j \left(\sum_k \text{loops}_{a_k}(\omega_j) \right) = \sum_j \text{loops}_{a_1,\dots,a_n}(\omega_j)$$

and note that $\text{loops}_{a_1,\dots,a_n}(\omega_j)$ is the number of branching points inside of ω_j.

We claim $(-1)^{\text{loops}_{a_1,\dots,a_n}(\omega_j)} = (-1)^{\omega_j\cdot\Theta}$. To prove this, consider each path Θ_l of Θ separately. If $\omega_j \cdot \Theta_l = 0 \mod 2$, then both endpoints of Θ_l must be on the inside or outside of ω_j; if $\omega_j \cdot \Theta_l = 1 \mod 2$, then one is in the inside and the other on the outside. Considering all Θ_l, we arrive at either $(-1)^{\text{loops}_{a_1,\dots,a_n,a_{\text{out}}}(\omega_j)} = (-1)^{\omega_j\cdot\Theta}$ or $(-1)^{\text{loops}_{a_1,\dots,a_n}(\omega_j)} = (-1)^{\omega_j\cdot\Theta}$, depending on whether a_{out} exists or not. If it does not, then we are done; if it does, since $a_{\text{out}} \in \partial\mathcal{F}_{\Omega_\delta}$ this point will always be on the outside of ω_j, implying $\text{loops}_{a_1,\dots,a_n,a_{\text{out}}}(\omega_j) = \text{loops}_{a_1,\dots,a_n}(\omega_j)$.

Therefore,

$$(-1)^{\text{loops}_{a_1,\dots,a_n}(\omega)} = \prod_j (-1)^{\text{loops}_{a_1,\dots,a_n}(\omega_j)} = \prod_j (-1)^{\omega_j\cdot\Theta} = (-1)^{\omega\cdot\Theta}.$$

□

Proposition 3.4. *Let $\omega \subseteq \mathcal{E}_{\Omega_\delta}$. Then, ω can be decomposed into a collection of edge-disjoint loops if and only if all $v \in \mathcal{V}_{\Omega_\delta}$ have even degree in ω.*

Proof. Implication ($\Rightarrow$) is straightforward, let us check ($\Leftarrow$) also holds by strong induction on $|\omega|$.

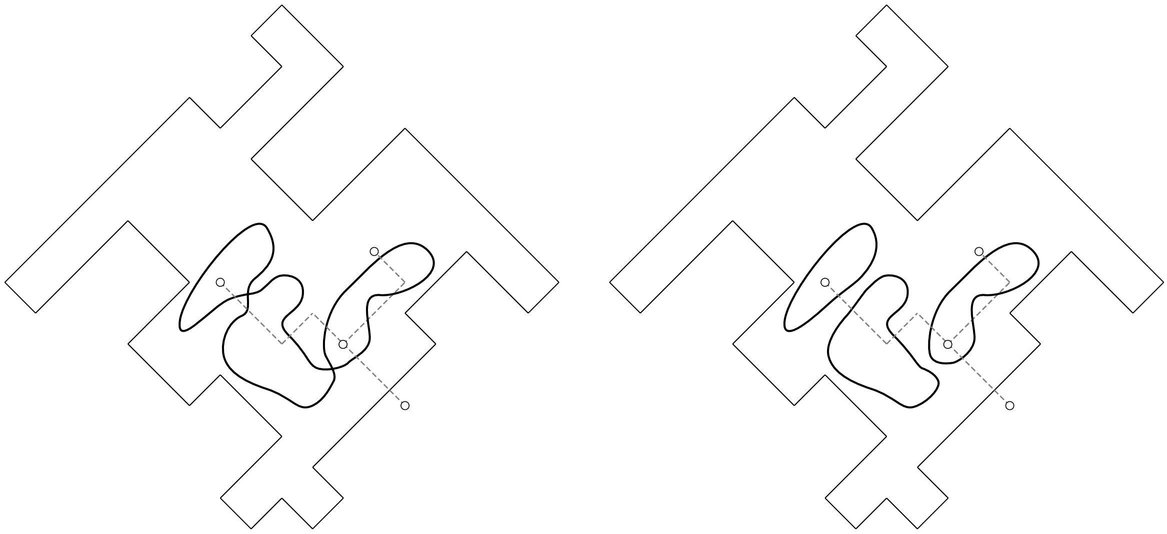

Figure 10: Example of a setting of Lemma 3.3 without and with crossings solved (left and right figures, respectively). Note the added a_{out} because there are an odd number of faces.

The case $|\omega| = 0$ is trivial since $\emptyset$ satisfies both properties. For the induction step, start at any vertex v_1 incident to at least one edge of ω, then build a path $\pi = v_1 \sim \cdots \sim v_n$ such that all $(v_i v_{i+1}) \in \omega$ and no edge is used twice, until it is no longer possible to continue.

After this process, all edges of ω incident in v_n must be used in π. If $v_n \neq v_1$, the number of such edges is $2|\{i : v_i = v_n\}| - 1$, which contradicts the fact that v_n has even degree on ω. Hence, $v_n = v_1$ and π is a loop. Since there is a valid decomposition for $\omega \setminus \pi$ by assumption, the proof is done. □

Proposition 3.5. *Let $v_1, \ldots, v_{2m} \in \operatorname{Int} \mathcal{V}_{\Omega_\delta}$ (not necessarily distinct), $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}] \subseteq \operatorname{Int} \mathcal{E}_{\Omega_\delta}$ and $P \subseteq \operatorname{Int} \mathcal{E}_{\Omega_\delta}$. Then, $P = \omega \oplus \Gamma$ for some $\omega \in \mathscr{C}_{\Omega_\delta}$ if and only if every v_k appearing an odd number of times in the list $v_1, \ldots, v_{2m}$ has odd degree in P and all other vertices have even degree.*

Proof. For any $A, B \subseteq \operatorname{Int} \mathcal{E}_{\Omega_\delta}$, the degree of a vertex v in $A \oplus B$ is

$$|\{e \in A : e \text{ is incident to } v\}| + |\{e \in B : e \text{ is incident to } v\}| - 2|\{e \in \omega_1 \cap \omega_2 : e \text{ is incident to } v\}|.$$

Thus, ($\Rightarrow$) follows from the fact every vertex has even degree in ω (Proposition 3.4) while ($\Leftarrow$) is a consequence of $P = (P \oplus \Gamma) \oplus \Gamma$ and $P \oplus \Gamma \in \mathscr{C}_{\Omega_\delta}$ because every vertex has even degree in $P \oplus \Gamma$ (Proposition 3.4). Note how the arguments hold even when □

Proposition 3.6. *For any $c_1, \ldots, c_{2m} \in \mathcal{C}_{\Omega_\delta}$ in even number and pairwise distinct and $Q \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$, there exists a decomposition*

$$Q = \omega \,\cup\, \gamma_1 \,\cup\, \cdots \,\cup\, \gamma_m$$

into a collection of loops ω and paths γ_k associated to a permutation $s \in S_{2m}$ such that

(i) Each path γ_k goes from $c_{s(2k-1)}$ to $c_{s(2k)}$.

(ii) If $u \sim v \sim w$ is part of the loop or path, then the edges (vu) and (vw) must be consecutive when rotating in v in either clockwise or counter-clockwise order.

Proof. We describe a procedure to generate such a decomposition. For each $v \in \mathcal{V}_{\Omega_\delta}$, let Q_v be the edges and half-edges of Q incident in v. Informally speaking, the idea is as follows: we pair the elements of Q_v, and then build paths and loops such that the exiting edge at each vertex v must be the pair in Q_v of the entering edge. If one pairs edges such that condition (ii) is verified, then the decomposition is valid.

We start by claiming that $|Q_v|$ is even: recall that Q is of the form

$$Q = (\omega \oplus \Gamma) \cup (v_1 c_1) \cup \cdots \cup (v_{2m} c_{2m})$$

where $v_k = v(c_k)$, $\omega \in \mathscr{C}_{\Omega_\delta}$ and $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}] \subseteq \operatorname{Int} \mathcal{E}_{\Omega_\delta}$. In addition, $|Q_v|$ is the number of edges and half-edges of Q incident to v, thus

$$\begin{aligned} |Q_v| = \Big(&\overbrace{|\{e \in \omega : \ e \text{ is incident to } v\}|}^{(a)} + \overbrace{|\{e \in \Gamma : \ e \text{ is incident to } v\}|}^{(b)} - \overbrace{2|\{e \in \omega \cap \Gamma : \ e \text{ is incident to } v\}|}^{(c)} \Big) \\ &+ \underbrace{|\{k : v = v_k\}|}_{(d)} \end{aligned}$$

and we know (a) and (c) are always even. If v appears an odd number of times on the list $v_1, \ldots, v_{2m}$, then both (b) and (d) are odd; otherwise, both are even.

For each $\{Q_v : v \in \mathcal{V}_{\Omega_\delta}, Q_v \neq \emptyset\}$ we fix a pairing in Q_v: that is, a function $p_v : Q_v \longrightarrow Q_v$ such that $p_v(p_v(e)) = e$ and $p_v(e) \neq e$ for every $e \in Q_v$ (at least one such pairing must exist because $|Q_v|$ is even). The choice of pairings induces a decomposition which is valid with the exception that condition (ii) may not be satisfied. We start by describing this decomposition, then see how an adequate choice of the pairings makes the induced decomposition satisfy (ii).

Given the pairings p_v, we start by building the path $\gamma_1 = s_1 \sim s_2 \sim \cdots s_n$ as follows: set $s_1 = c_1$ and take $(s_1 s_2)$ as the only half-edge incident to c_1, then iteratively take $(s_i s_{i+1}) = p_{s_i}\big((s_{i-1} s_i)\big)$ while $s_i \in \mathcal{V}_{\Omega_\delta}$, stopping when $s_i \in \mathcal{C}_{\Omega_\delta}$ — informally speaking, the path is building by starting at c_1, choosing the only half-edge possible and then iteratively choosing the exiting edge/half-edge at each vertex v as being the pair of the entering edge/half-edge in Q_v. Note that the only half-edges must be $(s_1 s_2)$ and, assuming the process ends, $(s_{n-1} s_n)$.

Let us build the pairings p_v so that γ_1 verifies condition (ii). Recall that the exiting edge chosen at each vertex v is the pair in Q_v of the entering edge. Therefore, if the edges are paired such that a pair of edges occurs consecutively when rotating in v in clockwise order, then (ii) is satisfied. The pairings are done as follows: for each v, stand in the vertex v and do a complete rotation clockwise, listing the edges $e_1, e_2, \ldots, e_{2l}$ in the order they are seen; set $p_v(e_{2i-1}) = e_{2i}$ and $p_v(e_{2i}) = e_{2i-1}$ for $1 \leq i \leq |Q_v|/2$.

Now we must check γ_1 does not repeat edges and ends at a corner. We start by claiming no edge is used twice in γ_1 in the same orientation, which in particular implies this process must stop eventually. By contradiction, let $i \geq 2$ be the smallest index such that $s_i = s_j$ and $s_{i+1} = s_{j+1}$ for some $j \neq k$. Let

$p = p_{s_i} = p_{s_j}$. Then,

$$\begin{aligned}(s_i s_{i+1}) = (s_j s_{j+1}) &\Rightarrow p\big((s_{i-1}s_i)\big) = p\big((s_{j-1}s_j)\big)\\ &\Rightarrow p\Big(p\big((s_{i-1}s_i)\big)\Big) = p\Big(p\big((s_{j-1}s_j)\big)\Big)\\ &\Rightarrow (s_{i-1}s_i) = (s_{j-1}s_j)\end{aligned}$$

and since $s_i = s_j$ we conclude $s_{i-1} = s_{j-1}$, contradicting the fact that i is the smallest index in γ_1 in these conditions.

We now claim no edge/half-edge is used twice in γ_1. Proceeding again by contradiction, let $i, j \geq 2$ be such that $s_i = s_{j+1}$ and $s_{i+1} = s_j$. Without loss of generality assume that these indexes are such that $|i-j|$ is the smallest possible, and further assumer that $i < j$. Let $p = p_{s_{i+1}} = p_{s_j}$. If $j = i+1$ then γ_1 would be of the form $\cdots s_i \sim s_{i+1} \sim s_i \sim \cdots$ in some section, implying $(s_{i+1}s_i) = p\big((s_{i+1}s_i)\big)$ and contradicting $p(e) \neq e$. If $j > i+1$,

$$\begin{aligned}(s_i s_{i+1}) = (s_{j+1} s_j) &\Rightarrow p\big((s_i s_{i+1})\big) = p\big((s_{j+1}s_j)\big)\\ &\Rightarrow (s_{i+1}s_{i+2}) = p\Big(p\big((s_{j-1}s_j)\big)\Big)\\ &\Rightarrow (s_{i+1}s_{i+2}) = (s_{j-1}s_j)\end{aligned}$$

and since $s_{i+1} = s_j$ we conclude $s_{i+2} = s_{j-1}$, contradicting the fact that i, j were the indices minimizing $|i-j|$.

The claim above implies the final site s_n is not the starting site s_1, therefore it must be one of the corners $c_2, \ldots, c_{2m}$. γ_2 is then built by choosing the first corner in $c_1, \ldots, c_{2m}$ that is not an endpoint of γ_1 and repeating the process, which is done iteratively until all paths $\gamma_1, \ldots, \gamma_m$ are obtained. The proofs as to why all these paths are edge-disjoint use the same arguments as above, with the difference that $(s_i s_{i+1})$ now belongs to some γ_k and $(s_j s_{j+1})$ to some γ_l. The permutation s is then defined so that condition (i) holds: $s(1) = 1$, $s(2)$ is the index of the ending corner of γ_1, $s(3)$ is the index of the starting corner of γ_1 and so on.

The remaining edges are decomposed into a collection of loops in a very similar way: pick a vertex $s_1 = v$ together with an edge $(s_1 s_2)$ that is not yet in the decomposition, then build a path ω_1 by choosing vertices in the same way until no more edges can be chosen. ω_1 repeats no edges nor does it share any edges with the previous paths, again due to very similar arguments to the ones above; in particular, ω_1 only goes through vertices. The finishing vertex v_f of ω_1 must be such that all edges of Q incident to v_f are in either $\gamma_1, \ldots, \gamma_m$ or ω_1. v_f must have even degree in each γ_k, and if v_f was not the starting vertex v_i of ω_1 then it would have odd degree in ω_1, contradicting $|Q_{v_f}|$ being even. Hence, $v_f = v_i$ and ω_1 is a loop. The procedure continues by creating loops $\omega_2, \ldots, \omega_{m'}$ in the same way until all edges of Q are somewhere in the decomposition, and one can prove all of these loops are edge-disjoint from each other and the paths γ_k using the arguments from before. $\square$

Proposition 3.7. *Let $c_1, \ldots, c_{2m-1} \in \mathcal{C}_{\Omega_\delta}$ be pairwise distinct and set $v_k = v(c_k)$. Additionally, let*

$e = (vu) \in \mathcal{E}_{\Omega_\delta}$ *be such that* $v, u \in \operatorname{Int} \mathcal{V}_{\Omega_\delta}$. *Then, the sets*

$$\mathscr{C}_v = \{P \cup (v_1 c_1) \cup \cdots \cup (v_{2m-1} c_{2m-1}) \oplus (ve) : P \in \mathscr{C}(v_1, \ldots, v_{2m-1}, v)\}$$

and

$$\mathscr{C}_u = \{P \cup (v_1 c_1) \cup \cdots \cup (v_{2m-1} c_{2m-1}) \oplus (ue) : P \in \mathscr{C}(v_1, \ldots, v_{2m-1}, u)\}$$

are equal.

Proof. It is enough to prove $\mathscr{C}_v \subseteq \mathscr{C}_u$, the other inclusion follows by symmetry of u and v. Given an element $Q \in \mathscr{C}_v$ we have

$$\begin{aligned} Q &= P \cup (v_1 c_1) \cup \cdots \cup (v_{2m-1} c_{2m-1}) \oplus (ve) \\ &= \big(P \oplus (vu)\big) \cup (v_1 c_1) \cup \cdots \cup (v_{2m-1} c_{2m-1}) \oplus (ue) \end{aligned}$$

for some $P \in \mathscr{C}(v_1, \ldots, v_{2m-1}, v)$, which means P is such that every vertex appearing an odd number of times in the list $v_1, \ldots, v_{2m-1}, v$ has odd degree in P while every other vertices have even degree (recall Proposition 3.5). Note that all vertices with the exception of v and u have the same degree in P and in $P \oplus (vu)$, and that the degrees of v and u are off by either 1 or -1. Thus, $P \oplus (vu) \subseteq \operatorname{Int} \mathcal{E}_{\Omega_\delta}$ is such that every vertex appearing an odd number of times in the list $v_1, \ldots, v_{2m-1}, u$ has odd degree in P while every other vertices have even degree, implying $P \oplus (vu) \in \mathscr{C}(v_1, \ldots, v_{2m-1}, u)$ and consequently $Q \in \mathscr{C}_u$. □

Proposition 3.8. *For any* $c_1, \ldots, c_{2m-1} \in \mathcal{C}_{\Omega_\delta}$ *and* $e \in \mathcal{E}_{\Omega_\delta}$, *Proposition 3.6 applies to elements of* $\mathscr{C}_\delta(c_1, \ldots, c_{2m-1}, e)$.

Proof. The proof of Proposition 3.6 applies here with little changes: for any $Q \in \mathscr{C}_\delta(c_1, \ldots, c_{2m-1}, e)$, $|Q_v|$ is still even for all vertices v and the sites with odd degree in Q are $c_1, \ldots, c_{2m-1}, e$. The paths and loops are drawn the same way with e taking the place of c_{2m}, and the arguments hold all the same. □

Proposition 3.9. *Let* γ *be a path in* $\mathcal{G}_{\Omega_\delta}$ *running from* c_1 *to* c_2. *Then,*

$$i\eta_{c_1}\overline{\eta}_{c_2} \exp\left(-\frac{i}{2}\operatorname{wind}(\gamma)\right) \in \{\pm 1\}.$$

Proof. Note that γ must start with the half-edge $\big(c_1 v(c_1)\big)$ and end with the half-edge $\big(v(c_2) c_2\big)$. Let $e^{i\alpha_1}$ and $e^{i\alpha_2}$ be the directions of $\big(v(c_1)c_1\big)$ and $\big(v(c_2)c_2\big)$, respectively. Then, $\eta_{c_1} = \pm e^{-i\frac{\alpha_1}{2}}$ and $\eta_{c_2} = \pm e^{-i\frac{\alpha_2}{2}}$. In addition, $\operatorname{wind}(\gamma) = \alpha_2 - \alpha_1 - \pi \mod 2\pi$. The argument of $i\eta_{c_1}\overline{\eta}_{c_2} \exp\left(-\frac{i}{2}\operatorname{wind}(\gamma)\right)$ is thus

$$\frac{\pi}{2} - \frac{\alpha_1}{2} + \frac{\alpha_2}{2} - \frac{\alpha_2 - \alpha_1 - \pi}{2} = 0 \mod \pi.$$

□

Proposition 3.10. *The definition (20) is independent of the orientation and numbering of the paths* γ_k.

Proof. It is enough to prove that the right-hand side of (20) stays constant if we take some γ_k to be in the opposite direction and if we swap the numbering of paths γ_k and γ_{k+1}.

For the first case, the factor $i\eta_{c_{s(2k-1)}}\overline{\eta}_{c_{s(2k)}} \exp\left(-\frac{i}{2}\operatorname{wind}(\gamma)\right) = \pm 1$ is replaced by

$$i\overline{\eta}_{c_{s(2k-1)}}\eta_{c_{s(2k)}} \exp\left(-\frac{i}{2}\operatorname{wind}(-\gamma)\right) = -\overline{i\eta_{c_{s(2k-1)}}\overline{\eta}_{c_{s(2k)}} \exp\left(-\frac{i}{2}\operatorname{wind}(\gamma)\right)} = \mp 1$$

while the values of s for $2k-1$ and $2k$ are swapped, so $\operatorname{sign}(s)$ becomes $-\operatorname{sign}(s)$.

For the second case, the product of all contributions of γ_k stays the same and s changes so that

$$2k-1 \mapsto s(2k+1) \qquad 2k \mapsto s(2k+2) \qquad 2k+1 \mapsto s(2k-1) \qquad 2k+2 \mapsto s(2k)$$

and the other values stay the same. Since the changes to s are described by the composition of 4 transpositions, $\operatorname{sign}(s)$ is also kept constant. □

Proposition 3.11. *The definition (20) is independent of the smoothing of Q.*

Proof. Let $Q \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$ and take two smoothings $Q = \omega^1 \dot{\cup} \left(\dot{\bigcup}_{k=1}^m \gamma_k^1\right)$ and $Q = \omega^2 \dot{\cup} \left(\dot{\bigcup}_{k=1}^m \gamma_k^2\right)$ with associated permutations $s^1, s^2 \in S_{2m}$. Push each path γ_k^2 slightly to its left, obtaining $\widetilde{\gamma}_k^2$. Note that computing (20) using $\widetilde{\gamma}_k^2$ instead of γ_k^2 gives the same value. Now, reorient and renumber the paths γ_k^1 and $\widetilde{\gamma}_k^2$ so that the union of all γ_k^1 and $\widetilde{\gamma}_k^2$ with the endpoints connected by a 180° turn yields a collection ω of m' oriented cycles $\omega = \bigcup_{j=1}^{m'} \omega_j$ where each ω_j is a succession of paths

$$\gamma_{k_j}^1, \widetilde{\gamma}_{k_j}^2, \gamma_{k_j+1}^1, \widetilde{\gamma}_{k_j+1}^2, \ldots, \gamma_{k_{j+1}-1}^1, \widetilde{\gamma}_{k_{j+1}-1}^2$$

(together with the 180° turns connecting them in between) with $k_1 = 1$ and the convention $k_{m'+1} = 2m+1$ (Figure 11). Note that the reorientation and renumbering of the paths does not change the value of the right-hand side of (20) using either smoothing (Proposition 3.10).

We start by claiming $\operatorname{sign}(s^1) \cdot \operatorname{sign}(s^2) = (-1)^{m'}$. Due to how the paths γ_k^1 and $\widetilde{\gamma}_k^2$ are connected and numbered, for each $j = 1, \ldots, m'$ the permutations s^1 and s^2 in the points $\{2k_j - 1, \ldots, 2k_{j+1} - 2\}$ are as follows:

s^1	s^2
$2k_j - 1 \mapsto s^1(2k_j - 1)$	$2k_j - 1 \mapsto s^1(2k_j)$
$2k_j \mapsto s^1(2k_j)$	$2k_j \mapsto s^1(2k_j + 1)$
$2k_j + 1 \mapsto s^1(2k_j + 1)$	$2k_j + 1 \mapsto s^1(2k_j + 2)$
$\vdots$	$\vdots$
$2k_{j+1} - 3 \mapsto s^1(2k_{j+1} - 3)$	$2k_{j+1} - 3 \mapsto s^1(2k_{j+1} - 2)$
$2k_{j+1} - 2 \mapsto s^1(2k_{j+1} - 2)$	$2k_{j+1} - 2 \mapsto s^1(2k_j - 1)$

(Informally speaking, the sequence of indexes ordered by s^1 are offset when compared to s^2, because the ending point of the first path of Q_1 is the starting point of the first path of Q_2 and so on). To make s^1

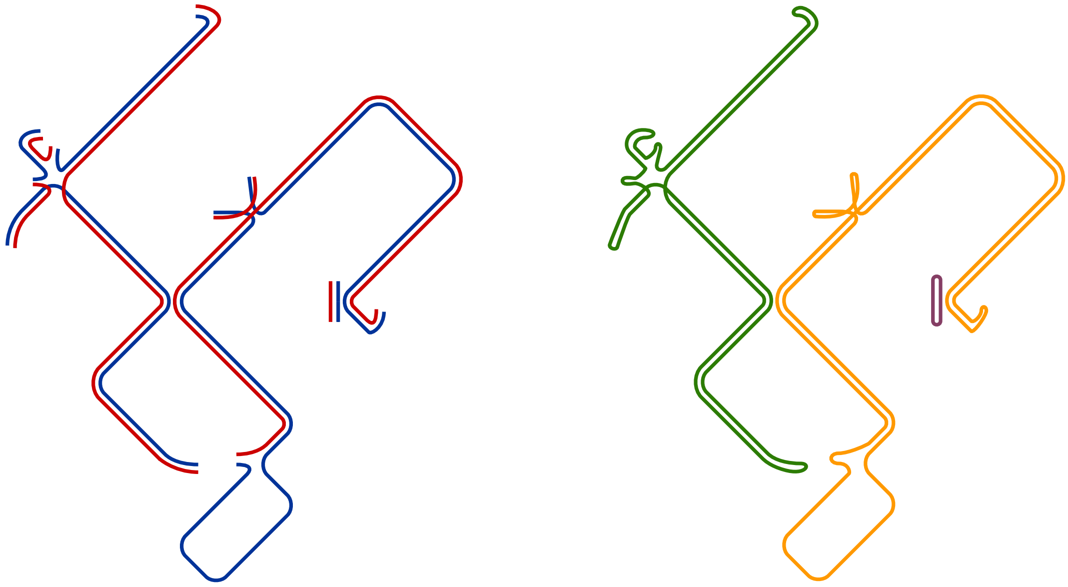

Figure 11: Union of paths of two smoothings as described in the proof of Proposition 3.11. The left figure shows the smoothing from Figure 8 in red (note how the loop is missing) together with a second smoothing in blue formed exclusively by paths. The right picture shows the paths of the two smoothings joined using 180° turns at the endpoints.

match s^2 in these points we can use $2k_{j+1}-2-(2k_j-1)$ transpositions, which is odd. Therefore, the number of transpositions needed for s^1 and s^2 to match is equal to m' mod 2, thus $\operatorname{sign}(s^2) = \operatorname{sign}(s^1)\cdot(-1)^{m'}$ and the claim is proved.

Take the definition (20) using the two smoothings and multiply them. We get

$$\begin{aligned}
&\operatorname{sign}(s^1)\cdot\operatorname{sign}(s^2)\cdot\left[\prod_{k=1}^{m}\exp\left(-\frac{i}{2}\operatorname{wind}(\gamma^1_k)\right)\cdot i\right]\cdot\left[\prod_{k=1}^{m}\exp\left(-\frac{i}{2}\operatorname{wind}(\widetilde{\gamma}^2_k)\right)\cdot i\right]=\\
&\qquad=(-1)^{m'}\cdot\prod_{j=1}^{m'}\exp\left(-\frac{i}{2}\operatorname{wind}(\omega_j)\right)\\
&\qquad=\prod_{j=1}^{m'}(-1)^{\omega_j\cdot\omega_j} && (22)\\
&\qquad=(-1)^{\omega\cdot\omega} && (23)
\end{aligned}$$

where (22) follows from applying Lemma 3.2 to all ω_j and (23) holds because all ω_j are closed curves and thus $\omega_j\cdot\omega_k = 0$ mod 2 if $k\neq j$. If we prove $(-1)^{\omega\cdot\omega}=1$, then we are done.

Push the collection of cycles ω^2 slightly to its left, obtaining $\widetilde{\omega}^2$. Since ω^1 and $\widetilde{\omega}^2$ do not self-intersect and $\omega\cdot\omega^1=\omega\cdot\widetilde{\omega}^2=\omega^1\cdot\widetilde{\omega}^2=0$ mod 2 because ω, ω^1 and $\widetilde{\omega}^2$ are collections of closed curves,

$$(-1)^{\omega\cdot\omega}=(-1)^{(\omega\cup\omega^1\cup\widetilde{\omega}^2)\cdot(\omega\cup\omega^1\cup\widetilde{\omega}^2)}$$

and this number does not change if we remove from ω the 180° turns used to connect the paths (since we can take them small enough). Let $\widehat{\omega}$ be ω without the added $180°$ turns. Then

$$\widehat{\omega} \cup \omega^1 \cup \widetilde{\omega}^2 = \left[\omega^1 \cup \left(\bigcup_{k=1}^{m} \gamma_k^1\right)\right] \cup \left[\widetilde{\omega}^2 \cup \left(\bigcup_{k=1}^{m} \widetilde{\gamma}_k^2\right)\right] = Q^1 \cup \widetilde{Q}^2$$

where Q^1 is the first smoothing and $\widetilde{Q}^2$ is the result of the second smoothing being pushed slightly to its left. The intersections of Q^1 and $\widetilde{Q}^2$ only occur around vertices, therefore it is enough to prove that the number of intersections around each vertex must be even.

To see this, take the parts of Q^1 and $\widetilde{Q}^2$ around a vertex v and consider one of the portions γ of $\widetilde{Q}^2$ traversing around v. Recall that γ was a result of a (smoothed out) lattice path pushed to its left; therefore, the number of parts of Q^1 that start to the left of γ when going around v must equal the number of parts of Q^1 that finish to the left of γ[6]. Consider a process where one walks along each part of Q^1 from start to finish and keep track of the number N of parts of Q^1 that are to the left of γ at each instant of the process. The previous observation implies N must be the same at the start and at the end of this process, and every time a part of Q^1 crosses γ, N either increases or decreases by 1. Hence, there must be an even number of crossings between γ and every other part of Q^1, and applying the same argument to all parts of $\widetilde{Q}^2$ gives the result. □

[6]Note that some of these parts may correspond to the same original path in Q^1, which traverses v multiple times. For this argument, they should be considered as separate.

4 The discrete spinor observables

In this section we define the discrete spinor observables, the main objects of interest, and prove their most relevant properties.

4.1 Motivating the spinors: fermionic variables

The spinor observables are rather technical objects and it is not obvious at first glance why they should be useful, so we start by giving some motivation for the spinor observables. The objective here is to define fermions which will then be used to give what we will call an "analytical description" of the spinor, useful to understand why this object is relevant. The technical results regarding spinor observables will be obtained using a "combinatorial description" (Definition 4.11). Therefore, this subsection is strictly conceptual and lighter on the details, not containing any results necessary for the present work.

In Physics, fermions are particles that follow Fermi-Dirac statistics and have half integer spin. Fermions are used in the continuous Ising model to formally define it, replacing the (ill-defined) spin field by more convenient fermionic field that codes the same information [Mus10]. Fermions are commonly introduced at the discrete level using Grassman variables, which are then used to express path integrals [Mus10, DFMS97, CCK17]. This work uses a second formulation of lattice fermions, more appropriate for the arguments that will follow. An overview of the two formalisms, together with a proof of the equivalence between them, can be found in [CCK17].

Informally speaking, a fermion is seen as the product of an order with a disorder variable that lives next to it. Since the former is associated to a site in $\mathcal{F}_{\Omega_\delta}$ and the latter is associated to a neighbour in $\mathcal{V}_{\Omega_\delta}$, it is natural to associate fermions to the intermediate objects $\mathcal{C}_{\Omega_\delta}$.

Definition 4.1. Let $c, d \in \mathcal{C}_{\Omega_\delta}$ be corners, define $v = v(c)$, $u = v(d)$, $p = f(c)$, $q = f(d)$ and fix a path $\gamma \subseteq \mathcal{E}_{\Omega_\delta}$ running from v to u. Set $\varrho = (pv) \oplus \gamma \oplus (uq)$. Then, the *fermion pair* $(\psi_c\psi_d)_\varrho$ is the random variable defined as

$$(\psi_c\psi_d)_\varrho := -\eta_c^2 \exp\Big(-\frac{i}{2}\operatorname{wind}(\varrho)\Big) \cdot \sigma_p\sigma_q(\mu_v\mu_u)_\gamma$$

where η_c^2 is the complex conjugate of the direction of the segment going from $v(c)$ to c.

A simple property separating fermion pairs from order and disorder pairs is their antisymmetry.

Proposition 4.2. $(\psi_c\psi_d)_\varrho = -(\psi_d\psi_c)_\varrho$.

Proof. Let $e^{i\alpha}$ and $e^{i\beta}$ be the directions of the segments $\big(v(c)c\big)$ and $\big(v(d)d\big)$. Then, $\eta_c^2 = e^{-i\alpha}$ and $\eta_d^2 = e^{-i\beta}$.

For a given path ϱ running from c to d, its winding must equal $\beta - (\alpha + \pi) + 2k\pi$ for some $k \in \mathbb{Z}$. Then,

$$\eta_c^2 \exp\Big(-\frac{i}{2}\operatorname{wind}(\varrho)\Big) = \exp\Big(-\frac{i\alpha}{2} - \frac{i\beta}{2} + \frac{i\pi}{2} - ik\pi\Big)$$

whereas if $-\varrho$ denotes the path ϱ running in the opposite direction

$$\eta_d^2 \exp\Big(-\frac{i}{2}\operatorname{wind}(-\varrho)\Big) = \exp\Big(-\frac{i\alpha}{2} - \frac{i\beta}{2} - \frac{i\pi}{2} + ik\pi\Big)$$

which are symmetric complex values. The rest of the fermion pair does not change when c and d are swapped. $\square$

The fermion operator shares the quasi-local behaviour of the disorder operator, in that it is independent of the path ϱ up to a sign. Because of the winding factor, a complete description of this sign dependence would be rather messy and not useful for our purposes, therefore we limit ourselves to the next result.

Proposition 4.3. *Let $c_1, c_2, c_3, c_4 \in \mathcal{C}_{\Omega_\delta}$ be corners. Let ϱ_{12} be a path on $\mathcal{G}_{\Omega_\delta}$ running from $v(c_1)$ to $v(c_2)$ to which the half edges $\big(c_1 v(c_1)\big)$ and $\big(v(c_2)c_2\big)$ are added, and similarly let ϱ_{34} be another path running from c_3 to c_4.*

Let l_1, l_2, l_3, l_4 be such that $\{l_1, l_2, l_3, l_4\} = \{1, 2, 3, 4\}$. Let $\varrho_{l_1 l_2}$ and $\varrho_{l_3 l_4}$ be paths running from c_{l_1} to c_{l_2} and from c_{l_3} to c_{l_4}, respectively, and defined analogously to the above. Then,

$$\mathbb{E}^+\Big[\mathbb{X} \cdot (\psi_{c_1}\psi_{c_2})_{\varrho_{12}}(\psi_{c_3}\psi_{c_4})_{\varrho_{34}}\Big] = \pm\,\mathbb{E}^+\Big[\mathbb{X} \cdot (\psi_{c_{l_1}}\psi_{c_{l_2}})_{\varrho_{l_1 l_2}}(\psi_{c_{l_3}}\psi_{c_{l_4}})_{\varrho_{l_3 l_4}}\Big]$$

where $\mathbb{X}$ is a product of order, disorder and fermion pairs.

Remark 4.4. We will be considering + boundary conditions because all future results are done in regards to this setting, but duality arguments imply the same holds for free boundary conditions.

Proof. Write the fermion pairs using order and disorder variables. Corollary 2.37 implies that the expected value of the random variables is the same up to a sign, so we are left with checking the multiplicative constants. This reduces to proving that

$$\eta_{c_1}^2\eta_{c_3}^2 \exp\bigg(-\frac{i}{2}\big(\mathrm{wind}(\varrho_{12}) + \mathrm{wind}(\varrho_{34})\big)\bigg) = \eta_{c_{l_1}}^2\eta_{c_{l_3}}^2 \exp\bigg(-\frac{i}{2}\big(\mathrm{wind}(\varrho_{l_1 l_2}) + \mathrm{wind}(\varrho_{l_3 l_4})\big)\bigg)$$

For each $j = 1, 2, 3, 4$, let $e^{i\alpha_j}$ be the direction of the segment $\big(v(c_j)c_j\big)$, which implies $\eta_{c_j}^2 = e^{-i\alpha_j}$. The winding of $\varrho_{l_1 l_2}$ is thus $\alpha_2 - (\alpha_1 + \pi) + 2k_{12}\pi$ for some $k_{12} \in \mathbb{Z}$ whereas the winding of $\varrho_{l_3 l_4}$ equals $\alpha_4 - (\alpha_3 + \pi) + 2k_{34}\pi$ for some $k_{34} \in \mathbb{Z}$. Hence,

$$\eta_{c_{l_1}}^2\eta_{c_{l_3}}^2 \exp\bigg(-\frac{i}{2}\big(\mathrm{wind}(\varrho_{l_1 l_2}) + \mathrm{wind}(\varrho_{l_3 l_4})\big)\bigg) = \exp\bigg(-\frac{i\alpha_1}{2} - \frac{i\alpha_2}{2} - \frac{i\alpha_3}{2} - \frac{i\alpha_4}{2} + i(k_{12} + k_{34} - 1)\pi\bigg)$$

which is symmetric on $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and changes sign depending on the value of $k_{12} + k_{34}$. $\square$

Consecutive applications of this Proposition imply that, up to a sign, the ψ commute and are independent of the paths chosen, which is also true for σ and μ (Corollary 2.37). For the sake of clarity, until the end of the subsection all further equalities will be written up to a sign. This allows us to drop the paths in the subscripts to simply write

$$\mathbb{E}^+\Bigg[\prod_j \sigma_{a_j} \prod_k \mu_{v_k} \prod_l \psi_{c_l}\Bigg]$$

for the expected value of these random variables, which can then be abstractly generalized to a product of formal variables of any number and in any order by imposing that they commute and setting it as 0 when there are an odd number of disorder and fermionic variables.

It should be clear to the reader that fermions by itself do not give any additional information regarding correlation of order and disorder variables, and in fact these objects have been studied as early as [KC71]. In [Smi06] it was observed that the function

$$F(z) = \mathbb{E}^+\left[\psi_c \psi_z\right]$$

for a fixed $c \in \mathrm{Int}\mathcal{C}_{\Omega_\delta}$ possesses properties (namely, it is s-holomorphic — see Definition 4.19 — as long as it is not defined on all corners around $f(c)$ at the same time) that allow a proof of the convergence to a continuous function. In fact, one can add as many σ_j and ψ_l and it still maintains such desirable properties. Because the product of a σ with a ψ is a μ (together with a multiplicative constant), disorder variables can be considered in some way. Hence, if we wanted to study

$$\mathbb{E}^+\left[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}\mu_{v_1}\mu_{v_2}\cdots\mu_{v_{2m}}\right]$$

for some faces $a_j \in \mathcal{F}_{\Omega_\delta}$ and vertices $v_k \in \mathcal{V}_{\Omega_\delta}$, we should replace μ_{v_k} with $\psi_{c_k}\sigma_{u_k}$ where c_k is a corner next to v_k and $u_k = f(c_k)$, leading us to

$$\mathbb{E}^+\left[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}\sigma_{u_1}\sigma_{u_2}\cdots\sigma_{u_{2m}}\psi_{c_1}\psi_{c_2}\cdots\psi_{c_{2m}}\right]$$

and now we add two extra ψ, one associated to the function input z and another associated to some fixed corner c to be paired with ψ_z. Taking $c = a_1 + \delta/2$ leaves us with the function

$$F(z) = \mathbb{E}^+\left[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}\sigma_{u_1}\sigma_{u_2}\cdots\sigma_{u_{2m}}\psi_{c_1}\psi_{c_2}\cdots\psi_{c_{2m}}\psi_{a_1+\delta/2}\psi_z\right] \tag{24}$$

which has the benefit that, when z is such that $v(z) = v(a_1 + \delta/2) = a_1 + \delta$, writing $\psi_{a_1+\delta/2}\psi_z$ as $\sigma_{a_1}\mu_{a_1+\delta} \cdot \sigma_{f(z)}\mu_{v(z)}$ makes the disorder part disappear and the order part of ψ_{a_1} cancel with the σ_{a_1} at the start. For example,

$$\begin{aligned} F(a_1 + 3\delta/2) &= \mathbb{E}^+\left[\sigma_{a_2}\cdots\sigma_{a_n}\sigma_{u_1}\cdots\sigma_{u_{2m}}\psi_{c_1}\cdots\psi_{c_{2m}}\sigma_{a_1+2\delta}\right] \\ &= \mathbb{E}^+\left[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}\mu_{v_1}\cdots\mu_{v_{2m}}\right] \end{aligned}$$

$$\begin{aligned} F(a_1 + \delta + i\delta/2) &= \mathbb{E}^+\left[\sigma_{a_2}\cdots\sigma_{a_n}\sigma_{u_1}\cdots\sigma_{u_{2m}}\psi_{c_1}\cdots\psi_{c_{2m}}\sigma_{a_1+(1+i)\delta}\right] \\ &= \mathbb{E}^+\left[\sigma_{a_1+(1+i)\delta}\sigma_{a_2}\cdots\sigma_{a_n}\mu_{v_1}\cdots\mu_{v_{2m}}\right] \end{aligned}$$

(ignoring the multiplicative constants that come with separating ψ into $\sigma\mu$). The function

$$\frac{1}{\mathbb{E}^+\left[\sigma_{a_1}\cdots\sigma_{a_n}\cdot\mu_{v_1}\cdots\mu_{v_{2m}}\right]}\mathbb{E}^+\left[\sigma_{a_1}\cdots\sigma_{a_n}\cdot\sigma_{u_1}\cdots\sigma_{u_{2m}}\cdot\psi_{c_1}\cdots\psi_{c_{2m}}\cdot\psi_{a_1+\delta/2}\psi_z\right] \tag{25}$$

thus encodes information about the multiplicative increment resulting from changing a_1 slightly at the points $a_1 + 3\delta/2$ and $a_1 + \delta + i\delta/2$, and has the discrete holomorphicity property.

Equation (24) encapsulates the crux of the analytical description of the spinor observable. In fact, the term "spinor observable" should be used for the function (24) to be accurate with literature. It matches the combinatorial description given in Definition 4.11 up to multiplicative constants[7], of which the most relevant is the one in (25) and the signs due to path choices of the disorder and fermionic variables. For a formal handling of the latter factor it is necessary to define the spinor on a double cover.

This section will focus on the statement and proof of the spinor observable properties, which will all be based on the combinatorial description. This description is necessary for the proof of the s-holomorphism (Proposition 4.20), which is fundamental to prove convergence.

4.2 Combinatorial interpretation of spin correlations

We start by assessing how to express spin correlations in a combinatorial way. In the Ising model with disorder lines $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}]$ and boundary conditions $+$, a spin configuration is represented by a set of domain walls $\omega \in \mathscr{C}_{\Omega_\delta}$, whose weight is the number of edges of $P = \omega \oplus \Gamma \in \mathscr{C}_{\Omega_\delta}(v_1, \ldots, v_{2m})$. It will be more convenient to consider instead $Q = P \cup (v_1 c_1) \cup \cdots \cup (v_{2m} c_{2m}) \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$, where c_k is a corner adjacent to v_k. Thus, for faces $a_1, \ldots, a_n$, we write

$$\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}] = \left(\mathcal{Z}^{\Gamma,+}_\delta\right)^{-1} \cdot \sum_{Q \in \mathscr{C}_{\Omega_\delta}(c_1,\ldots,c_{2m})} \alpha_c^{|Q|} \cdot [\text{Sign } \sigma_{a_1} \cdots \sigma_{a_n} \text{ in } \omega \in \mathscr{C}_{\Omega_\delta} \text{ associated with } Q]$$

with $|Q|$ being the number of full edges of Q and $\mathcal{Z}^{\Gamma,+}_\delta = \sum_{Q \in \mathscr{C}_{\Omega_\delta}(c_1,\ldots,c_{2m})} \alpha_c^{|Q|}$ being the partition function of the model. Our objective is to determine how to compute the sign on the right-hand side. We start with a special case where the faces are the ones that contain the corners c_k.

Proposition 4.5. *Consider the Ising model on the graph $\mathcal{G}^\dagger_{\Omega_\delta}$ at the critical temperature with disorder lines $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}]$ and boundary conditions $+$. Let $c_1, \ldots, c_{2m} \in \operatorname{Int}\mathcal{C}_{\Omega_\delta}$ be pairwise disjoint corners adjacent to $v_1, \ldots, v_{2m}$ and set $u_k = f(c_k)$. Then,*

$$\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{u_1} \cdots \sigma_{u_{2m}}] = \tau^0 \cdot \left(\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}\right)^{-1} \cdot \sum_{Q \in \mathscr{C}_{\Omega_\delta}(c_1,\ldots,c_{2m})} \alpha_c^{|Q|} \cdot \tau(Q)$$

where $\tau^0 \in \{\pm 1\}$ depends on the choice of Γ.

In addition, τ^0 can be computed as follows: let $\Theta^0 = \bigcup_{k=1}^m \theta^0_k$ be a collection of edge-disjoint oriented paths in $\mathcal{G}^\dagger_{\Omega_\delta}$ together with a permutation $s^0 \in S_{2m}$ such that θ^0_k is a path running from $u_{s^0(2k-1)}$ to $u_{s^0(2k)}$ to which the half-edges $\big(c_{s^0(2k-1)} u_{s^0(2k-1)}\big)$ and $\big(u_{s^0(2k)} c_{s^0(2k)}\big)$ are added. Then,

$$\tau^0 = (-1)^{\Theta^0 \cdot \Gamma} \cdot (-1)^{\Theta^0 \cdot \Theta^0} \cdot \operatorname{sign}\big(s^0\big) \cdot \prod_{k=1}^m -i\eta_{c_{2k-1}} \overline{\eta}_{c_{2k}} \exp\left(-\frac{i}{2}\operatorname{wind}(\theta^0_k)\right).$$

[7]Recall that in these computations, signs and multiplicative constants associated to writing $\psi \leftrightarrow \sigma\mu$ were neglected.

Remark 4.6. The argument from the proof of Proposition 3.8 can be used to check that $\tau^0 \in \{\pm 1\}$. In addition, note that the order of the c_k is relevant to compute τ^0. For instance, if we were to swap c_k and c_{k+1}, then $\tau(Q)$ would change sign for all Q (because the sign of a permutation associated to the smoothing would change) but τ^0 would also change (due to the $\operatorname{sign}(s^0)$ factor).

Remark 4.7. Recalling fermionic variables, this result shows how to compute $\mathbb{E}^+_{\Omega_\delta}[\psi_{c_1}\cdots\psi_{c_{2m}}]$ combinatorially.

Proof. It is enough to prove that, for any $Q \in \mathscr{C}_{\Omega_\delta}(c_1,\dots,c_{2m})$ associated to a configuration σ, $\sigma_{u_1}\cdots\sigma_{u_{2m}} = \tau^0 \cdot \tau(Q)$. Pick such a Q and take Θ^0 and s^0 as described. The associated element $\omega \in \mathscr{C}_{\Omega_\delta}$ is $\omega = Q \oplus \Gamma \oplus (v_1c_1) \oplus \cdots \oplus (v_{2m}c_{2m})$. Due to how the domain walls configuration is defined,

$$\begin{aligned}
\sigma_{u_1}\cdots\sigma_{u_{2m}} &= \prod_{k=1}^{m} \sigma_{u_{s^0(2k-1)}}\sigma_{u_{s^0(2k)}} \\
&= \prod_{k=1}^{m} (-1)^{\theta^0_k\cdot\omega} \qquad (26)\\
&= (-1)^{\Theta^0\cdot\omega} \\
&= (-1)^{\Theta^0\cdot\left(Q\oplus\Gamma\oplus(v_1c_1)\oplus\cdots\oplus(v_{2m}c_{2m})\right)} \\
&= (-1)^{\Theta^0\cdot Q}\cdot(-1)^{\Theta^0\cdot\Gamma}
\end{aligned}$$

where (26) holds because if $(-1)^{\theta^0_k\cdot\omega} = 1$ (the same argument holds when $(-1)^{\theta^0_k\cdot\omega} = -1$) means the path θ^0_k crosses an even number of edges of ω, and since ω is the domain walls configuration of σ this implies the spins at the endpoints of θ^0_k are equal and therefore $\sigma_{u_{s^0(2k-1)}}\sigma_{u_{s^0(2k)}} = 1$. Note how this expansion holds even if some u_k repeat. We are left with checking

$$(-1)^{\Theta^0\cdot Q} = \tau(Q)\cdot(-1)^{\Theta^0\cdot\Theta^0}\cdot\operatorname{sign}(s^0)\cdot\prod_{k=1}^{m} -i\eta_{c_{2k-1}}\overline{\eta}_{c_{2k}}\exp\left(-\frac{i}{2}\operatorname{wind}(\theta^0_k)\right).$$

Take a smoothing $Q = \omega\cup\gamma = \omega\cup\left(\bigcup_{k=1}^{m}\gamma_m\right)$ with associated $s\in S_{2m}$, which is used to compute $\tau(Q)$ according to (20). Reorient and renumber the paths γ_k and θ^0_k so that $\Delta \coloneqq \gamma\cup\Theta^0$ can be seen as a collection of m' oriented cycles $\Delta = \dot{\bigcup}_{j=1}^{m'}\Delta_j$ and each Δ_j is a succession of paths $\gamma_{k_j}, \Theta^0_{k_j}, \gamma_{k_j+1}, \Theta^0_{k_j+1}, \dots, \gamma_{k_{j+1}-1}, \Theta^0_{k_{j+1}-1}$ with $k_1 = 1$ and the convention $k_{m'+1} = 2m+1$. Note that reorienting and reordering the θ^0_k does not change the value of

$$\operatorname{sign}(s^0)\cdot\prod_{k=1}^{m} -i\eta_{c_{2k-1}}\overline{\eta}_{c_{2k}}\exp\left(-\frac{i}{2}\operatorname{wind}(\theta^0_k)\right)$$

(see the proof of Proposition 3.10 for more details). We have

$$
\begin{aligned}
\Theta^0 \cdot Q &= \Theta^0 \cdot (\omega \cup \gamma) \\
&= (\Theta^0 \cup \gamma) \cdot (\omega \cup \gamma) && (27)\\
&= \Delta \cdot (\omega \cup \gamma) \\
&= \Delta \cdot \gamma \mod 2 && (28)\\
&= \Delta \cdot (\Delta \setminus \Theta^0) \\
&= \Delta \cdot \Delta - \Theta^0 \cdot \Theta^0 \\
&= \Delta \cdot \Delta + \Theta^0 \cdot \Theta^0 \mod 2
\end{aligned}
$$

where (27) holds because ω and γ do not intersect or self-intersect and (28) holds because Δ and ω being collections of closed curves implies $\Delta \cdot \omega = 0 \mod 2$. Thus, $(-1)^{\Theta^0 \cdot Q} = (-1)^{\Theta^0 \cdot \Theta^0} \cdot (-1)^{\Delta \cdot \Delta}$.

On the other hand, we claim $\operatorname{sign}(s) \cdot \operatorname{sign}(s^0) = (-1)^{m'}$. Due to how the paths γ_k and θ^0_k are connected and numbered, for each $j = 1, \dots, m'$ the permutations s and s^0 in the points $\{2k_j - 1, \dots, 2k_{j+1} - 2\}$ are as follows:

$$
\begin{array}{cc}
s & s^0 \\
2k_j - 1 \mapsto s(2k_j - 1) & 2k_j - 1 \mapsto s(2k_j) \\
2k_j \mapsto s(2k_j) & 2k_j \mapsto s(2k_j + 1) \\
2k_j + 1 \mapsto s(2k_j + 1) & 2k_j + 1 \mapsto s(2k_j + 2) \\
\vdots & \vdots \\
2k_{j+1} - 3 \mapsto s(2k_{j+1} - 3) & 2k_{j+1} - 3 \mapsto s(2k_{j+1} - 2) \\
2k_{j+1} - 2 \mapsto s(2k_{j+1} - 2) & 2k_{j+1} - 2 \mapsto s(2k_j - 1)
\end{array}
$$

(informally speaking, the sequence of indexes ordered by s are offset when compared to s^0, because the ending point of γ_1 is the starting point of Θ^0_1 and so on). We can make s and s^0 match in the points $\{2k_j - 1, \dots, 2k_{j+1} - 2\}$ using $2k_{j+1} - 2 - (2k_j - 1)$ transpositions, which is odd. Therefore, the number of transpositions needed for s and s^0 to match is equal mod 2 to m', thus $\operatorname{sign}(s^0) = \operatorname{sign}(s) \cdot (-1)^{m'}$.

Knowing this and expanding $\tau(Q)$ using (20) with the current ordering and orientation of paths,

$$
\begin{aligned}
\tau(Q) \cdot \operatorname{sign}\big(s^0\big) \cdot \prod_{k=1}^{m} -i\eta_{c_{2k-1}}\overline{\eta}_{c_{2k}} \exp\left(-\frac{i}{2}\operatorname{wind}(\theta^0_k)\right) &= (-1)^{m'} \cdot \prod_{j=1}^{m'} \exp\left(-\frac{i}{2}\operatorname{wind}(\Delta_j)\right) \\
&= \prod_{j=1}^{m'} (-1)^{\Delta_j \cdot \Delta_j} && (29)\\
&= (-1)^{\Delta \cdot \Delta} && (30)
\end{aligned}
$$

where (29) follows from Lemma 3.2 and (30) uses the fact that all Δ_k are closed curves. □

We now generalize this statement to an arbitrary set of σ_{a_k} to obtain a result from [CCK17].

Proposition 4.8. *Consider the Ising model on the graph $\mathcal{G}^\dagger_{\Omega_\delta}$ at the critical temperature with disorder lines $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}]$ and boundary conditions $+$. Let $c_1, \ldots, c_{2m} \in \mathrm{Int}\mathcal{C}_{\Omega_\delta}$ be pairwise disjoint corners adjacent to $v_1, \ldots, v_{2m}$ and set $u_k = f(c_k)$. Then, for any faces $a_1, \ldots, a_n \in \mathrm{Int}\mathcal{F}_{\Omega_\delta}$,*

$$\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}] = \tau^0 \cdot \left(\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}\right)^{-1} \cdot \sum_{Q \in \mathscr{C}_{\Omega_\delta}(c_1,\ldots,c_{2m})} \alpha_c^{|Q|} \cdot (-1)^{\Theta\cdot Q} \cdot \tau(Q)$$

where Θ is a collection of edge-disjoint paths in $\mathcal{G}^\dagger_{\Omega_\delta}$ linking $u_1, \ldots, u_{2m}, a_1, \ldots, a_n$ and possibly $a_{\mathrm{out}} \in \partial\mathcal{F}_{\Omega_\delta}$, and $\tau^0 \in \{\pm 1\}$ depends on the choice of Γ and Θ.

In addition, τ^0 can be computed as follows: let $\Theta^0 = \bigcup_{k=1}^m \theta^0_k$ be a collection of edge-disjoint oriented paths in $\mathcal{G}^\dagger_{\Omega_\delta}$ together with a permutation $s^0 \in S_{2m}$ such that θ^0_k is a path running from $u_{s^0(2k-1)}$ to $u_{s^0(2k)}$ to which the half-edges $\left(c_{s^0(2k-1)}u_{s^0(2k-1)}\right)$ and $\left(u_{s^0(2k)}c_{s^0(2k)}\right)$ are added. Then,

$$\tau^0 = (-1)^{\Theta\cdot\Gamma} \cdot (-1)^{\Theta^0\cdot\Gamma} \cdot (-1)^{\Theta^0\cdot\Theta^0} \cdot \mathrm{sign}(s^0) \cdot \prod_{k=1}^m -i\eta_{c_{2k-1}}\overline{\eta}_{c_{2k}} \exp\left(-\frac{i}{2}\,\mathrm{wind}(\gamma^0_k)\right).$$

Remark 4.9. Regarding Proposition 4.8,

1. Some u_k may repeat or be equal to some a_k. Thus, some paths θ^0_k may be empty, in which case $\mathrm{wind}(\gamma^0_k) = 0$.
2. Θ and Θ^0 are not required to be edge-disjoint.

Proof. Note that

$$\sigma_{a_1}\cdots\sigma_{a_n} = \left(\sigma_{u_1}\cdots\sigma_{u_{2m}}\right)\cdot\left(\sigma_{u_1}\cdots\sigma_{u_{2m}}\sigma_{a_1}\cdots\sigma_{a_n}\right).$$

Thus, for $Q \in \mathscr{C}_{\Omega_\delta}(c_1, \ldots, c_{2m})$ with associated $\omega = Q \oplus \Gamma \oplus (v_1c_1) \oplus \cdots \oplus (v_{2m}c_{2m}) \in \mathscr{C}_{\Omega_\delta}$ which is the domain walls representation of some configuration σ,

$$\begin{aligned}\sigma_{u_1}\cdots\sigma_{u_{2m}}\sigma_{a_1}\cdots\sigma_{a_n} &= (-1)^{\Theta\cdot\omega}\\ &= (-1)^{\Theta\cdot Q}\cdot(-1)^{\Theta\cdot\Gamma}\end{aligned}$$

assuming a_{out} is not involved. If it were, we add a $\sigma_{a_{\mathrm{out}}} = +1$ factor at the beginning and the argument still holds. In addition, note this expansion is valid even if some u_k repeat or are equal to some a_k. The proof of Proposition 4.5 completes the argument. □

4.3 Defining the spinor observables

We introduce the discrete spinor observables, which will be defined on double covers. We start by stating the definitions, then proceed to check they are well defined. From now on to the end of this section, $a_k, c_k, u_k, v_k, \Gamma, \Theta$ are considered to be as described in Proposition 4.8. In addition, assume $a_1 + \delta \neq v_k$ for every v_k and $\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}] \neq 0$. We will denote $a_1^{\rightarrow} := a_1 + \frac{\delta}{2}$ and write $\mathbf{a} \equiv a_1, \ldots, a_n$, $\mathbf{c} \equiv c_1, \ldots, c_{2m}$, $\mathbf{u} \equiv u_1, \ldots, u_{2m}$ and $\mathbf{v} \equiv v_1, \ldots, v_{2m}$.

Definition 4.10. Given a lifted corner $\widetilde{z} \in [\mathcal{C}_{\Omega_\delta} \setminus \{\mathbf{c}, a_1^{\rightarrow}\}; \mathbf{a}; \mathbf{u}]$ and $Q \in \mathscr{C}_{\Omega_\delta}\left(\mathbf{c}, a_1 + \frac{\delta}{2}, z\right)$, the *complex phase* $\phi_{\mathbf{a}}^{\mathbf{c}}(Q, \widetilde{z})$ is defined by

$$\phi_{\mathbf{a}}^{\mathbf{c}}(Q, \widetilde{z}) := (-1)^{Q \setminus \mathrm{p}(Q) \cdot \Theta} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q), \widetilde{z}\big) \cdot \widetilde{\tau}(Q)$$

where $\widetilde{\tau}(Q)$ is the modified sign of Q given in (21) and the factor $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q), \widetilde{z}\big) \in \{\pm 1\}$ is defined as follows:

1. Fix forever a lift $\widetilde{a_1^{\rightarrow}} \in [\Omega_\delta; \mathbf{a}; \mathbf{u}]$ of the corner $a_1^{\rightarrow}$.
2. Consider the path $\mathrm{p}(Q)$ running from some corner c to z and take another smooth path π running from $a_1^{\rightarrow}$ to c such that $\pi \cdot \Theta = 0 \mod 2$ and π does not go through any of the points $\mathbf{a}$ and $\mathbf{u}$.
3. Let $\langle \pi, \mathrm{p}(Q) \rangle$ be the concatenation of the two paths and lift $\langle \pi, \mathrm{p}(Q) \rangle$ to the double cover $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ with starting point $\widetilde{a_1^{\rightarrow}}$. Such a path must end in one of the two lifts of z.
4. The factor is $+1$ if the lifted path ends in $\widetilde{z}$ and -1 otherwise.

Definition 4.11. Given a lifted corner $\widetilde{z} \in [\mathcal{C}_{\Omega_\delta} \setminus \{\mathbf{c}, a_1^{\rightarrow}\}; \mathbf{a}; \mathbf{u}]$, define

$$F^{\Gamma}_{[\Omega_\delta; \mathbf{a}; \mathbf{u}]}(\widetilde{z}) := \frac{1}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}]} \sum_{Q \in \mathscr{C}_{\Omega_\delta}\left(\mathbf{c}, a_1^{\rightarrow}, z\right)} \alpha_c^{|Q|} \cdot \phi_{\mathbf{a}}^{\mathbf{c}}(Q, \widetilde{z})$$

where the normalizing factor $\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}]$ is given by

$$\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}] := \mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}] \cdot \mathcal{Z}^{\Gamma,+}_{\Omega_\delta}$$

assuming $\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}] \neq 0$. Recall that $\mathcal{Z}^{\Gamma,+}_{\Omega_\delta} = \sum_{Q \in \mathscr{C}_{\Omega_\delta}(\mathbf{c})} \alpha_c^{|Q|}$ is the partition function of the Ising model with disorder lines Γ and boundary conditions $+$.

We will also consider these spinors defined in the edges.

Definition 4.12. Given a lifted edge $\widetilde{z} \in [\mathcal{E}_{\Omega_\delta}; \mathbf{a}; \mathbf{u}]$, define

$$F^{\Gamma}_{[\Omega_\delta; \mathbf{a}; \mathbf{u}]}(\widetilde{z}) := \frac{1}{\left(\cos \frac{\pi}{8}\right) \cdot \mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}]} \sum_{Q \in \mathscr{C}_{\Omega_\delta}\left(\mathbf{c}, a_1^{\rightarrow}, z\right)} \alpha_c^{|Q|} \cdot \phi_{\mathbf{a}}^{\mathbf{c}}(Q, \widetilde{z})$$

assuming $\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}] \neq 0$.

Unambiguity of the complex phase definition Consider the factor $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z})$ for a path γ running from some corner c to z. The definition requires another path π running smoothly from $a_1^{\rightarrow}$ to c that does not go through $\mathbf{a}$ or $\mathbf{u}$ and such that $\pi \cdot \Theta = 0 \mod 2$. It is trivial to check such a π exists: if $\pi \cdot \Theta = 1 \mod 2$ instead, then have π do a loop around the face $f(c)$ before ending in c. If the loop is small enough, then it will only intersect the path of Θ starting in $f(c)$.

The well-definedness of $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z})$ is a direct consequence of Lemma 3.3.

Proposition 4.13. *The factor* $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z})$ *as stated in Definition 4.10 is independent of the choice of* π.

Proof. If π_1 and π_2 are two appropriate paths, then $(-1)^{\langle \pi_1, \pi_2\rangle \cdot \gamma} = (-1)^{\pi_1 \cdot \gamma} \cdot (-1)^{\pi_2 \cdot \gamma} = 1$. Hence $\langle \pi_1, \pi_2 \rangle$ lifts to a loop in $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ by Lemma 3.3. Therefore the lift of $\langle \pi^1, \gamma\rangle$ and $\langle \pi^2, \gamma \rangle$ with the same start will end at the same point, concluding the proof. □

Remark 4.14. We can compute $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z})$ even if $\pi \cdot \Theta = 1 \mod 2$, if we make $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z}) = -1$ if the lifted path ends in $\widetilde{z}$ and $+1$ otherwise. The consistency of the definition is again a simple application of Lemma 3.3.

Regarding the definition of the complex phase, it requires a smoothing of Q to define $\mathrm{p}(Q)$, so we need to check different smoothings yield the same result.

Proposition 4.15. *The function* $\phi^{\mathfrak{c}}_{\mathbf{a}}(Q, \widetilde{z})$ *as stated in Definition 4.10 is independent of the choice of the smoothing of* Q.

Proof. Consider two smoothings of Q which originate two paths $\mathrm{p}_1(Q)$ and $\mathrm{p}_2(Q)$ ending in z. Let π_1 and π_2 be two adequate paths used to compute the $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}$ factor. Then,

$$\begin{aligned}\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(p_1(Q), \widetilde{z}\big) \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(p_2(Q), \widetilde{z}\big) = 1 \Leftrightarrow \\ &\Leftrightarrow \langle \pi_1, p_1(Q), p_2(Q), \pi_2 \rangle \text{ lifts to a loop in } [\Omega_\delta; \mathbf{a}; \mathbf{u}] \\ &\Leftrightarrow (-1)^{\langle \pi_1, p_1(Q), p_2(Q), \pi_2\rangle \cdot \Theta} = 1 \qquad (31) \\ &\Leftrightarrow (-1)^{p_1(Q)\cdot\Theta} \cdot (-1)^{p_2(Q)\cdot\Theta} = 1 \\ &\Leftrightarrow \Big((-1)^{p_1(Q)\cdot\Theta} \cdot (-1)^{Q\cdot\Theta}\Big) \cdot \Big((-1)^{p_2(Q)\cdot\Theta} \cdot (-1)^{Q\cdot\Theta}\Big) = 1 \\ &\Leftrightarrow (-1)^{Q\setminus \mathrm{p}_1(Q)\cdot\Theta} \cdot (-1)^{Q \setminus \mathrm{p}_2(Q)\cdot\Theta} = 1\end{aligned}$$

where Lemma 3.3 is used in step (31). Hence,

$$\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(p_1(Q), \widetilde{z}\big) \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(p_2(Q), \widetilde{z}\big) = (-1)^{Q\setminus \mathrm{p}_1(Q)\cdot\Theta} \cdot (-1)^{Q\setminus \mathrm{p}_2(Q)\cdot\Theta}$$

which can be rewritten as

$$(-1)^{Q\setminus \mathrm{p}_1(Q)\cdot\Theta} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(p_1(Q), \widetilde{z}\big) = (-1)^{Q\setminus \mathrm{p}_2(Q)\cdot\Theta} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(p_2(Q), \widetilde{z}\big).$$

□

4.4 Ratios of spin correlations in spinors observables

We state the connection between the combinatorial description of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ given in Definition 4.11 and the analytical description derived at the beginning of this section. The result itself is simple to deduce as the hard work has already been done in Proposition 4.8. Note that we consider two sets of disorder lines, Γ and $\widetilde{\Gamma}$, with the latter linking additional vertices.

Proposition 4.16. *For any corner z, the equality*

$$\frac{\mathbb{E}_{\Omega_\delta}^{\widetilde{\Gamma},+}[\sigma_{f(z)}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}_{\Omega_\delta}^{\Gamma,+}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} = \tau^0\overline{\eta}_z \mathrm{sheet}_{\boldsymbol{a},\boldsymbol{u}}(\varepsilon,\widetilde{z}) \cdot \frac{\mathcal{Z}_{\Omega_\delta}^{\Gamma,+}}{\mathcal{Z}_{\Omega_\delta}^{\widetilde{\Gamma},+}} \cdot F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}(\widetilde{z})$$

holds, where $\tau^0 \in \{\pm 1\}$ is as defined in Proposition 4.8 using $\widetilde{\Gamma} \equiv \widetilde{\Gamma}[\boldsymbol{v}, a_1+\delta, v(z)]$, Θ and some Θ^0 linking the faces $\boldsymbol{u}, a_1, f(z)$ and ε is the empty path, which is seen as connecting z to z.

Proof. Expanding the normalizing factor as defined in Definition 4.11, the claimed equality becomes

$$\begin{aligned}
&\mathbb{E}_{\Omega_\delta}^{\widetilde{\Gamma},+}\big[\sigma_{f(z)}\sigma_{a_2}\cdots\sigma_{a_n}\big] = \\
&\qquad = \tau^0 \mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{z}) \cdot \Big(\mathcal{Z}_{\Omega_\delta}^{\widetilde{\Gamma},+}\Big)^{-1} \cdot \sum_{Q\in\mathscr{C}_{\Omega_\delta}(\mathbf{c},a_1^{\rightarrow},z)} \alpha_c^{|Q|} \cdot \overline{\eta}_z \cdot \phi_{\mathbf{a}}^{\mathbf{c}}(Q,\widetilde{z}) \\
&\qquad = \tau^0 \mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{z}) \cdot \Big(\mathcal{Z}_{\Omega_\delta}^{\widetilde{\Gamma},+}\Big)^{-1} \cdot \sum_{Q\in\mathscr{C}_{\Omega_\delta}(\mathbf{c},a_1^{\rightarrow},z)} \alpha_c^{|Q|} \cdot (-1)^{Q\setminus \mathrm{p}(Q)\cdot\Theta} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q),\widetilde{z}\big) \cdot \tau(Q).
\end{aligned}$$

We now apply Proposition 4.8 to compute $\mathbb{E}_{\Omega_\delta}^{\widetilde{\Gamma},+}[\sigma_{f(z)}\sigma_{a_2}\cdots\sigma_{a_n}]$. For that, we need a collection of edge-disjoint paths in $\mathcal{G}_{\Omega_\delta}^{\dagger}$ linking $u_1,\dots,u_m,a_1,f(z),f(z),a_2,\dots,a_n$, so the Θ used in the definition of $\phi_{\mathbf{a}}^{\mathbf{c}}$ can be used. Hence, Proposition 4.8 yields

$$\mathbb{E}_{\Omega_\delta}^{\widetilde{\Gamma},+}\big[\sigma_{f(z)}\sigma_{a_2}\cdots\sigma_{a_n}\big] = \tau^0 \cdot \Big(\mathcal{Z}_{\Omega_\delta}^{\widetilde{\Gamma},+}\Big)^{-1} \cdot \sum_{Q\in\mathscr{C}_{\Omega_\delta}(\mathbf{c},a_1^{\rightarrow},z)} \alpha_c^{|Q|} \cdot (-1)^{Q\cdot\Theta} \cdot \tau(Q)$$

and thus the claim reduces to the fact

$$\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{z}) \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q),\widetilde{z}\big) = (-1)^{\mathrm{p}(Q)\cdot\Theta}$$

for every $Q \in \mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^{\rightarrow}, z)$.

Fix a path $\mathrm{p}(Q)$ running from some corner c to z and let π be a smooth path running from $a_1^{\rightarrow}$ to c such that $\pi\cdot\Theta = 0 \mod 2$ and π does not go though any of the points $\mathbf{a}$ and $\mathbf{u}$. Additionally, let π_0 be a path from $a_1^{\rightarrow}$ to z that lifts to a path running from $\widetilde{a_1^{\rightarrow}}$ to $\widetilde{z}$. Then

$$\begin{aligned}
\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q),\widetilde{z}\big) = 1 &\Leftrightarrow \big\langle \pi, \mathrm{p}(Q)\big\rangle \text{ lifts to a path running from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{z} \\
&\Leftrightarrow \big\langle \pi, \mathrm{p}(Q), \pi_0\big\rangle \text{ lifts to a loop} \\
&\Leftrightarrow (-1)^{\langle \pi,\mathrm{p}(Q),\pi_0\rangle\cdot\Theta} = 1 \qquad (32)\\
&\Leftrightarrow (-1)^{\pi_0\cdot\Theta}\cdot(-1)^{\mathrm{p}(Q)\cdot\Theta} = 1 \qquad (33)
\end{aligned}$$

where we use Lemma 3.3 in (32) and $\pi\cdot\Theta = 0 \mod 2$ in (33). The above implies

$$\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q),\widetilde{z}\big)\cdot(-1)^{\pi_0\cdot\Theta} = (-1)^{\mathrm{p}(Q)\cdot\Theta}$$

because all of the above factors are either $+1$ or -1. The $(-1)^{\pi_0\cdot\Theta}$ factor can be worked out using the same idea in reverse: setting π as smooth path running from $a_1^{\rightarrow}$ to z such that $\pi' \cdot \Theta = 0 \mod 2$ and π' does not go though any of the points $\mathbf{a}$ and $\mathbf{u}$, we have

$$\begin{aligned}(-1)^{\pi_0 \cdot \Theta} = 1 &\Leftrightarrow (-1)^{\langle \pi', \pi_0 \rangle \cdot \Theta} = 1 \\ &\Leftrightarrow \langle \pi', \pi_0 \rangle \text{ lifts to a loop} \\ &\Leftrightarrow \pi' \text{ lifts to a path running from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{z} \\ &\Leftrightarrow \langle \pi', \varepsilon \rangle \text{ lifts to a path running from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{z} \\ &\Leftrightarrow \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{z}) = 1.\end{aligned}$$

therefore $(-1)^{\pi_0 \cdot \Theta} = \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{z})$. □

In the particular case $v(z) = a_1 + \delta$, the second collection of disorder lines $\widetilde{\Gamma}$ should link the vertices $\mathbf{v}, a_1 + \delta, a_1 + \delta$. Therefore, we can take $\widetilde{\Gamma} = \Gamma$. This leads to the two following results that will be vital in extracting the information from the spinors. Note how they are expected given the analytical description of the spinor.

Proposition 4.17. *The equality*

$$\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} = \pm F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}\big(\widetilde{a}_1^{\rightarrow} + \delta\big)$$

holds.

The sign $\pm$ is equal to $\overline{\eta}_{a_1+\frac{3\delta}{2}}\tau^0$, where $\tau^0 \in \{\pm 1\}$ is as defined in Proposition 4.8 using Γ, Θ and some Θ^0 linking the faces $\boldsymbol{u}, a_1, a_1 + 2\delta$.

Proposition 4.18. *The equality*

$$\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+(1+i)\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} = \pm e^{\frac{\pi i}{4}} F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}\left(\widetilde{a}_1^{\rightarrow} + \frac{1+i}{2}\delta\right)$$

holds.

The sign $\pm$ is equal to $e^{-\frac{\pi i}{4}}\overline{\eta}_{a_1+(1+\frac{i}{2})\delta}\tau^0$, where $\tau^0 \in \{\pm 1\}$ is as defined in Proposition 4.8 using Γ, Θ and some Θ^0 linking the faces $\boldsymbol{u}, a_1, a_1 + (1+i)\delta$.

4.5 Properties of the spinor observables

Having the results needed to extract information about the spin correlations from the spinors, we now start tackling the problem of how to pass to the limit. To do this, we establish the properties of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ instrumental in both defining their continuous version and proving the convergence.

Multiplicative monodromy. The first property is related to the double cover setting: the monodromy of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$. If we take two different points $\widetilde{z}^1, \widetilde{z}^2$ with the same projection on Ω_δ, then $\text{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z}^1) = -\text{sheet}_{\mathbf{a},\mathbf{u}}(\gamma, \widetilde{z}^2)$ by definition. Therefore, Definitions 4.11 and 4.12 directly imply $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ has opposite signs on opposite sheets.

Argument and boundary condition. Consider the argument of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{z})$ when z is a corner. Following Definition 4.11, the argument of each term of the sum is dictated by the factor $\phi^{\mathbf{c}}_{\mathbf{a}}(Q,\widetilde{z})$, which depends on $\widetilde{\tau}(Q)$. By (21), $\widetilde{\tau}(Q) = \frac{\tau(Q)}{\overline{\eta}_z} = \tau(Q)\eta_z$ and noticing $\tau(Q) \in \{\pm 1\}$, we conclude each term has the same argument as $\pm\eta_z$, therefore the same is true for the whole sum.

When z is an edge midpoint, the argument of each term of the sum in Definition 4.12 is dictated by $\widetilde{\tau}(Q)$ in the same way. Decomposing it using (21) and (20) and using Proposition 3.4, the argument is given by $\pm i\eta_c \exp\Big(-\frac{i}{2}\operatorname{wind}(\gamma)\Big)$, with γ being a path from a corner c to z. If the velocity vector of γ starts with an argument of α and ends with an argument of β, then $\operatorname{wind}(\gamma) = \beta - \alpha$ and $\eta_c = \pm e^{-i\frac{\alpha+\pi}{2}}$. Therefore, the argument of the term is

$$\frac{\pi}{2} - \frac{\alpha+\pi}{2} - \frac{\beta-\alpha}{2} = -\frac{\beta}{2} \mod \pi$$

and since γ can reach z from either endpoint, different terms of the sum may have different arguments mod π. Thus, the argument of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{z})$ is not trivial to compute.

Due to the assumption that edges joining vertices of $\operatorname{Int}\mathcal{V}_{\Omega_\delta}$ are in $\operatorname{Int}\mathcal{E}_{\Omega_\delta}$, the edges in $\partial\mathcal{E}_{\Omega_\delta}$ are precisely those that can only be reached by exactly one of its endpoints. In this case, $e^{i\beta}$ is the direction of the edge z pointing towards the exterior of Ω_δ, so it is a discrete analogue of the outer normal to the boundary at z. If we denote this direction by $\nu_{\text{out}}(z)$, then we can write $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{z}) \cdot \sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}$, thus arriving at a boundary condition of sorts for $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$.

Holomorphicity. The third property is a discrete version of the holomorphicity property. Although these spinors verify the most common version of this property — referred to in the literature as the *discrete holomorphicity* —, we will require a stronger property, first introduced in [Smi10a, CS12]: the *s-holomorphicity*. This is crucial to deal with the passage to the scaling limit. We will leave an overview of discrete and s-holomorphicity and the technical details for Section 5, limiting ourselves for now to the proof that $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ is s-holomorphic.

To each corner $c \in \mathcal{C}_{\Omega_\delta}$ we associate the line $l(c) := \eta_c\mathbb{R}$ — seen as a subset of $\mathbb{C}$ — and denote by $\operatorname{Proj}_{l(c)}[w]$ the *projection* of a complex number w onto the line $l(c)$, which can be written as

$$\operatorname{Proj}_{l(c)}[w] = \Re\big(w\overline{\eta}_c\big)\eta_c = \frac{1}{2}\big(w + \eta_c^2\overline{w}\big)$$

Definition 4.19. A function $F : [C \cup \mathcal{E}_{\Omega_\delta}; \mathbf{a}; \mathbf{u}] \longrightarrow \mathbb{C}$ defined on the lifts of sets $C \subseteq \mathcal{C}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta}$ is *strongly holomorphic* at $c \in C$, or *s-holomorphic* for short, if for both $e \in \mathcal{E}_{\Omega_\delta}$ adjacent to c (that is, such that $|c - e| = \frac{\delta}{2}$)

$$F(\widetilde{c}) = \operatorname{Proj}_{l(c)}\big[F(\widetilde{e})\big]$$

for both lifts of c, with the lift of e taken to be on the same sheet of $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$. Moreover, F is s-holomorphic on C if it is s-holomorphic at each $c \in C$.

Proposition 4.20. *The function $F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}$ is s-holomorphic on its domain.*

Proof. We adapt the argument from Proposition 2.4 of [CHI15].

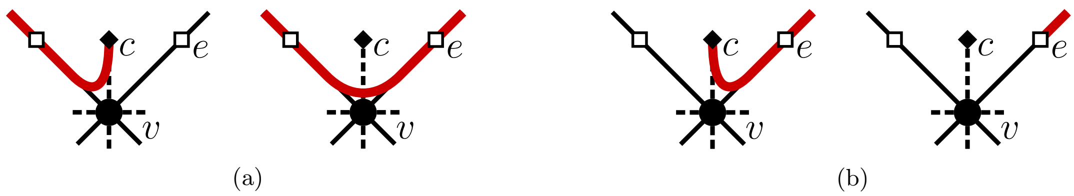

Figure 12: Auxiliary figure for the proof of Proposition 4.20, showing the transformation between $\mathrm{p}(Q_c)$ and $\mathrm{p}(Q_e)$, in the case where $(ve) \notin Q_c$ — subfigure (a) — and in the case where $(ve) \in Q_c$ — subfigure (b).

Let $c \in \mathcal{C}_{\Omega_\delta} \setminus \{\mathbf{c}, a_1^\rightarrow\}$ and $e \in \mathcal{E}_{\Omega_\delta}$ be adjacent points. The sums $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{c})$ and $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e})$ are defined over the sets $\mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^\rightarrow, c)$ and $\mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^\rightarrow, e)$, and there is a simple bijection between them: if $v = v(c)$, then we take the XOR difference of a configuration with the half-edges (cv) and (ve). We shall check the projections of corresponding terms in the sums coincide: that is, for any $Q_c \in \mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^\rightarrow, c)$ and $Q_e = Q_c \oplus (cv) \oplus (ve) \in \mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^\rightarrow, e)$, the equality

$$\alpha_c^{|Q_c|} \cdot \phi_{\mathbf{a}}^{\mathbf{c}}(Q_c, \widetilde{c}) = \mathrm{Proj}_{l(c)}\left[\frac{\alpha_c^{|Q_e|} \cdot \phi_{\mathbf{a}}^{\mathbf{c}}(Q_e, \widetilde{e})}{\cos\frac{\pi}{8}}\right] \tag{34}$$

holds, which implies the result.

There are two cases:

1. $(ve) \notin Q_c$ (subfigure (a) of Figure 12).

   Fix a smoothing of Q_c such that $\mathrm{p}(Q_c)$ ends in c. Then, take $\mathrm{p}(Q_c)$, remove (vc) and add (ve). The result is a valid smoothing of Q_e. We have:

   (i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q_c|}$

   (ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \ominus} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = (-1)^{Q_c \setminus \mathrm{p}(Q_c) \cdot \ominus} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_c), \widetilde{c}\big)$

   (iii) $\mathrm{wind}\big(\mathrm{p}(Q_e)\big) = \mathrm{wind}\big(\mathrm{p}(Q_c)\big) \pm \frac{\pi}{4}$, which implies $\widetilde{\tau}(Q_e) = e^{\mp\frac{\pi i}{8}} \cdot \widetilde{\tau}(Q_c)$

   Write $\widetilde{\tau}(Q_c) = \eta_c \tau(Q_c)$. Factoring the real terms out of $\mathrm{Proj}_{l(c)}$ — including $\widetilde{\tau}(Q_c)$ —, (34) reduces to

   $$\eta_c \cos\frac{\pi}{8} = \mathrm{Proj}_{l(c)}\left[e^{\mp\frac{\pi i}{8}}\eta_c\right]$$

   which is easily checked by expanding the right-hand side:

   $$\mathrm{Proj}_{l(c)}\left[e^{\mp\frac{\pi i}{8}}\eta_c\right] = \frac{1}{2}\left(e^{\mp\frac{\pi i}{8}}\eta_c + \eta_c^2 \cdot e^{\pm\frac{\pi i}{8}}\overline{\eta}_c\right) = \frac{\eta_c}{2}\left(e^{\mp\frac{\pi i}{8}} + e^{\pm\frac{\pi i}{8}}\right) = \eta_c \cos\frac{\pi}{8}$$

2. $(ve) \in Q_c$[8] (subfigure (b) of Figure 12).

   Fix a smoothing of Q_c such that $\mathrm{p}(Q_c)$ ends with the half-edges (ev) and (vc) (this is possible by using the argument of the proof of Proposition 3.6). Then, take $\mathrm{p}(Q_c)$ and remove those half-edges. The result is a valid smoothing of Q_e. We have:

   (i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q_c|-1}$

[8]This case is vacuous if $e \in \partial\mathcal{E}_{\Omega_\delta}$.

(ii) $(-1)^{Q_e\setminus \mathrm{p}(Q_e)\cdot\Theta}\cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e),\widetilde{e}\big) = (-1)^{Q_c\setminus \mathrm{p}(Q_c)\cdot\Theta}\cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_c),\widetilde{c}\big)$

(iii) $\mathrm{wind}\big(\mathrm{p}(Q_e)\big) = \mathrm{wind}\big(\mathrm{p}(Q_c)\big) \pm \frac{3\pi}{4}$, which implies $\widetilde{\tau}(Q_e) = e^{\mp\frac{3\pi i}{8}}\cdot\widetilde{\tau}(Q_c)$

Doing the same as before, (34) becomes

$$\eta_c\,\alpha_c\cos\frac{\pi}{8} = \mathrm{Proj}_{l(c)}\Big[e^{\mp\frac{3\pi i}{8}}\eta_c\Big]$$

and expanding the right-hand side now leads to

$$\alpha_c\cos\frac{\pi}{8} = \cos\frac{3\pi}{8}$$

which is true since

$$\left(\cos\frac{\pi}{8}\right)^{-1}\cos\frac{3\pi}{8} = \tan\frac{\pi}{8} = \sqrt{2}-1 = \alpha_c$$

□

Let us summarize these findings in a single statement.

Proposition 4.21. *Regarding the function* $F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}$,

1. *It has multiplicative monodromy* -1 *around each* $\boldsymbol{a}$ *and* $\boldsymbol{u}$.
2. *For every* $\widetilde{z}\in[\partial\mathcal{E}_{\Omega_\delta};\boldsymbol{a};\boldsymbol{u}]$, $F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}(\widetilde{z})\sqrt{\nu_{\mathrm{out}}(z)}\in\mathbb{R}$.
3. *It is s-holomorphic on* $[\mathcal{C}_{\Omega_\delta}\setminus\{\boldsymbol{c},a_1^{\rightarrow}\}\cup\mathcal{E}_{\Omega_\delta};\boldsymbol{a};\boldsymbol{u}]$.

Remark 4.22. The corners where $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ is not defined are the sites where two fermionic variables in (24) cancel each other out.

The spinor in the missing corners. We now study how $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ behaves in the lifts of corners $a_1^{\rightarrow}$ and $\mathbf{c}$. The argument is similar across all these cases: we prove the values of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ in the nearby edges have projections with opposite signs onto the line associated with the corner.

We start with $a_1^{\rightarrow}$, which is an adaptation of Lemma 3.2 of [CHI15].

Lemma 4.23. *The equality*

$$\mathrm{Proj}_{l(a_1^{\rightarrow})}\left[F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}\left(\widetilde{a}_1^{\rightarrow}\pm\frac{i}{2}\delta\right)\right] = \pm\tau^0\eta_{a_1^{\rightarrow}}$$

holds, with $\tau^0\in\{\pm1\}$ *as defined in Proposition 4.8 using* Γ, Θ *and some* Θ^0 *linking the faces* $\boldsymbol{u}$.

Remark 4.24. If the lifts of $a_1^{\rightarrow}\pm\frac{i}{2}\delta$ are on the other sheet, then it follows from the monodromy of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ that the projections have reversed signs.

Proof. Let $e = a_1 + \frac{1\pm i}{2}\delta$ and $v = a_1+\delta$. The sum $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e})$ is defined over the set $\mathscr{C}_{\Omega_\delta}(\mathbf{c},a_1^{\rightarrow},e)$. On the other hand, the normalizing factor can be written as

$$\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}] = \tau^0\cdot\sum_{Q\in\mathscr{C}_{\Omega_\delta}(\mathbf{c})}\alpha_c^{|Q|}\cdot(-1)^{Q\cdot\Theta}\cdot\tau(Q)$$

by Proposition 4.8. The XOR difference with the half-edges $(a_1^{\rightarrow} v)$ and (ve) yields a bijection between the sets $\mathscr{C}_{\Omega_\delta}(\mathbf{c})$ and $\mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^{\rightarrow}, e)$. If we prove that, for any $Q \in \mathscr{C}_{\Omega_\delta}(\mathbf{c})$ and $Q_e = Q \oplus (a_1^{\rightarrow} v) \oplus (ve) \in \mathscr{C}_{\Omega_\delta}(\mathbf{c}, a_1^{\rightarrow}, e)$, we have

$$\pm\, \eta_{a_1^{\rightarrow}} \cdot \alpha_c^{|Q|} \cdot (-1)^{Q \cdot \Theta} \cdot \tau(Q) = \mathrm{Proj}_{l(a_1^{\rightarrow})}\left[\frac{\alpha_c^{|Q_e|} \cdot \phi_{\mathbf{a}}^{\mathbf{c}}(Q_e, \widetilde{e})}{\cos\frac{\pi}{8}}\right] \tag{35}$$

we are done (note that the terms of the normalizing factor are real numbers).

As before, there are two cases:

1. $(ve) \notin Q$.

   Fix a smoothing of Q with permutation $s \in S_{2m}$ and add to it the path $a_1^{\rightarrow} \sim v \sim e$. This induces a valid smoothing of Q_e in which the added path is $\mathrm{p}(Q_e)$. Let $s_e \in S_{2m+2}$ be the associated permutation. We have:

   (i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q|}$.

   (ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \Theta} = (-1)^{Q \cdot \Theta}$.

   (iii) $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = 1$ because we are fixing the lift of e in the statement.

   (iv) $\mathrm{wind}\big(\mathrm{p}(Q_e)\big) = \pm\frac{3\pi}{4}$.

   (v) s_e matches s in the values $\{1, \ldots, 2m\}$ and additionally $s_e(2m+1) = 2m+1, s_e(2m+2) = 2m+2$. As such, $\mathrm{sign}(s_e) = \mathrm{sign}(s)$.

   (vi) The last two observations yield $\widetilde{\tau}(Q_e) = i\eta_{a_1^{\rightarrow}} e^{\mp\frac{3\pi i}{8}} \tau(Q)$.

   Factoring out the real terms, (35) reduces to

   $$\pm\eta_{a_1^{\rightarrow}} \cos\frac{\pi}{8} = \mathrm{Proj}_{l(a_1^{\rightarrow})}\left[i\eta_{a_1^{\rightarrow}} e^{\mp\frac{3\pi i}{8}}\right].$$

   This can be shown by expanding the right-hand side:

   $$\mathrm{Proj}_{l(a_1^{\rightarrow})}\left[i\eta_{a_1^{\rightarrow}} e^{\mp\frac{3\pi i}{8}}\right] = \frac{1}{2}\left(i\eta_{a_1^{\rightarrow}} e^{\mp\frac{3\pi i}{8}} - \eta_{a_1^{\rightarrow}}^2 \cdot i\overline{\eta}_{a_1^{\rightarrow}} e^{\pm\frac{3\pi i}{8}}\right) = \frac{i\eta_{a_1^{\rightarrow}}}{2}\left(e^{\mp\frac{3\pi i}{8}} - e^{\pm\frac{3\pi i}{8}}\right) = \pm\eta_{a_1^{\rightarrow}} \cos\frac{\pi}{8} \tag{36}$$

2. $(ve) \in Q$.

   Fix a smoothing of Q with permutation $s \in S_{2m}$ and let γ be the path or cycle that goes though (ev), which we assume without loss of generality that runs from e to v. Removing (ev) from γ and inserting $(a_1^{\rightarrow} e)$ in its place induces a valid smoothing of Q_e, with some associated permutation s_e. We now consider two subcases:

   2a. γ is a cycle (subfigure (a) of Figure 13).

   Then, replacing (ev) with $(a_1^{\rightarrow} e)$ in γ transforms it into a path, which is in fact $\mathrm{p}(Q_e)$. Then,

   (i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q|-1}$.

   (ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \Theta} = (-1)^{Q \setminus \gamma \cdot \Theta}$

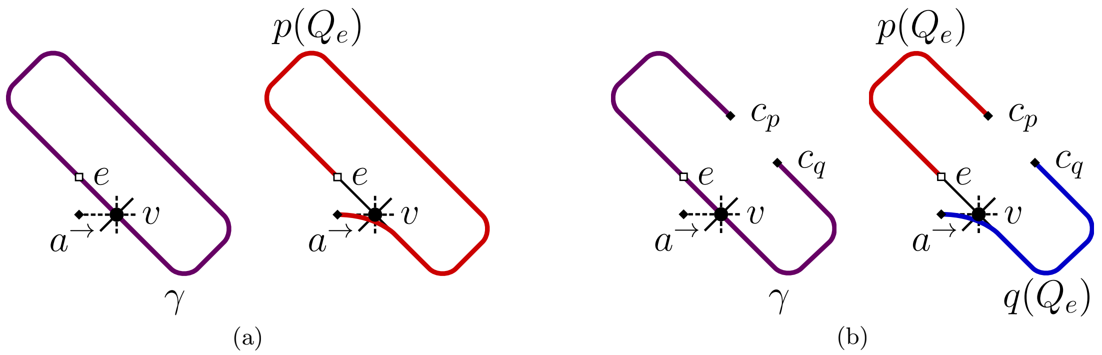

Figure 13: Auxiliary figure for the proofs of Lemmas 4.23 and 4.25 when $(ve) \in Q$, showing the transformation between $\mathrm{p}(Q_c)$ and $\mathrm{p}(Q_e)$ in the case where γ is a cycle — subfigure (a) — and in the case where γ is a path — subfigure (b).

(iii) Because of which lifting of e is considered, $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = 1$ if and only if γ lifts to a loop in $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$, which is equivalent to $(-1)^{\gamma \cdot \Theta} = 1$ by Lemma 3.3. Hence, $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = (-1)^{\gamma \cdot \Theta}$.

(iv) $\mathrm{wind}(\gamma)$ must be either 2π or -2π since it is a simple closed curve that does not intersect itself. In addition, $\mathrm{wind}\big(\mathrm{p}(Q_e)\big) = \mathrm{wind}(\gamma) \mp \frac{\pi}{4}$. Therefore, the path $\mathrm{p}(Q_e)$ contributes to $\widetilde{\tau}(Q_e)$ with a factor of $-i\eta_{a_1^\to} e^{\pm \frac{\pi i}{8}}$.

(v) s_e matches s in the values $\{1, \dots, 2m\}$ and additionally $s_e(2m+1) = 2m+1, s_e(2m+2) = 2m+2$. As such, $\mathrm{sign}(s_e) = \mathrm{sign}(s)$.

(vi) The last two observations yield $\widetilde{\tau}(Q_e) = i\eta_{a_1^\to} e^{\pm \frac{\pi i}{8}} \tau(Q)$.

Factoring out the real terms, (35) becomes

$$\pm \eta_{a_1^\to} \alpha_c \cos \frac{\pi}{8} = \mathrm{Proj}_{l(a_1^\to)} \Big[-i\eta_{a_1^\to} e^{\pm \frac{\pi i}{8}} \Big]$$

and expanding the right-hand side now leads to

$$\pm \eta_{a_1^\to} \alpha_c \cos \frac{\pi}{8} = \pm \eta_{a_1^\to} \sin \frac{\pi}{8}$$

which is true because $\alpha_c = \tan \frac{\pi}{8}$.

2b. γ is a path (subfigure (b) of Figure 13).

Say γ runs from c_p to c_q. Removing (ev) and adding $(a_1^\to e)$ breaks γ into two paths, one running from c_p to e — which becomes $\mathrm{p}(Q_e)$ — and another running from $a_1^\to$ to c_q, which we call $\mathrm{q}(Q_e)$. We have

(i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q|-1}$.

(ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \Theta} = (-1)^{Q \setminus \big(\mathrm{p}(Q_e) \cup (ev)\big) \cdot \Theta}$.

(iii) Let π be a smooth path running from $a_1^\to$ to c_p such that $\pi \cdot \Theta = 0 \mod 2$. Then, $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = 1$ if and only if $\big\langle \pi, \mathrm{p}(Q_e), (ev), (va_1^\to) \big\rangle$ lifts to a loop, which by Lemma 3.3 is equivalent to $(-1)^{\langle \pi, \mathrm{p}(Q_e), (ev), (va_1^\to) \rangle \cdot \Theta} = 1$. As such, $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = (-1)^{\big(\mathrm{p}(Q_e) \cup (ev)\big) \cdot \Theta}$.

(iv) If $\mathrm{wind}\big(\mathrm{p}(Q_e)\big) = \alpha$ and $\mathrm{wind}\big(\mathrm{q}(Q_e)\big) = \beta$, then $\mathrm{wind}(\gamma) = \alpha + \beta \pm \frac{\pi}{4}$. Therefore, $\mathrm{p}(Q_e)$ and $\mathrm{q}(Q_e)$ contribute to $\widetilde{\tau}(Q_e)$ with a joint factor of $i\eta_{c_p} e^{-\frac{\alpha i}{2}} \cdot i\eta_{a_1^{\rightarrow}} \overline{\eta}_{c_q} e^{-\frac{\beta i}{2}}$, whereas γ contributes to $\tau(Q)$ with a factor of $i\eta_{c_p} \overline{\eta}_{c_q} e^{-\frac{(\alpha+\beta)i}{2}} e^{\mp\frac{\pi i}{8}}$.

(v) If we assume without loss of generality that γ is the last path in the listing of path of Q, then s matches s_e in the values $\{1, \dots, 2m-2\}$. For the remaining values, we have

$$s(2m-1) = p \qquad s(2m) = q$$

on the one hand and

$$s_e(2m-1) = p \qquad s_e(2m) = 2m+2 \qquad s_e(2m+1) = 2m+1 \qquad s_e(2m+2) = q$$

on the other. Therefore, $\mathrm{sign}(s_e) = -\,\mathrm{sign}(s)$.

(vi) The last two observations yield $\widetilde{\tau}(Q_e) = -i\eta_{a_1^{\rightarrow}} e^{\pm\frac{\pi i}{8}} \tau(Q)$.

Factoring out the real terms, (35) becomes

$$\pm\eta_{a_1^{\rightarrow}} \alpha_c \cos\frac{\pi}{8} = \mathrm{Proj}_{l(a_1^{\rightarrow})}\Big[- i\eta_{a_1^{\rightarrow}} e^{\pm\frac{\pi i}{8}} \Big]$$

which is the same equality proven in the previous subcase.

□

Moving on to the remaining corners, the answer here is unfortunately not quite as nice. For distinct indices k, j, we employ the notation

$$[\mathbf{u}]_k \equiv u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_{2m}$$
$$[\mathbf{u}]_{k,j} \equiv^{9} u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_{j-1}, u_{j+1}, \dots, u_{2m}$$

and analogously write $[\mathbf{c}]_k$ and $[\mathbf{c}]_{k,j}$.

Lemma 4.25. *Fix two distinct indices k, j and suppose that $u_k + \delta = v_k$. Let $e_\pm = c_k \pm \frac{i}{2}\delta$ and let $\widehat{\Gamma}_\delta \equiv \widehat{\Gamma}_\delta^{\left[[\boldsymbol{v}]_{k,j}\right]}$ be an additional set of disorder lines. Then*

$$\mathrm{Proj}_{l(c_k)}\big[F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}(\widetilde{e}_\pm)\big] = \pm\Big((-1)^{k+1}\widetilde{\tau}^0 \mathrm{sheet}_{\boldsymbol{a},\boldsymbol{u}}(\varepsilon, \widetilde{c}_k)\eta_{c_k}\Big) \frac{F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;\boldsymbol{a},u_k,u_j;[\boldsymbol{u}]_{k,j}\right]}(\widetilde{c}_j)}{F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;u_k,\boldsymbol{a},u_j;[\boldsymbol{u}]_{k,j}\right]}(\widetilde{c}_j)}$$

holds, with $\widetilde{\tau}^0 \in \{\pm 1\}$ is as defined in Proposition 4.8 using some $\widetilde{\Gamma}_\delta \equiv \widetilde{\Gamma}_\delta^{\left[[\boldsymbol{v}]_k, a_1+\delta\right]}$, Θ and some $\widetilde{\Theta}^0 \equiv \big(\widetilde{\Theta}^0\big)^{\left[[\boldsymbol{u}]_k, a_1\right]}$, under the following assumptions:

1. $\mathbb{E}^{\widehat{\Gamma},+}_{\Omega_\delta}[\sigma_{a_1} \cdots \sigma_{a_n}\sigma_{u_k}\sigma_{u_j}] \neq 0$.
2. *The lifts of $e_\pm$ and c_k are on the same sheet of $[\Omega_\delta; \boldsymbol{a}; \boldsymbol{u}]$.*

[9] Assuming $k < j$; otherwise, this should be $u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_{k-1}, u_{k+1}, \dots, u_{2m}$.

3. *Both lifts of $\widetilde{c}_j$ are on the same sheet of* $\left[\Omega_\delta; \boldsymbol{a}, u_k, u_j; [\boldsymbol{u}]_{k,j}\right] = \left[\Omega_\delta; u_k, \boldsymbol{a}, u_j; [\boldsymbol{u}]_{k,j}\right]$.

Remark 4.26. The term between parenthesis on the right-hand side equals $\pm i$. In addition, note that the spinors involved are "less complex" in the sense that $\widehat{\Gamma}_\delta$ has 2 less endpoints.

Remark 4.27. A good intuition as to why one would expect an answer like this can be found using (25). At $z = c_k$, the function becomes

$$\frac{\mathbb{E}^+\left[\left(\prod_{l=1}^{n}\sigma_{a_l}\right)\left(\prod_{l=1}^{2m}\sigma_{u_l}\right)\left(\prod_{\substack{l=1\\l\neq k}}^{2m}\psi_{c_l}\right)\psi_{a_1+\delta/2}\right]}{\mathbb{E}^+\left[\left(\prod_{l=1}^{n}\sigma_{a_l}\right)\left(\prod_{l=1}^{2m}\sigma_{u_l}\right)\left(\prod_{l=1}^{2m}\psi_{c_l}\right)\right]}$$

(ignoring the extra constants that come with replacing ψ with $\sigma\mu$) which can be rewritten as

$$\frac{\mathbb{E}^+\left[\left(\prod_{l=1}^{n}\sigma_{a_l}\cdot\sigma_{u_k}\cdot\sigma_{u_j}\right)\left(\prod_{\substack{l=1\\l\neq k,j}}^{2m}\sigma_{u_l}\right)\left(\prod_{\substack{l=1\\l\neq k,j}}^{2m}\psi_{c_l}\right)\psi_{a_1+\delta/2}\psi_{c_j}\right]}{\mathbb{E}^+\left[\left(\sigma_{u_k}\cdot\prod_{l=1}^{n}\sigma_{a_l}\cdot\sigma_{u_j}\right)\left(\prod_{\substack{l=1\\l\neq k,j}}^{2m}\sigma_{u_l}\right)\left(\prod_{\substack{l=1\\l\neq k,j}}^{2m}\psi_{c_l}\right)\psi_{u_k+\delta/2}\psi_{c_j}\right]} \tag{37}$$

where it is used $\psi_{c_k} = \psi_{u_k+\delta/2}$ because $u_k + \delta = v_k$. Dividing both numerator and denominator by $\mathbb{E}_{\Omega_\delta}^{\widehat{\Gamma},+}[\sigma_{a_1}\cdots\sigma_{a_n}\sigma_{u_k}\sigma_{u_j}]$ makes the spinors come out.

Proof. Define

$$R^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{c}_k) := \frac{(-1)^{k+1}\eta_{c_k}}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\sum_{Q\in\mathscr{C}_{\Omega_\delta}\left([\mathbf{c}]_k, a_1^{\rightarrow}\right)}\alpha_c^{|Q|}\cdot(-1)^{Q\cdot\Theta}\cdot\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_k)\cdot\tau(Q)$$

We prove the equality

$$\mathrm{Proj}_{l(c_k)}\left[F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e}_\pm)\right] = \pm R^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{c}_k)$$

and leave for Lemma 4.28 the expansion of $R^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{c}_k)$.

For simplicity, write $c \equiv c_k$ and $e \equiv e_\pm$. We follow the same strategy as before: the XOR difference with the half edges (cv) and (ve) defines a bijection between the sets $\mathscr{C}_{\Omega_\delta}\left([\mathbf{c}]_k, a_1^{\rightarrow}\right)$ and $\mathscr{C}_{\Omega_\delta}\left(\mathbf{c}, a_1^{\rightarrow}, e\right)$; therefore it is enough to prove that, for any $Q \in \mathscr{C}_{\Omega_\delta}\left([\mathbf{c}]_k, a_1^{\rightarrow}\right)$ and $Q_e = Q\oplus(cv)\oplus(ve) \in \mathscr{C}_{\Omega_\delta}\left(\mathbf{c}, a_1^{\rightarrow}, e\right)$, we have

$$(-1)^{k+1}\eta_c\cdot\alpha_c^{|Q|}\cdot(-1)^{Q\cdot\Theta}\cdot\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c})\cdot\tau(Q) = \mathrm{Proj}_{l(c)}\left[\frac{\alpha_c^{|Q_e|}\cdot\phi^{\mathbf{c}}_{\mathbf{a}}(Q_e,\widetilde{e})}{\cos\frac{\pi}{8}}\right] \tag{38}$$

There are two cases:

1. $(ve) \notin Q$.

   Fix a smoothing of Q with permutation $s \in S_{2m}$ and add to it the path $c \sim v \sim e$. This induces a valid smoothing of Q_e in which the added path is $\mathrm{p}(Q_e)$. Let $s_e \in S_{2m+2}$ be the associated permutation. We have

(i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q|}$.

(ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \Theta} = (-1)^{Q \cdot \Theta}$.

(iii) $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = \mathrm{sheet}_{\mathbf{a},\mathbf{u}}\big(\varepsilon, \widetilde{c}\big)$.

(iv) $\mathrm{wind}\big(\mathrm{p}(Q_e)\big) = \pm\frac{3\pi}{4}$.

(v) Changing from $\mathscr{C}_{\Omega_\delta}\big([\mathbf{c}]_k, a_1^{\rightarrow}\big)$ to $\mathscr{C}_{\Omega_\delta}\left(\mathbf{c}, a_1^{\rightarrow}, e_\pm\right)$ causes some of the corners to have a different label — the index of the corners $c_{k+1}, \ldots, c_{2m}, a_1^{\rightarrow}$ goes up by one. This is something that will happen in every case checking of this proof, and the best way to compare the sign of the permutations in these cases is to use the number of *inversions*: for any permutation ρ,

$$\mathrm{sign}(\rho) = (-1)^{N(\rho)} \quad , \quad N(\rho) := \big\{(i,j) : i < j, \rho(i) > \rho(j)\big\}$$

For $j \in \{1, \ldots, 2m\}$ we have $s_e(j) = s(j)$ if $s(j) < k$ and $s_e(j) = s(j) + 1$ otherwise. In addition, $s_e(2m+1) = k$ and $s_e(2m+2) = 2m+2$. Using the number of inversions to compare the sign of the permutations, we conclude $\mathrm{sign}(s_e) = (-1)^{2m-(k-1)}\,\mathrm{sign}(s) = (-1)^{k+1}\,\mathrm{sign}(s)$.

(vi) The last two observations yield $\widetilde{\tau}(Q_e) = (-1)^{k+1} i\eta_c e^{\mp\frac{3\pi i}{8}}$.

Factoring out the real terms in (38), it remains to prove that

$$\pm\eta_c \cos\frac{\pi}{8} = \mathrm{Proj}_{l(c)}\left[i\eta_c e^{\mp\frac{3\pi i}{8}}\right]$$

which holds similarly to (36).

2. $(ve) \in Q$.

Fix a smoothing of Q with permutation $s \in S_{2m}$ and let γ be the path or cycle that goes though (ev), which we assume without loss of generality that runs from e to v. Removing (ev) from γ and inserting (ce) in its place induces a valid smoothing of Q_e, with some associated permutation s_e. Again consider two subcases:

2a. γ is a cycle (subfigure (a) of Figure 13).

Then, replacing (ev) with (cv) in γ transforms it into a path, which is in fact $\mathrm{p}(Q_e)$. Then,

(i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q|-1}$.

(ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \Theta} = (-1)^{Q \setminus \gamma \cdot \Theta}$

(iii) Let π be a smooth path running from $a_1^{\rightarrow}$ to c such that $\pi \cdot \Theta = 0 \mod 2$. Then,

$$\begin{aligned}
\text{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = 1 &\Leftrightarrow \big\langle \pi, \mathrm{p}(Q_e)\big\rangle \text{ lifts to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{e} \\
&\quad \begin{cases} \pi \text{ lifts to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{z} \\ \gamma \text{ lifts to a loop} \end{cases} \\
&\Leftrightarrow \qquad\qquad \text{OR} \\
&\quad \begin{cases} \pi \text{ does not lift to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{z} \\ \gamma \text{ does not lift to a loop} \end{cases} \\
&\Leftrightarrow \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{c}) \cdot (-1)^{\gamma \cdot \Theta} = 1
\end{aligned}$$

using Lemma 3.3. Therefore, $\text{sheet}_{\mathbf{a},\mathbf{u}}\big(\mathrm{p}(Q_e), \widetilde{e}\big) = \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{z}) \cdot (-1)^{\gamma \cdot \Theta}$.

(iv) $\text{wind}(\gamma)$ must be either 2π or -2π, since it is a simple closed curve that does not intersect itself. In addition, $\text{wind}\big(\mathrm{p}(Q_e)\big) = \text{wind}(\gamma) \mp \frac{\pi}{4}$. Therefore, the path $\mathrm{p}(Q_e)$ contributes to $\widetilde{\tau}(Q_e)$ with a factor of $-i\eta_c e^{\pm \frac{\pi i}{8}}$.

(v) For $j \in \{1, \ldots, 2m\}$ we have $s_e(j) = s(j)$ if $s_e(j) < k$ and $s_e(j) = s(j)+1$ otherwise. In addition, $s_e(2m+1) = k$ and $s_e(2m+2) = 2m+2$. Hence, $\text{sign}(s_e) = (-1)^{2m-(k-1)} \text{sign}(s) = (-1)^{k+1} \text{sign}(s)$.

(vi) The last two observations yield $\widetilde{\tau}(Q_e) = -(-1)^{k+1} i\eta_c e^{\pm \frac{\pi i}{8}}$.

Factoring out the real factors in (38), we are left with proving

$$\pm \eta_c \alpha_c \cos \frac{\pi}{8} = \text{Proj}_{l(c)} \Big[- i\eta_c e^{\pm \frac{\pi i}{8}} \Big] \Leftrightarrow \pm \eta_c \alpha_c \cos \frac{\pi}{8} = \pm \eta_c \sin \frac{\pi}{8}$$

which holds because $\alpha_c = \tan \frac{\pi}{8}$.

2b. γ is a path (subfigure (b) of Figure 13).

Say γ runs from c_p to c_q. Removing (ev) and adding (ce) breaks γ into two paths, one running from c_p to e — which becomes $\mathrm{p}(Q_e)$ — and another running from c to c_q, which we call $\mathrm{q}(Q_e)$. We have

(i) $\alpha_c^{|Q_e|} = \alpha_c^{|Q|-1}$.

(ii) $(-1)^{Q_e \setminus \mathrm{p}(Q_e) \cdot \Theta} = (-1)^{Q \setminus \left(\mathrm{p}(Q_e) \cup (ev)\right)}$.

(iii) Let π_{c_p} and π_c be smooth paths running from $a_1^{\rightarrow}$ to c_p and c, respectively, such that

$\pi_{c_p} \cdot \Theta = \pi_c \cdot \Theta = 0 \mod 2$. Then,

$$\begin{aligned}
\text{sheet}_{\mathbf{a},\mathbf{u}}\big(\text{p}(Q_e), \widetilde{e}\big) = 1 &\Leftrightarrow \big\langle \pi_{c_p}, \text{p}(Q_e) \big\rangle \text{ lifts to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{e} \\
&\Leftrightarrow \big\langle \pi_{c_p}, \text{p}(Q_e), (ev), (vc) \big\rangle \text{ lifts to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } c \\
&\quad \begin{cases} \pi_c \text{ lifts to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{c} \\ \big\langle \pi_{c_p}, \text{p}(Q_e), (ev), (vc), \pi_c \big\rangle \text{ lifts to a loop} \end{cases} \\
&\Leftrightarrow \qquad\qquad \text{OR} \\
&\quad \begin{cases} \pi_c \text{ does not lift to a path from } \widetilde{a_1^{\rightarrow}} \text{ to } \widetilde{c} \\ \big\langle \pi_{c_p}, \text{p}(Q_e), (ev), (vc), \pi_c \big\rangle \text{ does not lift to a loop} \end{cases} \\
&\Leftrightarrow \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{c}) \cdot (-1)^{\langle \pi_{c_p}, \text{p}(Q_e), (ev), (vc), \pi_c \rangle \cdot \Theta} = 1 \\
&\Leftrightarrow \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{c}) \cdot (-1)^{\big(\text{p}(Q_e) \cup (ev)\big) \cdot \Theta} = 1
\end{aligned}$$

where we used Lemma 3.3. Therefore,

$$\text{sheet}_{\mathbf{a},\mathbf{u}}\big(\text{p}(Q_e), \widetilde{e}\big) = \text{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{c}) \cdot (-1)^{\big(\text{p}(Q_e) \cup (ev)\big) \cdot \Theta}$$

(iv) If $\text{wind}\big(\text{p}(Q_e)\big) = \alpha$ and $\text{wind}\big(\text{q}(Q_e)\big) = \beta$, then $\text{wind}(\gamma) = \alpha + \beta \pm \frac{\pi}{4}$. Therefore, $\text{p}(Q_e)$ and $\text{q}(Q_e)$ contribute to $\widetilde{\tau}(Q_e)$ with a factor of $i\eta_{c_p} e^{-\frac{\alpha i}{2}} \cdot i\eta_{a_1^{\rightarrow}} \overline{\eta}_{c_q} e^{-\frac{\beta i}{2}}$, whereas γ contributes to $\tau(Q)$ with a factor of $i\eta_{c_p} \overline{\eta}_{c_q} e^{-\frac{(\alpha+\beta)i}{2}} e^{\mp \frac{\pi i}{8}}$.

(v) Assume without loss of generality that γ is the last path in the listing of path of Q. To compare $\text{sign}(s)$ and $\text{sign}(s_e)$, we use an auxiliary permutation. Let $s' \in S_{2m+2}$ be defined as follows: for $j \in \{1, \ldots, 2m\}$, we have $s'(j) = s(j)$ if $s(j) < k$ and $s'(j) = s(j) + 1$ otherwise; in addition, $s'(2m+1) = k, s'(2m+2) = 2m+2$[10]. Then, $\text{sign}(s') = (-1)^{2m-(k-1)} \text{sign}(s) = (-1)^{k+1} \text{sign}(s)$. Comparing $\text{sign}(s')$ and $\text{sign}(s_e)$, we see that they match in $\{1, \ldots, 2m-2\}$ and

$$s'(2m-1) = p \qquad s'(2m) = q \qquad s'(2m+1) = k \qquad s'(2m+2) = 2m+2$$

whereas

$$s_e(2m-1) = p \qquad s_e(2m) = 2m+2 \qquad s_e(2m+1) = k \qquad s_e(2m+2) = q$$

which implies $\text{sign}(s_e) = -\text{sign}(s')$. Putting everything together, we find out that $\text{sign}(s_e) = -(-1)^{k+1} \text{sign}(s)$.

– The last two observations yield $\widetilde{\tau}(Q_e) = -(-1)^{k+1} i\eta_{a_1^{\rightarrow}} e^{\pm \frac{\pi i}{8}} \tau(Q)$.

[10] This corresponds to the change of labels when going from $\mathscr{C}_{\Omega_\delta}\big([\mathbf{c}]_k, a_1^{\rightarrow}\big)$ to $\mathscr{C}_{\Omega_\delta}\big(\mathbf{c}, a_1^{\rightarrow}, e\big)$, assuming the new path runs from c to e.

Factoring out the real factors in (38) leads to

$$\pm\eta_c\alpha_c\cos\frac{\pi}{8} = \mathrm{Proj}_{l(c)}\Big[-i\eta_c e^{\pm\frac{\pi i}{8}}\Big]$$

as was the case previously.

□

Lemma 4.28. *If $u_k+\delta=v_k$ then*

$$R^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}(\widetilde{c}_k) = \Big((-1)^{k+1}\widetilde{\tau}^0\mathrm{sheet}_{\boldsymbol{a},\boldsymbol{u}}(\varepsilon,\widetilde{c}_k)\eta_{c_k}\Big)\frac{F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;\boldsymbol{a},u_k,u_j;[\boldsymbol{u}]_{k,j}\right]}(\widetilde{c}_j)}{F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;u_k,\boldsymbol{a},u_j;[\boldsymbol{u}]_{k,j}\right]}(\widetilde{c}_j)}$$

under the assumptions of Lemma 4.25.

Proof. Using Propositions 4.8 and 4.16,

$$\begin{aligned}
F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{c}_k) &= \frac{(-1)^{k+1}\eta_{c_k}}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\sum_{Q\in\mathscr{C}_{\Omega_\delta}([\mathbf{c}]_k,a_1^{\rightarrow})}\alpha_c^{|Q|}\cdot(-1)^{Q\cdot\Theta}\cdot\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_k)\cdot\tau(Q)\\
&= \frac{(-1)^{k+1}\eta_{c_k}\,\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_k)}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\cdot\widetilde{\tau}^0\mathcal{Z}^{\widetilde{\Gamma}_\delta,+}_{\Omega_\delta}\cdot\mathbb{E}^{\widetilde{\Gamma}_\delta,+}_{\Omega_\delta}[\sigma_{a_2}\cdots\sigma_{a_n}\sigma_{u_k}]\\
&\equiv \frac{(-1)^{k+1}\eta_{c_k}\,\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_k)}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\cdot\widetilde{\tau}^0\mathcal{Z}^{\widetilde{\Gamma}_\delta,+}_{\Omega_\delta}\cdot\mathbb{E}^{\widetilde{\Gamma}_\delta^{\left[[\mathbf{v}]_{k,j},a_1+\delta,v(c_j)\right]},+}_{\Omega_\delta}[\sigma_{f(c_j)}\cdot\sigma_{a_2}\cdots\sigma_{a_n}\sigma_{u_k}\sigma_{u_j}]\\
&= \frac{(-1)^{k+1}\eta_{c_k}\,\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_k)}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\cdot\widetilde{\tau}^0\mathcal{Z}^{\widetilde{\Gamma}_\delta,+}_{\delta}\cdot\widetilde{\tau}^1\overline{\eta}_{c_j}\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_j)\frac{\mathcal{Z}^{\widehat{\Gamma}_\delta,+}}{\mathcal{Z}^{\widetilde{\Gamma}_\delta,+}}\cdot\mathbb{E}^{\widehat{\Gamma}_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}\sigma_{u_k}\sigma_{u_j}]\cdot\\
&\quad\cdot F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;\mathbf{a},u_k,u_j;[\mathbf{u}]_{k,j}\right]}(\widetilde{c}_j)\\
&= \Big[(-1)^{k+1}\widetilde{\tau}^0\widetilde{\tau}^1\eta_{c_k}\overline{\eta}_{c_j}\,\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_k)\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_j)\Big]\frac{\mathcal{Z}^{\widehat{\Gamma}_\delta,+}}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]}\mathbb{E}^{\widehat{\Gamma}_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}\sigma_{u_k}\sigma_{u_j}]\cdot\\
&\quad\cdot F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;\mathbf{a},u_k,u_j;[\mathbf{u}]_{k,j}\right]}(\widetilde{c}_j)
\end{aligned}$$

with $\widetilde{\tau}^1\in\{\pm1\}$ as defined in Proposition 4.8 using $\widetilde{\Gamma}_\delta$, Θ and some $\widetilde{\Theta}^1$ linking the faces $[\mathbf{u}]_{k,j},a_1,v_j$[11].

Now, let us expand the normalizing factor. Assuming $u_k+\delta=v_k$ and applying Proposition 4.16,

$$\begin{aligned}
\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}] &= \mathcal{Z}^{\Gamma,+}_{\Omega_\delta}\cdot\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]\\
&\equiv \mathcal{Z}^{\Gamma,+}_{\Omega_\delta}\cdot\mathbb{E}^{\Gamma^{\left[[\mathbf{v}]_{k,j},u_k+\delta,v(c_j)\right]},+}_{\Omega_\delta}[\sigma_{f(c_j)}\cdot\sigma_{a_1}\cdots\sigma_{a_n}\sigma_{u_j}]\\
&= \mathcal{Z}^{\Gamma,+}_{\Omega_\delta}\cdot\widetilde{\tau}^1\overline{\eta}_{c_j}\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_j)\cdot\frac{\mathcal{Z}^{\widehat{\Gamma}_\delta,+}_{\delta}}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}}\cdot\mathbb{E}^{\widehat{\Gamma}_\delta,+}_{\delta}[\sigma_{a_1}\cdots\sigma_{a_n}\sigma_{u_k}\sigma_{u_j}]\cdot F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;u_k,\mathbf{a},u_j;[\mathbf{u}]_{k,j}\right]}(\widetilde{c}_j)\\
&= \Big[\widetilde{\tau}^1\overline{\eta}_{c_j}\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_j)\Big]\mathcal{Z}^{\widehat{\Gamma}_\delta,+}_{\delta}\mathbb{E}^{\widehat{\Gamma}_\delta,+}_{\delta}[\sigma_{a_1}\cdots\sigma_{a_n}\sigma_{u_k}\sigma_{u_j}]\cdot F^{\widehat{\Gamma}_\delta}_{\left[\Omega_\delta;u_k,\mathbf{a},u_j;[\mathbf{u}]_{k,j}\right]}(\widetilde{c}_j)
\end{aligned}$$

[11]We may not have $\widetilde{\tau}^0=\widetilde{\tau}^1$ due to the order of the corners — Remark 4.6.

and the third assumption of Lemma 4.25 allows us to cancel the $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon, \widetilde{c}_j)$ factor, yielding the stated. □

Remark 4.29. The s-holomorphicity of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ and Lemmas 4.23 and 4.25 are the only statements relying on the Ising model being at the critical temperature.

PART III

PASSING TO THE SCALING LIMIT

5 S-holomorphicity in square lattices

This section focuses on the concept of s-holomorphicity, a vital tool to pass the discrete spinors to the scaling limit. The exposition that follows is heavily motivated by the previous section but does not require any of its results. Our main reference will be [CS12], and we warn that the definitions are equivalent after the multiplication by $\sqrt{i}$. Other references are [CS11] and the survey [Smi10b].

5.1 Motivating s-holomorphicity: strategy for the convergence proof

Given some domain Ω, we considered some discretization Ω_δ of Ω with a square grid and studied the Ising model in the discrete setting. In the previous section, we defined the discrete spinors $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ o the canonical double cover $[\Omega_\delta;\mathbf{a};\mathbf{u}]$ with some branching points $\mathbf{b} \equiv b_1, \ldots, b_r$ (which are $\mathbf{a}$ and $\mathbf{u}$). From these spinors we can extract information regarding the Ising model (namely, Propositions 4.17 and 4.18). To progress, we need to pass to the scaling limit: given a family of discrete domains Ω_δ such that $\Omega_\delta \xrightarrow{\delta\to 0} \Omega$ in some sense, we want to prove $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]} \xrightarrow{\delta\to 0} f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}$ for a complex function $f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}$ defined on $[\Omega;\mathbf{a};\mathbf{u}]$.

Intuitively, this $f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}$ should share some of the properties of the $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ passed to the limit. Considering Proposition 4.21[12], these would be:

1. It is holomorphic on $[\Omega;\mathbf{a};\mathbf{u}]$[13].

2. It has multiplicative monodromy -1 around each branching point $b_1, \ldots, b_r$.

3. For every $\widetilde{z} \in [\partial\Omega;\mathbf{a};\mathbf{u}]$, $f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}$.

This reasoning faces a difficulty: the boundary condition is not robust enough to pass to the limit. Because Ω_δ is always a square grid rotated by an angle of $\frac{\pi}{4}$, the discrete version of $\nu_{\text{out}}(z)$ can only take the values of $e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}$ and $e^{\frac{7\pi i}{4}}$, hence we may not have $\nu^{\Omega_\delta}_{\text{out}}(z) \xrightarrow{\delta\to 0} \nu^{\Omega}_{\text{out}}(z)$.

A solution to this problem is to integrate the square of $f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}$. Note that

$$f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R} \Leftrightarrow \left(f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}\right)^2(z) \cdot i\nu_{\text{out}}(z) \in i\mathbb{R}^+_0$$

and $i\nu_{\text{out}}(\widetilde{z})$ is now tangent to Ω at z. Consider an antiderivative h of $(f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]})^2$, which verifies

$$h(v) - h(u) = \int_\gamma \left(f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}\right)^2(z)\, dz$$

for any path γ running from u to v. If $\gamma \subseteq \partial\Omega$ then the integrand must be imaginary. Hence, h verifies the boundary condition $\Re(h) \equiv Cte$, which passes to the limit naturally and is generally more pleasant to deal with.

This strategy runs into a technical issue: it requires a definition of a "discrete primitive" of $(F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]})^2$. This is problematic because, under usual definitions of discrete holomorphicity, there is no guarantee that

[12] We will require some additional knowledge of $f^\Gamma_{[\Omega;\mathbf{a};\mathbf{u}]}$ around the branching points, which is given by Lemmas 4.23 and 4.25; see Section 6.

[13] We ignore for now that the s-holomorphicity of the discrete version fails at some points.

the square of a discrete holomorphic function is discrete holomorphic, and so there may not be a well-defined primitive.

The solution for this is a rather astounding observation first described in [Smi06]. By requiring that $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ is s-holomorphic, a stronger version of the usual discrete holomorphicity, it is possible to provide a suitable definition of $\Re \int \left(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}\right)^2$. In addition, this function shares many of the properties of discrete harmonic functions, as one would expect.

5.2 Discrete holomorphic and harmonic functions

Our setup is a discrete domain Ω_δ lifted to the canonical double cover $[\Omega_\delta; \mathbf{b}]$ with some branching points $\mathbf{b} \in \operatorname{Int} \mathcal{F}_{\Omega_\delta}$ (recall that we treat the double cover locally as $\mathbb{C}$). In Ω_δ, we take the points $\mathcal{V}_{\Omega_\delta}$, $\mathcal{F}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta}$ (and in the near future, $\mathcal{C}_{\Omega_\delta}$) as is exemplified in Figure 6. This lattice will be considered from a different perspective: we will focus on the edge midpoints $\mathcal{E}_{\Omega_\delta}$ — note that these form a non-rotated square lattice of side δ —, where the functions are defined. From this perspective, $\mathcal{V}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta}$ are the faces of the dual lattice coloured in a chequerboard fashion, and this is where primitives will be defined (Figure 14). Finally, we will restrict ourselves to functions that have opposite signs in different sheets.

An intuitive way of defining a discrete version of a holomorphic function is to ask for a discrete version of the Cauchy-Riemann equations $\partial_{i\alpha} F = i\partial_\alpha F$, and there are multiple ways of doing so. For $e_{NW}, e_{SW}, e_{SE}, e_{NE} \in \mathcal{E}_{\Omega_\delta}$ which are vertices of a square of side δ starting in the upper left corner and going counter-clockwise (Figure 15), one possibility is to require that

$$\frac{F(\widetilde{e}_{NW}) - F(\widetilde{e}_{SW})}{\widetilde{e}_{NW} - \widetilde{e}_{SW}} = \frac{F(\widetilde{e}_{SE}) - F(\widetilde{e}_{SW})}{\widetilde{e}_{SE} - \widetilde{e}_{SW}} \Leftrightarrow F\big(\widetilde{e}_{NW}\big) - F\big(\widetilde{e}_{SW}\big) = i\Big[F\big(\widetilde{e}_{SE}\big) - F\big(\widetilde{e}_{SW}\big)\Big],$$

a definition first proposed in [Isa41]. However, this identity is not symmetric with respect to lattice rotations. The same author also proposed another definition that does not have this issue and appears more commonly in probabilistic arguments, including our own. This is the one we consider.

Definition 5.1. A function $F : [E; \mathbf{b}] \longrightarrow \mathbb{C}$ defined on the lift of a set $E \subseteq \mathcal{E}_{\Omega_\delta}$ is *discrete holomorphic* or *preholomorphic* if, for every $e_{NW}, e_{SW}, e_{SE}, e_{NE} \in E$ that are vertices of a square of side δ (starting in the upper left corner and going counter-clockwise) such that no branching point $\mathbf{b}$ is in the center of the square, we have

$$\frac{F(\widetilde{e}_{NW}) - F(\widetilde{e}_{SE})}{\widetilde{e}_{NW} - \widetilde{e}_{SE}} = \frac{F(\widetilde{e}_{NE}) - F(\widetilde{e}_{SW})}{\widetilde{e}_{NE} - \widetilde{e}_{SW}} \Leftrightarrow F\big(\widetilde{e}_{NW}\big) - F\big(\widetilde{e}_{SE}\big) = i\Big[F\big(\widetilde{e}_{NE}\big) - F\big(\widetilde{e}_{SW}\big)\Big] \tag{39}$$

for both lifts of the edge midpoints, as long as all four are on the same sheet of $[\Omega_\delta; \mathbf{b}]$.

Note that the double cover setting is mostly a technicality, given that we only work with functions that have opposite signs on opposite sheets. This definition, as well as future ones, can (and will) be considered on the usual plane.

Remark 5.2. Similar to the Cauchy-Riemann equations, (39) translates into two restrictions (its real and imaginary parts). By colouring $\mathcal{E}_{\Omega_\delta}$ in a chequerboard fashion, we see that one restriction relates

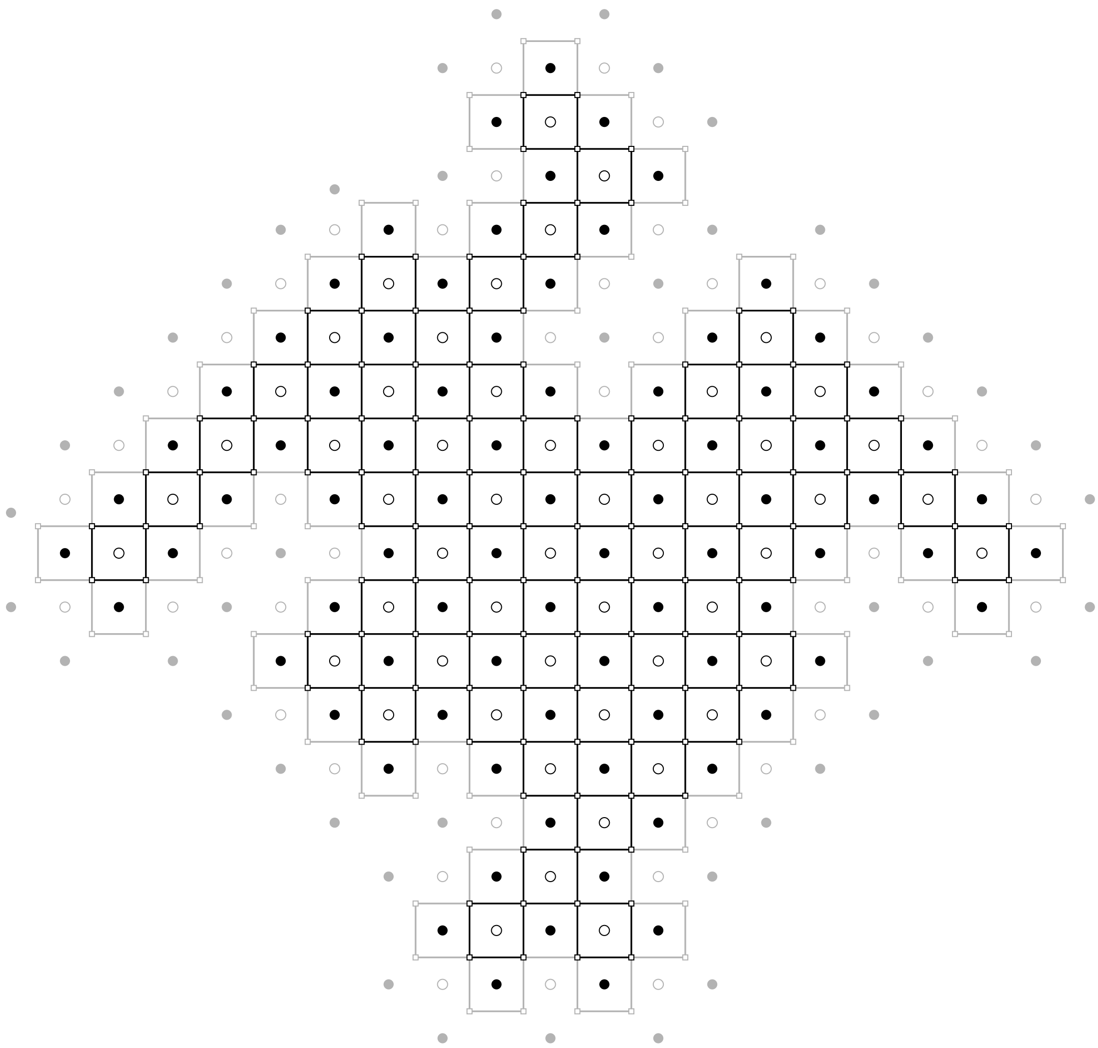

Figure 14: Example of the sites $\mathcal{V}_{\Omega_\delta}$, $\mathcal{F}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta}$ from Figure 6 seen in the new perspective: the edges drawn here connect sites of $\mathcal{E}_{\Omega_\delta}$. The coloring of the edges depends on $\mathcal{C}_{\Omega_\delta}$ (not shown), as explained in Figure 17.

the real part at the white vertices and imaginary part at the black vertices, and vice-versa for the other. Hence, in a sense, F splits into these two functions, each verifying a separate set of equations from the other.

There is an equivalent way of stating (39). If $z \in \mathcal{F}_{\Omega_\delta}$ ($\in \mathcal{V}_{\Omega_\delta}$) is the site in the center of the square $[e_{NW}e_{SW}e_{SE}e_{NE}]$, let $w_N, w_W, w_S, w_E \in \mathcal{V}_{\Omega_\delta}$ ($\in \mathcal{F}_{\Omega_\delta}$) be the sites directly above, to the left, below and to the right of z, respectively. Then, (39) is equivalent to

$$(\widetilde{w}_W - \widetilde{w}_N)F(\widetilde{e}_{NW}) + (\widetilde{w}_S - \widetilde{w}_W)F(\widetilde{e}_{SW}) + (\widetilde{w}_E - \widetilde{w}_S)F(\widetilde{e}_{SE}) + (\widetilde{w}_N - \widetilde{w}_E)F(\widetilde{e}_{NE}) = 0 \qquad (40)$$

and the left-hand side can be interpreted as a discrete version of the counter-clockwise path integral around the square $[\widetilde{w}_N\widetilde{w}_W\widetilde{w}_S\widetilde{w}_E]$. Furthermore, (40) generalizes nicely to non-square lattices [CS12].

The sum of discrete holomorphic functions is trivially discrete holomorphic, but the same is not obvious for products (with product defined pointwise). If we try to plug some common holomorphic functions in (39), we quickly find out that not all of them satisfy it. For instance, restrictions of 1, z and

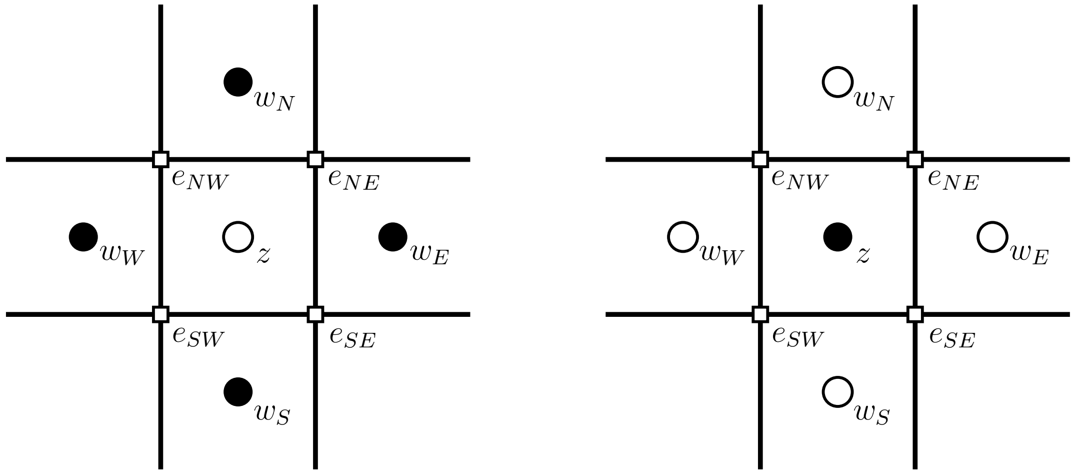

Figure 15: Setting for discrete holomorphism definitions, depending on whether $z \in \mathcal{F}_{\Omega_\delta}$ (left) or $z \in \mathcal{V}_{\Omega_\delta}$ (right).

z^2 to $\mathcal{E}_{\Omega_\delta}$ are discrete holomorphic, whereas z^3 and further powers are not — as it turns out, restrictions of holomorphic functions may verify (39) only up to an error of $O(\delta^3)$. This illustrates the main problem with this definition and similar variants: the product of two discrete holomorphic functions may not be discrete holomorphic. Nevertheless, we can still mimic some of the usual complex analysis theory.

We define the discrete version of the Laplacian operator. Again, several different possibilities exist, but one stands out as the simplest and most intuitive.

Definition 5.3. Given a function $H : [V \cup F; \mathbf{b}] \longrightarrow \mathbb{C}$ defined on the lifts of sets $V \subseteq \mathcal{V}_{\Omega_\delta}$ and $F \subseteq \mathcal{F}_{\Omega_\delta}$, define its *discrete Laplacian* wherever possible as

$$[\Delta_\delta H](\widetilde{z}) := \frac{1}{2\delta^2} \sum_{y \sim z} H(\widetilde{y}) - H(\widetilde{z}), \qquad z \in \operatorname{Int} \mathcal{V}_{\Omega_\delta} \cup \operatorname{Int} \mathcal{F}_{\Omega_\delta} \setminus \{\mathbf{b}\}$$

where the neighbours y are of the same type as z: if $z \in \mathcal{V}_{\Omega_\delta} (\in \mathcal{F}_{\Omega_\delta})$ then all $y \in \mathcal{V}_{\Omega_\delta}$ $(\in \mathcal{F}_{\Omega_\delta})$.

Definition 5.4. A function $H : [V \cup F; \mathbf{b}] \longrightarrow \mathbb{C}$ is *discrete harmonic* or *preharmonic* if $\Delta_\delta H(\widetilde{z}) = 0$ at all points where its discrete Laplacian is defined.

Remark 5.5. The $2\delta^2$ factor corresponds to the area of the square formed by the neighbours of z.

Remark 5.6. The Laplacian accounting only for the difference of H with respect to neighbours of the same type reflects the two independent parts of an antiderivative, which will be seen in Proposition 5.13.

We now define the discrete versions of the Wirtinger derivatives $\partial/\partial z$ and $\partial/\partial \overline{z}$.

Definition 5.7. Let $H : [V \cup F; \mathbf{b}] \longrightarrow \mathbb{C}$ be a function defined on the lifts of sets $V \subseteq \mathcal{V}_{\Omega_\delta}$ and $F \subseteq \mathcal{F}_{\Omega_\delta}$. For $e \in \operatorname{Int} \mathcal{E}_{\Omega_\delta}$, let $v_1, v_2 \in \operatorname{Int} \mathcal{V}_{\Omega_\delta}$ be its endpoints and $f_1, f_2 \in \operatorname{Int} \mathcal{F}_{\Omega_\delta}$ the faces it separates. Define

$$[\partial_\delta H](\widetilde{e}) := \frac{1}{2}\left(\frac{H(\widetilde{f}_2) - H(\widetilde{f}_1)}{f_2 - f_1} + \frac{H(\widetilde{v}_2) - H(\widetilde{v}_1)}{v_2 - v_1} \right)$$

wherever possible, with all lifts on the same sheet of $[\Omega_\delta; \mathbf{b}]$.

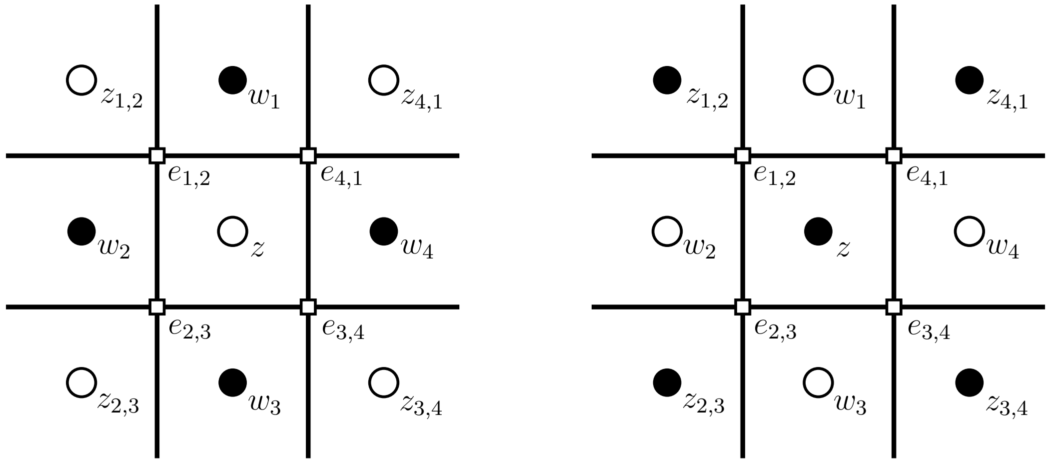

Figure 16: Setting for proofs, depending on whether $z \in \mathcal{F}_{\Omega_\delta}$ (left) or $z \in \mathcal{V}_{\Omega_\delta}$ (right).

Remark 5.8. The motivation for this definition may not be clear at first glance, especially since there are multiple dispositions of the vertices and faces. Consider the configuration provided by Figure 16 on the left, taking $e \equiv e_{4,1}, v_1 \equiv w_4, v_2 \equiv w_2, f_1 \equiv z, f_2 \equiv z_{4,1}$. Then,

$$\begin{aligned}
\frac{\partial H}{\partial z}(\widetilde{e}) &\equiv \frac{1}{2}\left(\frac{\partial H}{\partial x}(\widetilde{e}) - i\frac{\partial H}{\partial y}(\widetilde{e})\right) \\
&\equiv e^{-i\frac{\pi}{4}}\frac{1}{2}\left(\frac{H(\widetilde{f}_2) - H(\widetilde{f}_1)}{\sqrt{2}\delta} - i\frac{H(\widetilde{v}_2) - H(\widetilde{e}_1)}{\sqrt{2}\delta}\right) \\
&= e^{-i\frac{\pi}{4}}\frac{1}{2}\left(\frac{H(\widetilde{f}_2) - H(\widetilde{f}_1)}{e^{-i\frac{\pi}{4}}(f_2 - f_1)} + \frac{H(\widetilde{v}_2) - H(\widetilde{e}_1)}{e^{-i\frac{\pi}{4}}(v_2 - v_1)}\right) \\
&= \frac{1}{2}\left(\frac{H(\widetilde{f}_2) - H(\widetilde{f}_1)}{f_2 - f_1} + \frac{H(\widetilde{v}_2) - H(\widetilde{v}_1)}{v_2 - v_1}\right)
\end{aligned}$$

and note how this definition is invariant under the changes $v_1 \leftrightarrow v_2$ and $f_1 \leftrightarrow f_2$. The other configurations yield the same result.

Definition 5.9. Let $F : [E; \mathbf{b}] \longrightarrow \mathbb{C}$ be a function defined on the lift of a set $E \subseteq \mathcal{E}_{\Omega_\delta}$. For $z \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ ($\operatorname{Int}\mathcal{F}_{\Omega_\delta} \setminus \{\mathbf{b}\}$), let $w_1, w_2, w_3, w_4 \in \mathcal{F}_{\Omega_\delta}$ ($\mathcal{V}_{\Omega_\delta}$) be the sites directly above, to the left, below and to the right of z, respectively. Define

$$[\overline{\partial}_\delta F](\widetilde{z}) := -\frac{i}{4\delta^2}\sum_{k=1}^{4}(w_{k+1} - w_k)F\left(\frac{\widetilde{w}_{k+1} + \widetilde{w}_k}{2}\right)^{14}$$

wherever possible, with all lifts on the same sheet of $[\Omega_\delta; \mathbf{b}]$.

Note that a function $F : [E; \mathbf{b}] \longrightarrow \mathbb{C}$ is discrete holomorphic if $\overline{\partial}_\delta F(\widetilde{z}) = 0$ in all points where $\overline{\partial}_\delta F$ is defined.

Remark 5.10. Let us also motivate this definition. Setting $e_{k,k+1} = \frac{1}{2}(w_k + w_{k+1})$ (Figure 16) and

[14] Recall the abuse of notation to denote the lift of $\frac{w_{k+1}+w_k}{2}$ that is in the same sheet as $\widetilde{z}_{k+1}$ and $\widetilde{z}_k$.

taking the real axis in the direction $\overrightarrow{e_{23}e_{41}}$ and the imaginary axis in the direction $\overrightarrow{e_{34}e_{12}}$, we have

$$\begin{aligned}\frac{\partial F}{\partial \overline{z}}(\widetilde{z}) &\equiv \frac{1}{2}\left(\frac{\partial F}{\partial x}(\widetilde{z}) + i\frac{\partial F}{\partial y}(\widetilde{z})\right)\\ &\equiv \frac{1}{2}\left(\frac{F(\widetilde{e}_{41}) - F(\widetilde{e}_{23})}{\sqrt{2}\delta} + i\frac{F(\widetilde{e}_{12}) - F(\widetilde{e}_{34})}{\sqrt{2}\delta}\right)\\ &= \frac{1}{2\sqrt{2}\delta}\Big(iF(\widetilde{e}_{12}) - F(\widetilde{e}_{23}) - iF(\widetilde{e}_{34}) + F(\widetilde{e}_{41})\Big)\\ &= \frac{1}{2\sqrt{2}\delta}\left(iF(\widetilde{e}_{12})\frac{w_2 - w_1}{-\sqrt{2}\delta} - F(\widetilde{e}_{23})\frac{w_3 - w_2}{-i\sqrt{2}\delta} - iF(\widetilde{e}_{34})\frac{w_4 - w_3}{\sqrt{2}\delta} + F(\widetilde{e}_{41})\frac{w_1 - w_4}{i\sqrt{2}\delta}\right)\\ &= -\frac{i}{4\delta^2}\sum_{k=1}^{4}(w_{k+1} - w_k)F(\widetilde{e}_{k,k+1})\end{aligned}$$

We check the discrete version of the factorization $\Delta = 4\partial/\partial\overline{z} \cdot \partial/\partial z$ holds.

Proposition 5.11. $\Delta_\delta = 4\overline{\partial}_\delta\partial_\delta$ *whenever the right-hand side is defined.*

Proof. We prove $\Delta H(\widetilde{z}) = 4\overline{\partial}_\delta\partial_\delta H(\widetilde{z})$ for $z \in \operatorname{Int}\mathcal{F}_{\Omega_\delta}$ $(\operatorname{Int}\mathcal{V}_{\Omega_\delta})$. Let $w_1, w_2, w_3, w_4 \in \mathcal{V}_{\Omega_\delta}$ $(\mathcal{F}_{\Omega_\delta})$ be the sites directly above, to the left, below and to the right of z, respectively. For $k = 1, \ldots, 4$ let $e_{k,k+1} = (w_k w_{k+1})$ and $z_{k,k+1}$ the face (vertex) adjacent to $e_{k,k+1}$ different from z (Figure 16). Assume H is defined at all $\widetilde{z}_{k,k+1}$, $\widetilde{w}_{k,k+1}$ and $\widetilde{z}$ and these lifts are on the same sheet of $[\Omega_\delta; \mathbf{b}]$.

Since $z_{k,k+1} - z = -i(w_{k+1} - w_k)$, we have

$$\begin{aligned}\partial_\delta H(\widetilde{e}_{k,k+1}) &= \frac{1}{2}\left(\frac{H(\widetilde{w}_{k+1}) - H(\widetilde{w}_k)}{w_{k+1} - w_k} + \frac{H(\widetilde{z}_{k,k+1}) - H(z)}{\widetilde{z}_{k,k+1} - z}\right)\\ &= \frac{\big(H(\widetilde{w}_{k+1}) - H(\widetilde{w}_k)\big) + i\big(H(\widetilde{z}_{k,k+1}) - H(\widetilde{z})\big)}{2(w_{k+1} - w_k)}\end{aligned}$$

and so

$$\begin{aligned}\overline{\partial}_\delta\partial_\delta H(\widetilde{z}) &= -\frac{i}{4\delta^2}\sum_{k=1}^{4}(w_{k+1} - w_k)\cdot\partial_\delta H(\widetilde{e}_{k,k+1})\\ &= -\frac{i}{8\delta^2}\left(\sum_{k=1}^{4} H(\widetilde{z}_{k+1}) - H(\widetilde{z}_k)\right) + \frac{1}{8\delta^2}\left(\sum_{k=1}^{4} H(\widetilde{z}_{k,k+1}) - H(\widetilde{z})\right)\\ &= \frac{1}{4}\Delta_\delta H(\widetilde{z})\end{aligned}$$

□

From this result follows that if H is discrete harmonic then $F = \partial_\delta H$ is discrete holomorphic. Conversely, in simply connected domains, if F is discrete holomorphic then we can define a discrete harmonic "antiderivative" H, which is composed of two parts: one defined on $\operatorname{Int}\mathcal{V}_{\Omega_\delta}$ and another defined on $\operatorname{Int}\mathcal{F}_{\Omega_\delta}$.

Definition 5.12. Given a function $F : [E; \mathbf{b}] \longrightarrow \mathbb{C}$ defined on the lift of a set $E \subseteq \mathcal{E}_{\Omega_\delta}$ and $\widetilde{\gamma} = \widetilde{z}_1 \sim \ldots \sim \widetilde{z}_n$ the lift of a path of $\mathcal{G}^\dagger_{\Omega_\delta}$ or $\mathcal{G}_{\Omega_\delta}$, define the *line integral* of F along $\widetilde{\gamma}$ as

$$\int_{\widetilde{\gamma}} F(\widetilde{z})\,d\widetilde{z} := \sum_{k=1}^{n-1} F\left(\frac{\widetilde{z}_{k+1} + \widetilde{z}_k}{2}\right)(z_{k+1} - z_k)$$

Proposition 5.13. *If Ω_δ is simply connected and $F : [\mathcal{E}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ is discrete holomorphic, then there exist two discrete harmonic functions $H|_{\mathcal{V}_{\Omega_\delta}} : [\mathcal{V}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ and $H|_{\mathcal{F}_{\Omega_\delta}} : [\mathcal{F}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$, each unique up to an additive constant, such that*

$$H(\widetilde{z}_2) - H(\widetilde{z}_1) = \int_{\widetilde{z}_1}^{\widetilde{z}_2} F(\widetilde{z})\, d\widetilde{z}$$

where either $z_1, z_2 \in \mathcal{V}_{\Omega_\delta}$ or $z_1, z_2 \in \mathcal{F}_{\Omega_\delta}$ and the integral is computed along the lift of any path in $\mathcal{G}^\dagger_{\Omega_\delta}$ or $\mathcal{G}_{\Omega_\delta}$ running from $\widetilde{z}_1$ to $\widetilde{z}_2$.

Proof. We start by proving the integral is independent of the path chosen, which amounts to checking the integral of F along any closed path $\widetilde{\gamma}$ is 0. Assuming for now $\widetilde{\gamma}$ does not repeat vertices (faces) and there are no branching points in its interior, let I be the set of lifted faces (vertices) in the interior of $\widetilde{\gamma}$ and note that $I \subseteq [\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}; \mathbf{b}]$ $\big([\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}; \mathbf{b}]\big)$ because Ω_δ is simply connected. Then,

$$\int_{\widetilde{\gamma}} F(\widetilde{z})\, d\widetilde{z} = \pm \sum_{\widetilde{z} \in I} \overline{\partial}_\delta F(\widetilde{z})$$

where the sign depends on the orientation of $\widetilde{\gamma}$. To see that this is indeed true, expand the sum and group the contributions of each edge; the ones from edges inside γ are cancelled out. Using the discrete holomorphism hypothesis,

$$\pm \sum_{\widetilde{z} \in I} \overline{\partial}_\delta F(\widetilde{z}) = 0.$$

In addition, if $z_1, z_2, z_3, z_4 \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ $(\mathrm{Int}\,\mathcal{F}_{\Omega_\delta})$ are the four sites adjacent to a branching point b in counter-clockwise and

$$\widetilde{\gamma} = \widetilde{z}_1^1 \sim \cdots \sim \widetilde{z}_4^1 \sim \widetilde{z}_1^2 \sim \cdots \widetilde{z}_4^2 \sim \widetilde{z}_1^1$$

where $\{\widetilde{z}_k^1, \widetilde{z}_k^2\}$ are the two lifts of z_k (in other words, $\widetilde{\gamma}$ is the smallest loop around b), then $\int_{\widetilde{\gamma}} F(\widetilde{z})\, d\widetilde{z} = 0$ because F has opposite signs in different sheets. Combining these arguments, one concludes that $\int_{\widetilde{\gamma}} F(\widetilde{z})\, d\widetilde{z} = 0$ for any generic curve that does not repeat sites. If $\widetilde{\gamma}$ does repeat sites, divide it into a collection of curves that do not repeat sites and use the same arguments.

It is easy to see $H|_{\mathcal{V}_{\Omega_\delta}}$ $(H|_{\mathcal{F}_{\Omega_\delta}})$ is well-defined after we fix any value at a single point. Finally, for $z \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ $(\mathrm{Int}\,\mathcal{F}_{\Omega_\delta})$, define $w_k, e_{k,k+1}$ and $z_{k,k+1}$ as shown in Figure 16. As before, $z_{k,k+1} - z = -i(w_{k+1} - w_k)$ and thus

$$\begin{aligned}
\Delta_\delta H(\widetilde{z}) &= \frac{1}{2\delta^2} \sum_{k=1}^{4} H(\widetilde{z}_{k,k+1}) - H(\widetilde{z}) \\
&= \frac{1}{2\delta^2} \sum_{k=1}^{4} (\widetilde{z}_{k,k+1} - z) F(\widetilde{e}_{k,k+1}) \\
&= -\frac{i}{2\delta^2} \sum_{k=1}^{4} (\widetilde{w}_{k+1} - w_k) F(\widetilde{e}_{k,k+1}) \\
&= 0
\end{aligned}$$

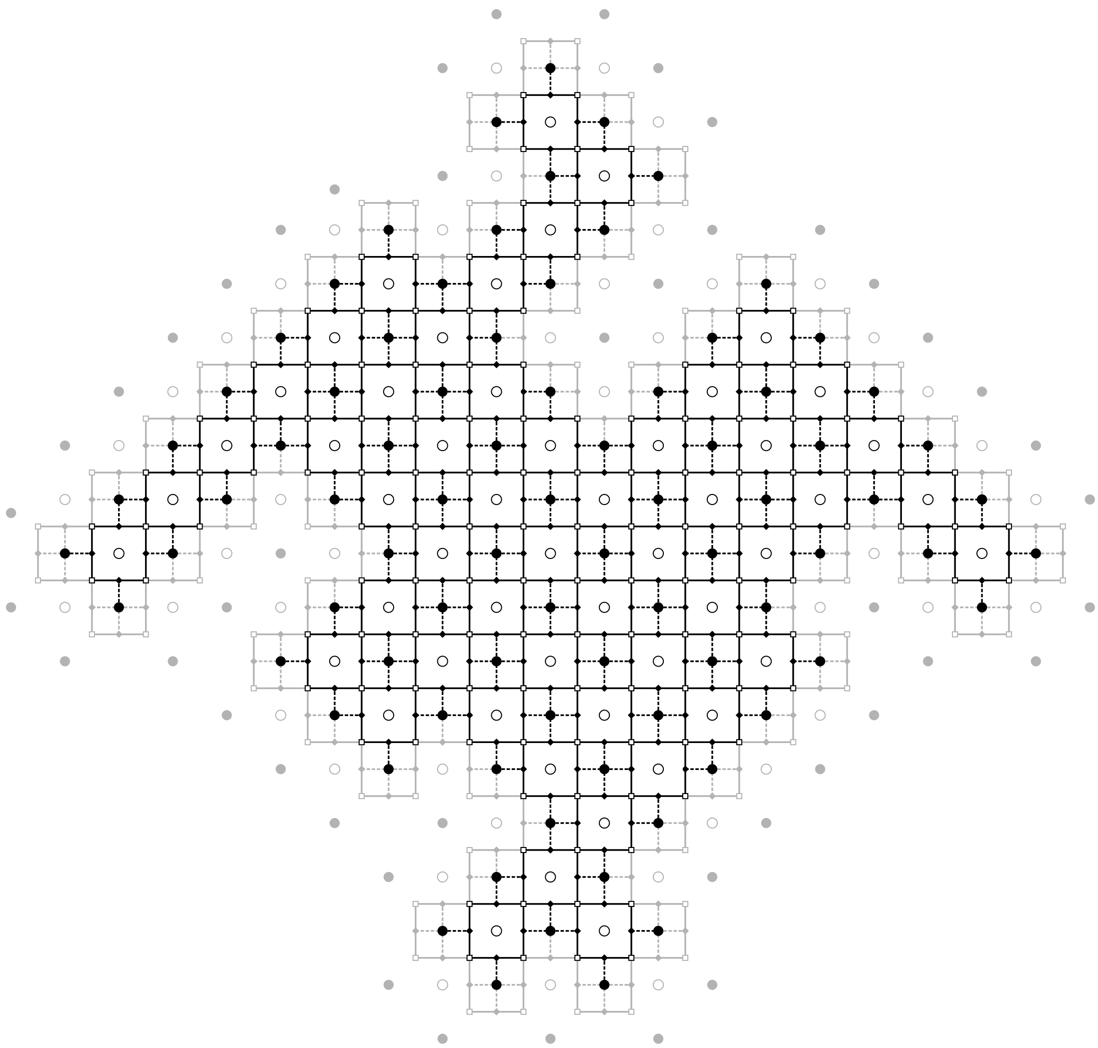

Figure 17: Example of the sites $\mathcal{V}_{\Omega_\delta}$, $\mathcal{F}_{\Omega_\delta}$, $\mathcal{E}_{\Omega_\delta}$ and $\mathcal{C}_{\Omega_\delta}$ from Figure 6 seen in the new perspective: the edges drawn here connect sites of $\mathcal{E}_{\Omega_\delta}$. The coloring of the edges depends on whether the respective corner belongs to $\mathrm{Int}\mathcal{C}_{\Omega_\delta}$ or $\partial\mathcal{C}_{\Omega_\delta}$

by using the discrete holomorphicity of F at $\widetilde{z}$ as written in (40). □

Remark 5.14. The well definition of the primitives does not require the discrete holomorphicity at all points. For instance, for some $b \in \{\mathbf{b}\}$, to define $H|_{\mathcal{F}_{\Omega_\delta}}$ the argument does not need $\overline{\partial}_\delta F\big(\widetilde{b}+\delta\big) = 0$ (assuming $b+\delta \neq \mathbf{b}$). This is because all loops around $b+\delta$ must go around the branching point b, therefore the spinor property will always cover these cases.

5.3 S-holomorphicity

We now consider the corners $\mathcal{C}_{\Omega_\delta}$ as exemplified in Figure 17. Note that between two connected sites of $\mathcal{E}_{\Omega_\delta}$ there corresponds an element of $\mathcal{C}_{\Omega_\delta}$, which is emanated from nearest vertex.

To every corner $c \in \mathcal{C}_{\Omega_\delta}$ we associate the line $l(c) := \eta_c\mathbb{R}$ seen as a subset of $\mathbb{C}$, and denote by $\mathrm{Proj}_{l(c)}[w]$ the *projection* of a complex number w onto the line $l(c)$, which can be written as

$$\mathrm{Proj}_{l(c)}[w] = \Re\big(w\overline{\eta}_c\big)\eta_c = \frac{1}{2}\big(w + \eta_c^2\overline{w}\big)$$

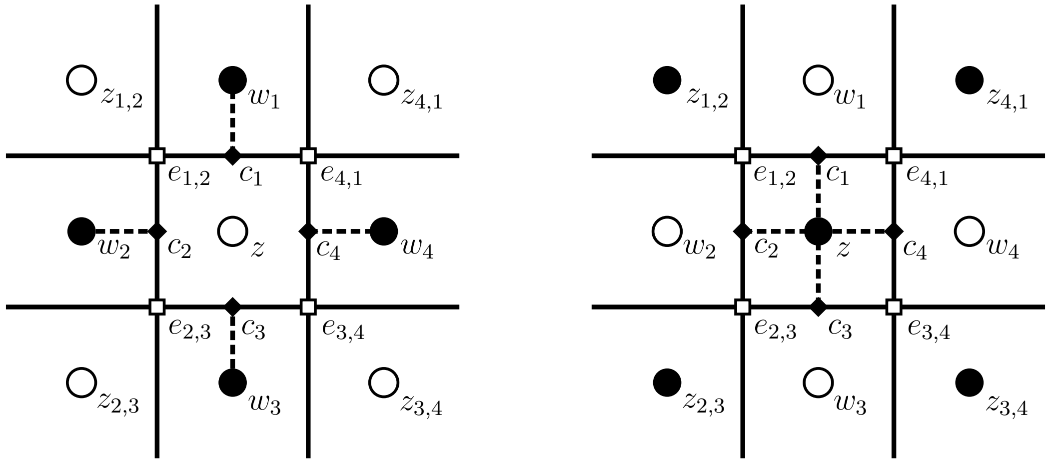

Figure 18: Setting of Figure 16 with added corners, depending on whether $z \in \mathcal{F}_{\Omega_\delta}$ (left) or $z \in \mathcal{V}_{\Omega_\delta}$ (right).

Definition 5.15. A function $F : [C \cup \mathcal{E}_{\Omega_\delta}; \mathbf{b}] \longrightarrow \mathbb{C}$ defined on the lifts of sets $C \subseteq \mathcal{C}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta}$ is *strongly holomorphic* at $c \in C$, or *s-holomorphic* for short, if for both $e \in \mathcal{E}_{\Omega_\delta}$ adjacent to c (that is, such that $|c - e| = \frac{\delta}{2}$)

$$F(\widetilde{c}) = \mathrm{Proj}_{l(c)}\big[F(\widetilde{e})\big] \tag{41}$$

for both lifts of c, with the lift of e on the same sheet of $[\Omega_\delta; \mathbf{b}]$ as that of c. Moreover, F is s-holomorphic in C if it is s-holomorphic at each $c \in C$.

Remark 5.16. This definition differs from [CS12] in two fundamental points. For one, the projections are done onto different lines that differ by a $\sqrt{i}$ factor. In addition, the s-holomorphicity in [CS12] is assigned to edge midpoints and no corners are mentioned. Note that, for two adjacent edges e_1 and e_2 sharing a vertex v, if we know

$$\mathrm{Proj}_{\frac{1}{\sqrt{\alpha}}\mathbb{R}}\big[F(\widetilde{e_1})\big] = \mathrm{Proj}_{\frac{1}{\sqrt{\alpha}}\mathbb{R}}\big[F(\widetilde{e_2})\big] \tag{42}$$

where α is the unit bisector of the angle $(e_1 v e_2)$, then we can extend F to c so that (41) holds. Therefore, the restriction imposed on $F|_{\mathcal{E}_{\Omega_\delta}}$ is the same. In fact, this is how s-holomorphicity should be seen: as restriction on a function defined on the edge midpoints, rather than on both the edge midpoints and corners. Each corner acts as a representative of the two edges adjacent to it; if F is s-holomorphic at a corner, then the values of F at the two adjacent edges verify (42).

We start by studying the relation between s-holomorphicity and discrete holomorphicity.

Proposition 5.17. *If $F : [C \cup \mathcal{E}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ is s-holomorphic at the four corners surrounding $z \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \cup \mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$, then $\overline{\partial}_\delta F(\widetilde{z}) = 0$.*

Proof. Consider $w_k, e_{k,k+1}$ and $z_{k,k+1}$ as shown in Figure 16 and let c_k be the corner between $e_{k-1,k}$ and $e_{k,k+1}$ (Figure 18). The s-holomorphicity implies

$$F(\widetilde{e}_{k,k+1}) - F(\widetilde{e}_{k-1,k}) = \eta_{c_k}^2 \left(\overline{F(\widetilde{e}_{k-1,k})} - \overline{F(\widetilde{e}_{k,k+1})}\right)$$

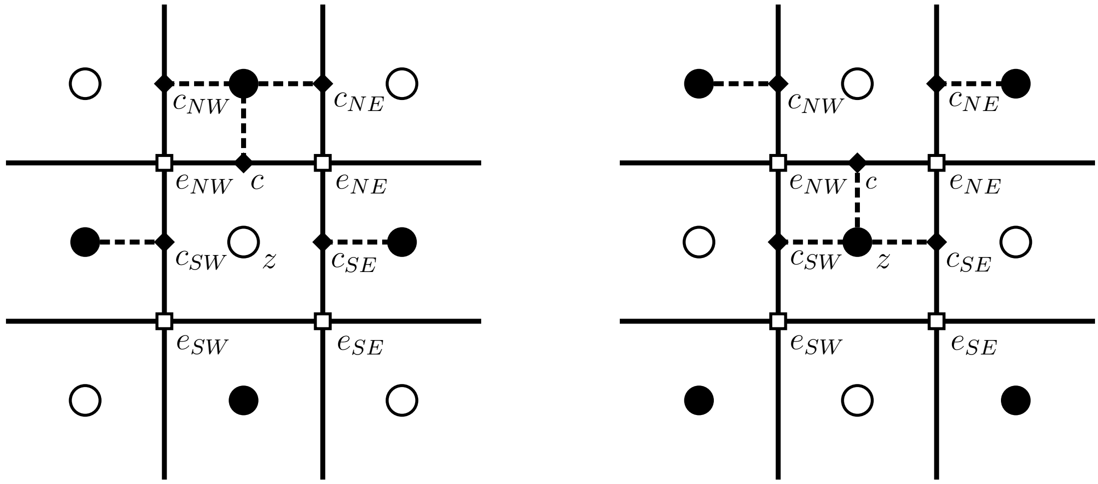

Figure 19: Setting for equation (43), depending on whether $z \in \mathcal{F}_{\Omega_\delta}$ (left) or $z \in \mathcal{V}_{\Omega_\delta}$ (right).

and by the definition of η_{c_k} we get

$$\overline{\eta}^2_{c_k} = \pm\frac{1}{\delta}(w_k - z)$$

where the $\pm$ sign depends on whether $z \in \mathcal{V}_{\Omega_\delta}$ or $z \in \mathcal{F}_{\Omega_\delta}$. Thus,

$$\begin{aligned}
\overline{\partial}_\delta F(\widetilde{z}) &= -\frac{i}{4\delta^2}\sum_{k=1}^{4}(w_{k+1} - w_k)F(\widetilde{e}_{k,k+1}) \\
&= -\frac{i}{4\delta^2}\sum_{k=1}^{4} w_k\big(F(\widetilde{e}_{k-1,k}) - F(\widetilde{e}_{k,k+1})\big) \\
&= -\frac{i}{4\delta^2}\sum_{k=1}^{4}\big(w_k - z\big)\big(F(\widetilde{e}_{k-1,k}) - F(\widetilde{e}_{k,k+1})\big) \\
&= \mp\frac{i}{4\delta}\sum_{k=1}^{4}\overline{\eta}^2_{c_k}\big(F(\widetilde{e}_{k-1,k}) - F(\widetilde{e}_{k,k+1})\big) \\
&= \mp\frac{i}{4\delta}\sum_{k=1}^{4}\overline{F(\widetilde{e}_{k,k+1})} - \overline{F(\widetilde{e}_{k-1,k})} \\
&= 0
\end{aligned}$$

□

Recall Remark 5.2, where it was explored the idea that (39) can be seen as two restrictions. Now that we are considering corners, the values at $\mathcal{C}^1_{\Omega_\delta}$ and $\mathcal{C}^i_{\Omega_\delta}$ correspond to taking the real and imaginary parts, respectively. This means the discrete holomorphicity on four edge midpoints $e_{NE}, e_{SE}, e_{SW}, e_{NW}$ can be translated to the four corners $c_{NW}, c_{SW}, c_{SE}, c_{NE}$ directly above (Figure 19): F verifies

$$F\big(\widetilde{c}_{NW}\big) - F\big(\widetilde{c}_{SE}\big) = i\Big[F\big(\widetilde{c}_{NE}\big) - F\big(\widetilde{c}_{SW}\big)\Big] \tag{43}$$

with the lifts taken to be on the same sheet. Conversely, a function that verifies (43) can be extended to the other corners and edge midpoints in an s-holomorphic way.

Proposition 5.18. *For sets $C^1 \subseteq \mathcal{C}^1_{\mathbb{C}_\delta}$ and $C^i \subseteq \mathcal{C}^i_{\mathbb{C}_\delta}$, let $F : [C^1 \cup C^i; \boldsymbol{b}] \longrightarrow \mathbb{C}$ be such that $F\big([C^1; \boldsymbol{b}]\big) \subset \mathbb{R}$*

and $F\big([C^i;\boldsymbol{b}]\big)\subset i\mathbb{R}$. *Set* $E=\big\{e\in\mathcal{E}_{\mathbb{C}_\delta}: e\pm\frac{i}{2}\delta\in C^1\cup C^i\big\}$ *and let* $C^{\lambda,\overline{\lambda}}\subseteq\mathcal{C}^{\lambda}_{\mathbb{C}_\delta}\cup\mathcal{C}^{\overline{\lambda}}_{\mathbb{C}_\delta}$ *be the set of corners* c *for which the following property holds: the corners* $c_{NW},c_{SW},c_{SE},c_{NE}\in\mathcal{C}^{1}_{\mathbb{C}_\delta}\cup\mathcal{C}^{i}_{\mathbb{C}_\delta}$ *that form a square of side* δ *starting in the upper left corner and going counter-clockwise with* c *in the center (Figure 19) belong to* $C^1\cup C^i$ *and verify*

$$F\big(\widetilde{c}_{NW}\big)-F\big(\widetilde{c}_{SE}\big)=i\Big[F\big(\widetilde{c}_{NE}\big)-F\big(\widetilde{c}_{SW}\big)\Big] \tag{44}$$

for all lifts on the same sheet of $[\mathbb{C};\boldsymbol{b}]$.

Then, F *can be (uniquely) extended in an s-holomorphic function to* $[E\cup C^1\cup C^i\cup C^{\lambda,\overline{\lambda}};\boldsymbol{b}]$.

Proof. If such an extension exists, then for $e\in E$ one must have $F(\widetilde{e})=F\big(\widetilde{e}+\frac{i}{2}\delta\big)+F\big(\widetilde{e}+\frac{i}{2}\delta\big)$ and for $c\in C^{\lambda,\overline{\lambda}}$ one must have $F(\widetilde{c})=\mathrm{Proj}_{l(c)}\big[F\big(\widetilde{c}+\frac{1}{2}\delta\big)\big]=\mathrm{Proj}_{l(c)}\big[F\big(\widetilde{c}-\frac{1}{2}\delta\big)\big]$. If we prove the two projections coincide, we are done.

We prove the case $c\in\mathcal{C}^{\lambda}_{\mathbb{C}_\delta}$, the other is similar. Let $e_W,e_E\in\mathcal{E}_{\mathbb{C}_\delta}$ be the edge midpoints directly to the left and right of c, respectively. Let c_{NW} and c_{SW} (c_{NE} and c_{SE}) be the corners directly above and below z_W (z_E), respectively. Since $c\in\mathcal{C}^{\lambda}_{\mathbb{C}_\delta}$, we know $c_{SW},c_{NE}\in\mathcal{C}^{1}_{\mathbb{C}_\delta}$ and $c_{NW},c_{SE}\in\mathcal{C}^{i}_{\mathbb{C}_\delta}$. Thus,

$$\begin{aligned}\mathrm{Proj}_{\lambda\mathbb{R}}\big[F(\widetilde{e}_W)\big]&=F(\widetilde{e}_W)+i\overline{F(\widetilde{e}_W)}\\&=\Big(F\big(\widetilde{c}_{NW}\big)+iF\big(\widetilde{c}_{SW}\big)\Big)(1-i)\\&=\Big(F\big(\widetilde{c}_{SE}\big)+iF\big(\widetilde{c}_{NE}\big)\Big)(1-i)\\&=F(\widetilde{e}_E)+i\overline{F(\widetilde{e}_E)}\\&=\mathrm{Proj}_{\lambda\mathbb{R}}\big[F(\widetilde{e}_E)\big]\end{aligned}$$

□

The main interest of s-holomorphicity is that it is a sufficient condition for discrete holomophicity to be partially preserved across products.

Proposition 5.19. *If* $F:[C\cup\mathcal{E}_{\Omega_\delta};\boldsymbol{b}]\longrightarrow\mathbb{C}$ *is s-holomorphic at the four corners surrounding* $z\in\mathcal{V}_{\Omega_\delta}\cup\mathcal{F}_{\Omega_\delta}$, *then* $\Re\big(\big[\overline{\partial}_\delta F^2\big](\widetilde{z})\big)=0$.

Proof. The argument is very similar to that of Proposition 5.17. Define w_k, $e_{k,k+1}$, $z_{k,k+1}$ and c_k as shown in Figure 18. As before,

$$F(\widetilde{e}_{k,k+1})-F(\widetilde{e}_{k-1,k})=\eta^2_{c_k}\Big(\overline{F(\widetilde{e}_{k-1,k})}-\overline{F(\widetilde{e}_{k,k+1})}\Big)\quad\text{and}\quad\overline{\eta}^2_{c_k}=\pm\frac{1}{\delta}(w_k-z)$$

where the $\pm$ sign depends on whether $z \in \mathcal{V}_{\Omega_\delta}$ or $z \in \mathcal{F}_{\Omega_\delta}$. Thus,

$$\begin{aligned}
\overline{\partial}_\delta F(\widetilde{z}) &= -\frac{i}{4\delta^2}\sum_{k=1}^{4}(w_{k+1}-w_k)F^2(\widetilde{e}_{k,k+1}) \\
&= -\frac{i}{4\delta^2}\sum_{k=1}^{4} w_k\big(F^2(\widetilde{e}_{k-1,k}) - F^2(\widetilde{e}_{k,k+1})\big) \\
&= -\frac{i}{4\delta^2}\sum_{k=1}^{4} \big(w_k - z\big)\big(F^2(\widetilde{e}_{k-1,k}) - F^2(\widetilde{e}_{k,k+1})\big) \\
&= \mp\frac{i}{4\delta}\sum_{k=1}^{4} \overline{\eta}_{c_k}^2 \big(F(\widetilde{e}_{k-1,k}) - F(\widetilde{e}_{k,k+1})\big)\big(F(\widetilde{e}_{k-1,k}) + F(\widetilde{e}_{k,k+1})\big) \\
&= \mp\frac{i}{4\delta}\sum_{k=1}^{4} \Big(\overline{F(\widetilde{e}_{k,k+1})} - \overline{F(\widetilde{e}_{k-1,k})}\Big)\Big(F(\widetilde{e}_{k,k+1}) + F(\widetilde{e}_{k-1,k})\Big) \in i\mathbb{R}
\end{aligned}$$

□

A generalization to products of distinct spinors follows easily. Despite not being used in the future, we state and prove this fact.

Proposition 5.20. *If $F_1, F_2 : [C \cup \mathcal{E}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ are s-holomorphic at the four corners surrounding $z \in \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta}$, then $\Re\big(\big[\overline{\partial}_\delta F_1 F_2\big](\widetilde{z})\big) = 0$.*

Proof. Note that $F_1 + F_2$ and $F_1 - F_2$ must be s-holomorphic at the four corners surrounding z and $4\big[\overline{\partial}_\delta F_1 F_2\big](\widetilde{z}) = \big[\overline{\partial}_\delta (F_1+F_2)^2\big](\widetilde{z}) - \big[\overline{\partial}_\delta (F_1-F_2)^2\big](\widetilde{z})$. The result now follows from Proposition 5.19. □

5.4 Integrating F^2

Following Remark 5.2, the condition $\Re\big(\overline{\partial}_\delta G(\widetilde{z})\big) = 0$ means only half the equations from (39) are verified. This implies it is possible to define a discrete version of $H = \Re\int G$ on the original, not lifted domain Ω_δ. Similar to Proposition 5.13, this primitive has two parts — one defined on $\mathcal{V}_{\Omega_\delta}$ and another defined on $\mathcal{F}_{\Omega_\delta}$ — which are independent from each other. It so happens one can define H on both $\mathcal{V}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta}$ using

$$H(f) - H(v) = 2\delta\left|F\left(\frac{\widetilde{f}+\widetilde{v}}{2}\right)\right|^2 \tag{45}$$

with $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ and $f \in \mathcal{F}_{\Omega_\delta}$ lifted to the same sheet. Note that $c = \frac{1}{2}(f+v)$ does not exist if $v \in \partial\mathcal{V}_{\Omega_\delta}$, and how the right-hand side is independent of the lift of c.

Since we want to apply this result to the spinors of Section 4, and Lemmas 4.23 and 4.25 state that the s-holomorphicity fails in some corners east of branching points because of a sign, we will account for such a possibility. Since a change of sign in the value of F on the right-hand side of (45) makes no difference, this is mostly a technicality. We will let $\mathbf{s}$ be some of the branching points $\mathbf{b}$ and denote $\mathbf{s}^{\rightarrow} := \mathbf{s} + \frac{\delta}{2}$.

Proposition 5.21. *Let $F : [\mathcal{C}_{\Omega_\delta} \setminus \{\boldsymbol{s}^{\rightarrow}\} \cup \mathcal{E}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ be an s-holomorphic spinor such that*

$$\operatorname{Proj}_{l(s^{\rightarrow})}\left[F\left(\widetilde{s}^{\rightarrow} - \frac{i}{2}\delta\right)\right] = -\operatorname{Proj}_{l(s^{\rightarrow})}\left[F\left(\widetilde{s}^{\rightarrow} + \frac{i}{2}\delta\right)\right]$$

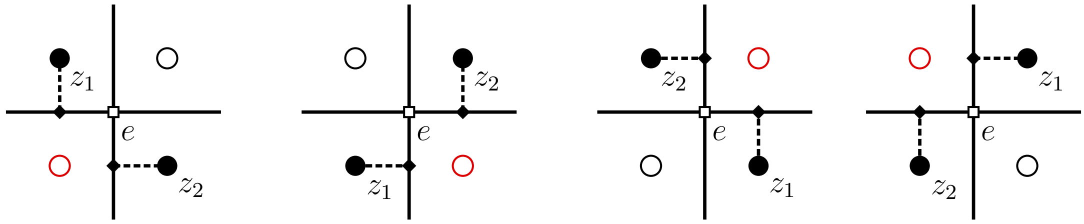

Figure 20: The face chosen to compute $H(z_2) - H(z_1)$ in the proof of (i) of Proposition 5.21 is shown for all possible configuration of z_1 and z_2.

for both lifts of all $s \in \{\boldsymbol{s}\}$. Then, the function $H : \operatorname{Int}\mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta} \longrightarrow \mathbb{R}$ given by (45), after setting

$$\left|F\left(\frac{\widetilde{f}+\widetilde{v}}{2}\right)\right| := \left|\operatorname{Proj}_{l(f^{\rightarrow})}\left[F\left(\widetilde{f}+\frac{1+i}{2}\delta\right)\right]\right|$$

if $f \in \{\boldsymbol{s}\}$ and $v = f + \delta$, is well-defined up to a constant. In addition,

(i) For $z_1, z_2 \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ or $z_1, z_2 \in \mathcal{F}_{\Omega_\delta}$ adjacent,

$$H(z_2) - H(z_1) = \Re\left(F^2\left(\frac{\widetilde{z}_2+\widetilde{z}_1}{2}\right)(z_2 - z_1)\right)$$

with z_1 and z_2 lifted to the same sheet. This implies that, for any $z_1, z_2 \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ or $z_1, z_2 \in \mathcal{F}_{\Omega_\delta}$,

$$H(z_2) - H(z_1) = \Re\int_{z_1}^{z_2} F^2(\widetilde{z})dz \tag{46}$$

with the integral computed along any path γ in $\mathcal{G}^{\dagger}_{\Omega_\delta}$ or $\mathcal{G}_{\Omega_\delta}$ running from z_1 to z_2.

(ii) H is discrete superharmonic on $\operatorname{Int}\mathcal{V}_{\Omega_\delta} \setminus \big(\{\boldsymbol{s}\}+\delta\big)$ and discrete subharmonic on $\operatorname{Int}\mathcal{F}_{\Omega_\delta} \setminus \{\boldsymbol{b}\}$:

$$[\Delta_\delta H](v) \leq 0 \qquad\qquad [\Delta_\delta H](f) \geq 0$$

for every $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta} \setminus \big(\{\boldsymbol{s}\}+\delta\big)$ and $f \in \operatorname{Int}\mathcal{F}_{\Omega_\delta} \setminus \{\boldsymbol{b}\}$ such that H is defined on all 4 neighbours.

Proof. We first prove H is well-defined, which amounts to checking the sum of the increments along closed paths in $\operatorname{Int}\mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta}$ is 0. Following the argument from Proposition 5.13, it is enough to prove this when γ is the smallest loop around some $e \in \mathcal{E}_{\Omega_\delta}$.

For $e \in \mathcal{E}_{\Omega_\delta}$, let $c_N, c_W, c_S, c_E \in \mathcal{C}_{\Omega_\delta} \setminus \{\mathbf{s}^{\rightarrow}\}$ be the corners directly above, to the left, below and to the right of e, respectively. The increment of H when going around e is

$$\pm\left(\left|F(\widetilde{c}_N)\right|^2 - \left|F(\widetilde{c}_W)\right|^2 + \left|F(\widetilde{c}_S)\right|^2 - \left|F(\widetilde{c}_E)\right|^2\right). \tag{47}$$

Note that the values of F at these corners are projections of $F(\widetilde{e})$ at the lines associated to the corners. In addition, the lines associated to c_N and c_S are perpendicular, so $\left|F(\widetilde{c}_N)\right|^2 + \left|F(\widetilde{c}_S)\right|^2 = \left|F(\widetilde{e})\right|^2$. Likewise, $\left|F(\widetilde{c}_W)\right|^2 + \left|F(\widetilde{c}_E)\right|^2 = \left|F(\widetilde{e})\right|^2$, therefore (47) equals 0. Finally, if one of the corners is $\mathbf{s}^{\rightarrow}$, the projection may fail by a sign, which makes no difference when plugged in to (45).

We proceed to the proof of (i). For $z_1, z_2 \in \mathcal{V}_{\Omega_\delta}$ adjacent, let $e = \frac{1}{2}(z_1 + z_2)$ be the edge connecting z_1 and z_2. Now, there are 4 possible configurations for z_1 and z_2 and 2 ways of choosing the face adjacent to z_1 and z_2 in order to compute $H(\widetilde{z}_2) - H(\widetilde{z}_1)$. If one uses the face found when rotating about e from z_1 to z_2 in the counterclockwise direction (Figure 20), then we arrive at

$$H(z_2) - H(z_1) = 2\delta\Big|\mathrm{Proj}_{\eta_1\mathbb{R}}\big[F(\widetilde{e})\big]\Big|^2 - 2\delta\Big|\mathrm{Proj}_{\eta_2\mathbb{R}}\big[F(\widetilde{e})\big]\Big|^2$$

with $\eta_1^2 = \frac{\lambda}{\sqrt{2\delta}}\overline{(z_2 - z_1)}$ and $\eta_2^2 = -\frac{\overline{\lambda}}{\sqrt{2\delta}}\overline{(z_2 - z_1)}$ (if we were to choose the other face, then we would swap $\lambda \leftrightarrow \overline{\lambda}$), and again note how the exception in the corners $\mathbf{s}^{\rightarrow}$ makes no difference. Writing $F \equiv F(\widetilde{e})$ and expanding,

$$\begin{aligned}
H(z_2) - H(z_1) &= 2\delta\left[\left|\frac{1}{2}\left(F + \frac{\lambda}{\sqrt{2\delta}}\overline{(z_2 - z_1)}\cdot\overline{F}\right)\right|^2 - \left|\frac{1}{2}\left(F - \frac{\overline{\lambda}}{\sqrt{2\delta}}\overline{(z_2 - z_1)}\cdot\overline{F}\right)\right|^2\right] \\
&= \frac{\delta}{2}\left[\left(F + \frac{\lambda}{\sqrt{2\delta}}\overline{(z_2 - z_1)}\cdot\overline{F}\right)\left(\overline{F} + \frac{\overline{\lambda}}{\sqrt{2\delta}}(z_2 - z_1)\cdot F\right) - \right. \\
&\qquad \left. - \left(F - \frac{\overline{\lambda}}{\sqrt{2\delta}}\overline{(z_2 - z_1)}\cdot\overline{F}\right)\left(\overline{F} - \frac{\lambda}{\sqrt{2\delta}}(z_2 - z_1)\cdot F\right)\right] \\
&= \frac{\delta}{2}\left[\frac{\lambda + \overline{\lambda}}{\sqrt{2\delta}}(z_2 - z_1)\cdot F^2 + \frac{\lambda + \overline{\lambda}}{\sqrt{2\delta}}\overline{(z_2 - z_1)}\cdot\overline{F^2}\right] \\
&= \frac{1}{2}\left[(z_2 - z_1)\cdot F^2 + \overline{(z_2 - z_1)\cdot F^2}\right] \\
&= \Re\Big((z_2 - z_1)\cdot F^2\Big)
\end{aligned}$$

(note how swapping $\lambda \leftrightarrow \overline{\lambda}$ does not change the result, which is yet another proof that H is well-defined).

Finally, we prove the superharmonicity of H on $\mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \setminus \big(\{\mathbf{s}\} + \delta\big)$ (the subharmonicity on $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$ follows similarly). For $z \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \setminus \big(\{\mathbf{s}\} + \delta\big)$, define w_k, $e_{k,k+1}$, $z_{k,k+1}$ and c_k as shown in Figure 18. Because $z \in \mathcal{V}_{\Omega_\delta}$, we know $F(\widetilde{c}_k) = \overline{\lambda}^k F_k$ for some $F_k \in \mathbb{R}$. Beware looping indices: $F(\widetilde{c}_5) \equiv F(\widetilde{c}_1)$ but $\overline{\lambda}^5 F_5 = -\overline{\lambda} F_5$.

We start by writing $F(\widetilde{e}_{k,k+1})$ in function of $F(\widetilde{c}_k)$ and $F(\widetilde{c}_{k+1})$, its projections onto different lines. Note that $\eta_{c_{k+1}} = \pm\overline{\lambda}\eta_{c_k}$, thus

$$\begin{cases}
\frac{1}{2}\Big(F(\widetilde{e}_{k,k+1}) + \eta_{c_k}^2\overline{F(\widetilde{e}_{k,k+1})}\Big) = F(\widetilde{c}_k) \\
\frac{1}{2}\Big(F(\widetilde{e}_{k,k+1}) - i\eta_{c_k}^2\overline{F(\widetilde{e}_{k,k+1})}\Big) = F(\widetilde{c}_{k+1})
\end{cases}$$

(recall that $z \in \{\mathbf{s}\} + \delta$ is excluded) and note that, for all these equalities to hold simultaneously, z cannot be a branching point (which is always true when $z \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \setminus \big(\{\mathbf{s}\} + \delta\big)$ but excludes the points $\mathbf{b}$ when $z \in \mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$). These equations yield

$$\frac{i+1}{2}F(\widetilde{e}_{k,k+1}) = iF(\widetilde{c}_k) + F(\widetilde{c}_{k+1}) \Rightarrow F(\widetilde{e}_{k,k+1}) = (1-i)\big(iF(\widetilde{c}_k) + F(\widetilde{c}_{k+1})\big)$$

therefore

$$
\begin{aligned}
F^2(\widetilde{e}_{k,k+1}) &= 2i\Big(F^2(\widetilde{c}_k) - F^2(\widetilde{c}_{k+1}) - 2iF(\widetilde{c}_k)F(\widetilde{c}_{k+1})\Big) \\
&= 2i\Big((-i)^k F_k^2 - (-i)^{k+1}F_{k+1}^2 \pm 2\overline{\lambda}(-i)^{k+1}F_kF_{k+1}\Big)
\end{aligned}
$$

where the $\pm$ sign is $+$ for $k = 1, 2, 3$ and $-$ for $k = 4$ (since $F(\widetilde{c}_5) = -\overline{\lambda}^5 F_5$). In addition, $z_{k,k+1} - z = \sqrt{2}\delta\lambda i^k$. Hence,

$$
\begin{aligned}
[\Delta_\delta H](z) &= \frac{1}{2\delta^2}\sum_{k=1}^{4} H(z_{k,k+1}) - H(z) \\
&= \frac{1}{2\delta^2}\Re\sum_{k=1}^{4} F^2(\widetilde{e}_{k,k+1})(z_{k,k+1} - z) \\
&= \frac{\sqrt{2}}{\delta}\Re\sum_{k=1}^{4} \lambda i F_k^2 - \lambda F_{k+1}^2 \pm 2F_kF_{k+1} \\
&= \frac{1}{\delta}\sum_{k=1}^{4} -F_k^2 - F_{k+1}^2 \pm 2\sqrt{2}F_kF_{k+1} \\
&= -\frac{1}{\delta}\Big[2\big(F_1^2 + F_2^2 + F_3^2 + F_4^2\big) - 2\sqrt{2}\big(F_1F_2 + F_2F_3 + F_3F_4 - F_4F_1\big)\Big] \\
&= -\frac{1}{\delta}\Big[\Big(F_1 - \sqrt{2}F_2 + F_3\Big)^2 + \Big(F_1 - F_3 + \sqrt{2}F_4\Big)^2\Big] \\
&\le 0
\end{aligned}
$$

□

Remark 5.22. Note that

$$F_1 - 2\sqrt{2}F_2 + F_3 = 0 \Leftrightarrow i\big(\overline{\lambda}F_1\big) + \big(\overline{\lambda}^2 F_2\big) = i\big(\overline{\lambda}^2 F_2\big) + \big(\overline{\lambda}^3 F_3\big) \Leftrightarrow F(\widetilde{e}_{1,2}) = F(\widetilde{e}_{2,3})$$

$$F_1 - F_3 + 2\sqrt{2}F_4 = 0 \Leftrightarrow i\big(\overline{\lambda}^3 F_3\big) + \big(\overline{\lambda}^4 F_4\big) = i\big(\overline{\lambda}^4 F_4\big) + \big(-\overline{\lambda}^5 F_1\big) \Leftrightarrow F(\widetilde{e}_{3,4}) = F(\widetilde{e}_{4,1})$$

which means both $\delta\big|[\Delta_\delta H](z)\big|$ and $\big|F(\widetilde{e}_{2,3}) - F(\widetilde{e}_{1,2})\big|^2 + \big|F(\widetilde{e}_{4,1}) - F(\widetilde{e}_{3,4})\big|^2$, as functions of (F_1, F_2, F_3, F_4), are non-negative quadratic forms with the same two-dimensional kernel. Hence,

$$Const_1 \cdot \delta\big|[\Delta_\delta H](z)\big| \le \big|F(\widetilde{e}_{2,3}) - F(\widetilde{e}_{1,2})\big|^2 + \big|F(\widetilde{e}_{4,1}) - F(\widetilde{e}_{3,4})\big|^2 \le Const_2 \cdot \delta\big|[\Delta_\delta H](z)\big|$$

for some positive constants $Const_1, Const_2$ independent of all other variables.

5.5 The boundary modification trick

As stated in the beginning of this section, we are particularly interested in the case where F satisfies the Riemann boundary condition $F(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}$. Following Proposition 5.21, we define the discrete

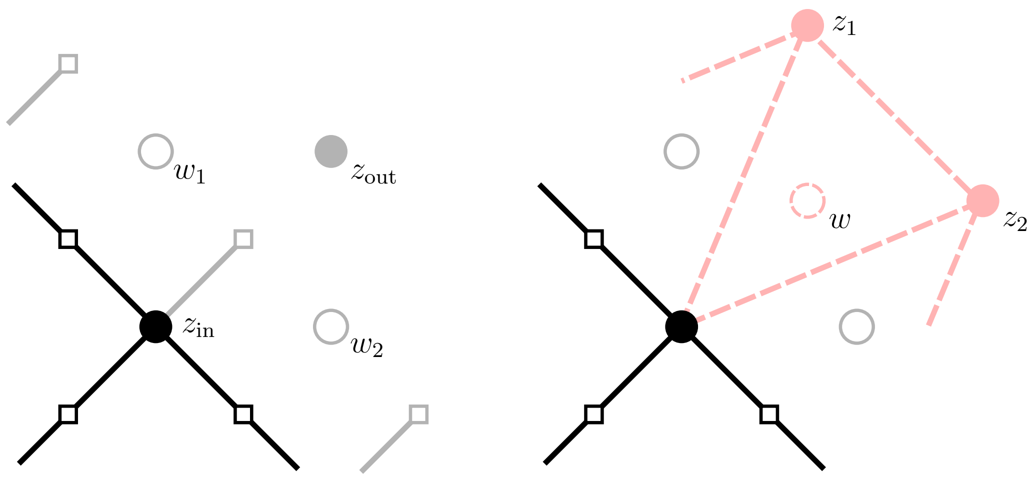

Figure 21: The boundary modification trick: the domain is modified so that the new triangles become the boundary. The red edges are the ones added to $\mathcal{G}_{\Omega_\delta}$. Note that the edges in black are from $\mathcal{G}_{\Omega_\delta}$.

version of $\Re \int F^2$ and statement (i) implies

$$H(z) \equiv Const \qquad z \in \partial\mathcal{F}_{\Omega_\delta}$$

with H undefined in $\partial\mathcal{V}_{\Omega_\delta}$. Although it is not possible to extend H using (45), one could do it according to (i) of Proposition 5.21. This would lead to a more complicated boundary condition for $\partial\mathcal{V}_{\Omega_\delta}$ and require more work in the sequel, where an estimative of $H|_{\partial\mathcal{V}_{\Omega_\delta}} - H|_{\partial\mathcal{F}_{\Omega_\delta}}$ would be needed (namely, Onsager's magnetization estimate, see [Smi06] for more details). To extend H to $\partial\mathcal{V}_{\Omega_\delta}$ while keeping agreement with the boundary condition, we would prefer

$$H(z) \equiv Const \qquad z \in \partial\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{F}_{\Omega_\delta}$$

and, as pointed out in [CS12], one can do exactly that by modifying the boundary.

For every $z_{\text{out}} \in \partial\mathcal{V}_{\Omega_\delta}$, we consider the square $[z_{\text{out}}w_1z_{\text{in}}w_2]$ of side δ with $z_{\text{in}} \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ and $w_1, w_2 \in \partial\mathcal{F}_{\Omega_\delta}$ (one of w_1, w_2 may not exist, we add it if needed). Take the point w in the line segment $[z_{\text{in}}z_{\text{out}}]$ such that the distance between z_{in} and w is δ. Two points z_1 and z_2 are added such that $[z_{in}wz_1w_1]$ and $[z_{in}wz_2w_2]$ are rhombi (Figure 21). The points z_i created become the new $\partial\mathcal{V}_{\Omega_\delta}$, all the sites w are added to $\partial\mathcal{F}_{\Omega_\delta}$ and all edges $(z_{in}z_1), (z_{in}z_2), (z_1z_2)$ are added to $\mathcal{G}_{\Omega_\delta}$. Note that these edges may intersect, but that is not relevant for arguments and self-overlapping regions can be handled by placing them on local Riemann surfaces. The modified domain is still a valid discretization of the original Ω in the sense that it approximates Ω when $\delta \to 0$.

We now set $H(z_1) = H(z_2) = H(w_1) = H(w_2)$ and this will make it so that H is discrete superharmonic at z_{in}, although we have to take some care with the definition of the discrete Laplacian. For isoradial graphs — that is, where all faces can be inscribed into circles of equal radii —, the generalization is given by

$$[\Delta_\delta H](z) := \frac{1}{A} \sum_{w \sim z} \tan\theta_w \cdot \big[H(w) - H(z)\big]$$

where A is the area of the polygon formed by the neighbours of z and θ_w is equal to half the angle $(u_1 z u_2)$ with u_1 and u_2 being the faces on each side of the edge (zw). In the usual lattice square, $\theta_w = \pi/4$; for the added vertices, $\theta_w = \pi/8$.

For future computations, it is not necessary to consider the actually modified boundary: we can have z_{out} act as a stand-in for the two z_i. In practice, this entails setting H constant in $\partial\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{F}_{\Omega_\delta}$ and modifying the definition of the discrete Laplacian so that

$$\big[\widetilde{\Delta}_\delta H\big](z) := \frac{1}{2\delta^2} \sum_{w\sim z} c_{zw} \cdot \big[H(w) - H(z)\big] \tag{48}$$

where $c_{zw} = 2\tan\pi/8 = 2(\sqrt{2}-1)$ if $z \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ and $w \in \partial\mathcal{V}_{\Omega_\delta}$, and $c_{zw} = 1$ otherwise. The new symbol is used to reinforce the tweak.

Remark 5.23. To be completely correct, one should also update the area factor that normalizes (48). Since it makes no difference for our computations, for the sake of simplicity we leave it as it is.

Proposition 5.21 is reformulated below to account for this modification and also fix the value of H on the boundary as 0. Its proof does not require knowledge of the boundary modification trick, just the definition (48).

Proposition 5.24. *Let $F : [\mathcal{C}_{\Omega_\delta} \setminus \{\boldsymbol{s}^{\rightarrow}\} \cup \mathcal{E}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ be an s-holomorphic spinor such that*

$$\operatorname{Proj}_{l(b^{\rightarrow})}\left[F\left(\widetilde{b}^{\rightarrow} - \frac{i}{2}\delta\right)\right] = -\operatorname{Proj}_{l(b^{\rightarrow})}\left[F\left(\widetilde{b}^{\rightarrow} + \frac{i}{2}\delta\right)\right]$$

for both lifts of all $s = \{\boldsymbol{s}\}$ and verifying the Riemann boundary condition

$$F(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}, \quad z \in \partial\mathcal{E}_{\Omega_\delta}$$

Then, there is a function $H : \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta} \longrightarrow \mathbb{R}$ such that

(i) For $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ and $f \in \mathcal{F}_{\Omega_\delta}$, H satisfies (45), after setting

$$\left|F\left(\frac{\widetilde{f}+\widetilde{v}}{2}\right)\right| := \left|\operatorname{Proj}_{l(f^{\rightarrow})}\left[F\left(\widetilde{f} + \frac{1+i}{2}\delta\right)\right]\right|$$

if $f \in \{\boldsymbol{s}\}$ and $v = f + \delta$.

(ii) For $z \in \partial\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{F}_{\Omega_\delta}$, $H(z) = 0$.

(iii) For any $z_1, z_2 \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ or $z_1, z_2 \in \mathcal{F}_{\Omega_\delta}$,

$$H(z_2) - H(z_1) = \Re \int_{z_1}^{z_2} F^2(\widetilde{z})dz$$

with the integral computed along any path γ in $\mathcal{G}^{\dagger}_{\Omega_\delta}$ or $\mathcal{G}_{\Omega_\delta}$ running from z_1 to z_2.

(iv) H has a nonpositive derivative in the inner normal direction: for $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ that are adjacent to boundary vertices, $H(v) \leq 0$.

(v) H is discrete superharmonic on $\mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \setminus \big(\{\boldsymbol{s}\} + \delta\big)$ *and discrete subharmonic on* $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta} \setminus \{\boldsymbol{b}\}$*:*

$$\big[\widetilde{\Delta}_\delta H\big](v) \leq 0 \qquad\qquad \big[\widetilde{\Delta}_\delta H\big](f) \geq 0$$

for every $v \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \setminus \big(\{\boldsymbol{s}\} + \delta\big)$ *and* $f \in \mathrm{Int}\,\mathcal{F}_{\Omega_\delta} \setminus \{\boldsymbol{b}\}$.

Proof. For (iv), if $v' \in \partial\mathcal{V}_{\Omega_\delta}$ is the adjacent boundary vertex then the two faces bounded by the edge $\frac{1}{2}(v+v')$ must be in $\partial\mathcal{F}_{\Omega_\delta}$ (otherwise $v' \in \mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$), and the statement follows from (ii) and (45).

To prove the superharmonicity at $z \in \mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ adjacent to a boundary vertex, define w_k, $e_{k,k+1}$, $z_{k,k+1}$ and c_k as in Proposition 5.21 and suppose $z_{k,k+1} \in \partial\mathcal{V}_{\Omega_\delta}$ for some index k. We compute the contribution of $z_{k,k+1}$ to $\big[\widetilde{\Delta}_\delta H\big](z)$:

$$\begin{aligned}
\frac{1}{2\delta^2}\cdot 2(\sqrt{2}-1)\big[H(z_{k,k+1}) - H(z)\big] &= \frac{\sqrt{2}-1}{\delta^2}\big[H(w_k) - H(z)\big] \\
&= \frac{2(\sqrt{2}-1)}{\delta}\big|F(\widetilde{c}_k)\big|^2 \\
&= \frac{2(\sqrt{2}-1)}{\delta}\Big|\mathrm{Proj}_{\eta_{c_k}\mathbb{R}}\big(F(\widetilde{e}_{k,k+1})\big)\Big|^2
\end{aligned}$$

and note that $\overline{\eta}_{c_k}^2 = \overline{\lambda}\nu_{\mathrm{out}}(z)$, which together with $F(\widetilde{e}_{k,k+1})\sqrt{\nu_{\mathrm{out}}(\widetilde{z})} \in \mathbb{R}$ implies $\arg\big(\eta_{c_k}\big) = \arg\big(F(\widetilde{e}_{k,k+1})\big) + \pi/8$, therefore

$$\begin{aligned}
\frac{2(\sqrt{2}-1)}{\delta}\Big|\mathrm{Proj}_{\eta_{c_k}\mathbb{R}}\big(F(\widetilde{e}_{k,k+1})\big)\Big|^2 &= \frac{2(\sqrt{2}-1)}{\delta}\cos^2\frac{\pi}{8}\Big|F(\widetilde{e}_{k,k+1})\Big|^2 \\
&= \frac{\sqrt{2}}{2\delta}\Big|F(\widetilde{e}_{k,k+1})\Big|^2
\end{aligned}$$

If the contribution of $z_{k,k+1}$ to $\big[\widetilde{\Delta}_\delta H\big](z)$ were computed as in Proposition 5.21, then it would be

$$\begin{aligned}
\frac{1}{2\delta^2}\cdot\Re\Big(F^2(\widetilde{e}_{k,k+1})(z_{k,k+1}-z)\Big) &= \frac{1}{2\delta^2}\Big|F(\widetilde{e}_{k,k+1})(z_{k,k+1}-z)^{\frac{1}{2}}\Big|^2 \\
&= \frac{1}{2\delta^2}\Big|F(\widetilde{e}_{k,k+1})\Big|^2\cdot\sqrt{2}\delta \\
&= \frac{\sqrt{2}}{2\delta}\Big|F(\widetilde{e}_{k,k+1})\Big|^2
\end{aligned}$$

where we use $F^2(\widetilde{e}_{k,k+1})(z_{k,k+1}-z) \in \mathbb{R}^+$ because $(z_{k,k+1}-z)\,||\,\nu_{\mathrm{out}}(\widetilde{z})$. Since the contribution does not change, the proof of Proposition 5.21 concludes the argument. □

5.6 Harmonic properties of $\Re\int F^2$

The last subsections were dedicated to the definition of the discrete analogue of $H = \Re\int F^2$, which is done using (45) and a slight modification on the boundary. According to statement (v) of Proposition 5.24, $H|_{\mathcal{V}_{\Omega_\delta}}$ is superharmonic and $H|_{\mathcal{F}_{\Omega_\delta}}$ is subharmonic on (most of) the interior of the domain. We now show some properties that allow H to be treated as if it were a harmonic function: minimum and maximum principles and uniform comparability in adjacent sites. Finally, we state a result concerning the regularity of the original function F.

Extremes Equation (45) implies $H(v) \leq H(f)$ for adjacent $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta}$ and $f \in \operatorname{Int}\mathcal{F}_{\Omega_\delta}$. Together with the super and subharmonicity, we arrive at

$$\min_{\Omega'_\delta} H = \min_{\partial\Omega'_\delta} H|_{\mathcal{V}_{\Omega_\delta}} \qquad \max_{\Omega'_\delta} H = \max_{\partial\Omega'_\delta} H|_{\mathcal{F}_{\Omega_\delta}} \tag{49}$$

for any bounded subset $\Omega'_\delta \subseteq \Omega_\delta$, with the added restrictions $\mathbf{b} \notin \Omega'_\delta$ for the maximum principle and $\mathbf{s} + \delta \notin \Omega'_\delta$ for the minimum principle. To prove this for vertices (for faces the argument is the same), note that all coefficients of the modified Laplacian (48) are positive. Hence, given $v \in \Omega'_\delta \setminus \partial\Omega'_\delta$, either there is a neighbour vertex v' such that $H(v') < H(v)$ or for all neighbours v' we have $H(v') = H(v)$. Using the same argument for the neighbouring sites, the neighbours of the neighbouring sites and so on, we must eventually reach the boundary.

Uniform comparability Let us start by comparing local values of $H|_{\mathcal{V}_{\Omega_\delta}}$ and $H|_{\mathcal{F}_{\Omega_\delta}}$. From this point forward, $Const$ will represent some positive constant independent of other variables.

Proposition 5.25. *Let $H : \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta} \longrightarrow \mathbb{R}$ be defined according to Proposition 5.24. Let $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta} \setminus \{\boldsymbol{s} + \delta\}$ surrounded by inner faces $f_1, f_2, f_3, f_4 \in \operatorname{Int}\mathcal{F}_{\Omega_\delta} \setminus \{\boldsymbol{b}\}$ and vertices $v_{1,2}, v_{2,3}, v_{3,4}, v_{4,1}$. Then,*

$$\max_k H(f_k) - H(v) \leq Const \cdot \left(H(v) - \min_k H(v_{k,k+1})\right)$$

Remark 5.26. Switching the roles of vertices and faces, one also has

$$H(f) - \min_k H(v_{k,k+1}) \leq Const \cdot \left(\max_k H(f_k) - H(f)\right)$$

for $f \in \operatorname{Int}\mathcal{F}_{\Omega_\delta} \setminus \{\mathbf{b}\}$ surrounded by inner vertices $v_1, v_2, v_3, v_4 \in \operatorname{Int}\mathcal{V}_{\Omega_\delta} \setminus (\{\mathbf{s}\} + \delta)$ and faces $f_{1,2}, f_{2,3}, f_{3,4}, f_{4,1}$

Proof. Consider the disposition of the faces and vertices as shown in the right side of Figure 18 with $z = v$, $w_k = f_k$ and $z_{k,k+1} = v_{k,k+1}$.

By subtracting a constant, we can assume without loss of generality that $H \geq 0$ at all of these sites and neighbours of all f_k. Note that this modification does not change the statement of the Proposition and we will not use statements (ii) or (iv) of Proposition 5.24. Under this assumption, the superharmonicity of H at v implies that

$$H(v) \geq \frac{1}{\sum_k c_{vv_{k,k+1}}} \sum_k c_{vv_{k,k+1}} H(v_{k,k+1}) \geq Const \cdot H(v_{k,k+1})$$

for all k, and in particular $H(v) \geq Const \cdot \min_k H(v_{k,k+1})$. If we prove that $\max_k H(f_k) \leq Const \cdot H(v)$, then

$$\max_k H(f_k) - H(v) \leq Const \cdot H(v) \leq Const \cdot \left(H(v) - \min_k H(v_{k,k+1})\right)$$

and we are done.

Let $M = \max_k H(f_k)$ and suppose by contradiction we can take $K = M/H(v)$ to be as large as we want. One would now expect to proceed by taking the face $f_{\max}$ where the maximum occurs and use the fact

$H(f_{\max})/H(v)$ is as large as we want. However, we will use a stronger fact: that $H(f_k)/H(v) \geq Const \cdot M$ for all indices k, and not just the one where the maximum occurs. Using the subharmonicity at each f_k and proceeding similarly to before, we deduce that $H(f_k) \leq Const \cdot H(f_{k+1})$ for any k, and all 4 inequalities taken together yield

$$Const \cdot M \leq H(f_k) \leq M$$

for any k, which means that in our setup we can take $H(f_k)/H(v)$ to behave essentially as $Const \cdot K$. The assumption thus applies to all four indices k and not just the one where the maximum occurs. We will ignore the extra $Const$ factor, since it is not relevant.

Fix an index k and recall that $H(v_{k,k+1}) \leq Const \cdot H(v)$. We have

$$1 + \frac{\pm Const}{K - Const} = \frac{K \cdot H(v) - Const \cdot H(v)}{K \cdot H(v) - Const \cdot H(v)} \leq \frac{H(f_k) - H(v_{k,k+1})}{H(f_k) - H(v)} \leq \frac{K \cdot H(v)}{K \cdot H(v) - H(v)} = 1 + \frac{1}{K-1}$$

therefore

$$\frac{H(f_k) - H(v_{k,k+1})}{H(f_k) - H(v)} = 1 + O\left(\frac{1}{K}\right).$$

Note that the differences in the numerator and denominator are given by (45) as the square of values of F in corners, which by the s-holomorphicity are projections of values of F in edges. Let $e_{k,k+1}$ be the edge $(vv_{k,k+1})$ and c_k the corner between $v_{k,k+1}$ and f_k. Then,

$$\frac{\left|\mathrm{Proj}_{\eta_{c_k}\mathbb{R}}\left[F(\widetilde{e}_{k,k+1})\right]\right|}{\left|\mathrm{Proj}_{\overline{\lambda}\eta_{c_k}\mathbb{R}}\left[F(\widetilde{e}_{k,k+1})\right]\right|} = \sqrt{1 + O\left(\frac{1}{K}\right)} = 1 + O\left(\frac{1}{K}\right)$$

which implies

$$\arg F(\widetilde{e}_{k,k+1}) = \arg \eta_{c_k} - \frac{\pi}{8} + O\left(\frac{1}{K}\right) \mod \frac{\pi}{2}$$

(we leave the details for Lemma 5.27). Considering all 4 indices k and the fact $\arg \eta_{c_k} = (1-k)\frac{\pi}{4} \mod \pi$, this means $F(\widetilde{e}_{k-1,k})$ and $F(\widetilde{e}_{k,k+1})$ have arguments that differ by at least $\frac{\pi}{4}$, up to an error of $O(K^{-1})$.

Let us now analyse the norm of $F(\widetilde{e}_{k,k+1})$. We can write it using its projections:

$$F(\widetilde{e}_{k,k+1}) = (1-i)\Big(i\mathrm{Proj}_{\eta_{c_k}\mathbb{R}}\left[F(\widetilde{e}_{k,k+1})\right] + \mathrm{Proj}_{\overline{\lambda}\eta_{c_k}\mathbb{R}}\left[F(\widetilde{e}_{k,k+1})\right]\Big)$$

(see proof of Proposition 5.21 for more details). On one hand, $2\delta\left|\mathrm{Proj}_{\eta_{c_k}\mathbb{R}}\left[F(\widetilde{e}_{k,k+1})\right]\right|^2 = H(f_k) - H(v_{k,k+1})$ and

$$\left(1 - \frac{Const}{K}\right) \cdot M = M - Const \cdot H(v_{k,k+1}) \leq H(f_k) - H(v_{k,k+1}) \leq M;$$

on the other hand,

$$2\delta\left|\mathrm{Proj}_{\overline{\lambda}\eta_{c_k}\mathbb{R}}\left[F(\widetilde{e}_k)\right]\right|^2 = H(f_k) - H(v) = \left(1 - \frac{1}{K}\right)M.$$

Taking everything together,

$$\delta\left|F(\widetilde{e}_{k,k+1})\right|^2 = Const \cdot M + O\left(\frac{1}{K}\right).$$

Using that $F(\widetilde{e}_{k-1,k})$ and $F(\widetilde{e}_{k,k+1})$ have different enough arguments and the estimates on their absolute values, we arrive at

$$\delta\big|F(\widetilde{e}_{k,k+1}) - F(\widetilde{e}_{k-1,k})\big|^2 = Const \cdot M + O\bigg(\frac{1}{K}\bigg)$$

which means

$$\begin{aligned} M + O\bigg(\frac{1}{K}\bigg) &\leq Const \cdot \delta\Big(\big|F(\widetilde{e}_{2,3}) - F(\widetilde{e}_{1,2})\big|^2 + \big|F(\widetilde{e}_{4,1}) - F(\widetilde{e}_{3,4})\big|^2\Big) \\ &\leq Const \cdot \delta^2\big|[\Delta_\delta H](v)\big| \qquad (50) \\ &= Const \cdot \sum_k c_{vv_{k,k+1}}\big(H(v) - H(v_{k,k+1})\big) \\ &\leq Const \cdot H(v) \end{aligned}$$

where (50) follows from Remark 5.22. This is contradictory with $K = M/H(v)$ being as large as we want. □

Lemma 5.27. *For any complex z,*

$$\frac{\big|\mathrm{Proj}_{\eta\mathbb{R}}[z]\big|}{\big|\mathrm{Proj}_{\overline{\lambda}\eta\mathbb{R}}[z]\big|} = 1 + O\bigg(\frac{1}{K}\bigg) \Rightarrow \arg z = \arg\eta - \frac{\pi}{8} + O\bigg(\frac{1}{K}\bigg) \mod \frac{\pi}{2}$$

Remark 5.28. We are stating that if the projections of z to two lines have very close lengths, then z should be close to the bisector of one of the angles formed by the lines (hence the mod $\frac{\pi}{2}$).

Proof. Let $\mathrm{Proj}_{\eta\mathbb{R}}[z] = \eta r$ and $\mathrm{Proj}_{\overline{\lambda}\eta\mathbb{R}}[z] = \overline{\lambda}\eta s$. Then,

$$\begin{cases} \frac{1}{2}\big(z + \eta^2\overline{z}\big) = \eta r \\ \frac{1}{2}\big(z - i\eta^2\overline{z}\big) = \overline{\lambda}\eta s \end{cases} \Rightarrow z = \eta(1-i)(ir + \overline{\lambda}s)$$

and we are left with computing the argument of $ir + \overline{\lambda}s$. Assume $r, s > 0$ for now and consider the triangle $[ABC]$ where $A = 0$, $B = ir$ and $C = ir + \overline{\lambda}s$. Set $\alpha = \arg(ir + \overline{\lambda}s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and note that $B\widehat{A}C = \frac{\pi}{2} - \alpha$. The law of cosines yields $\overline{AC} = \sqrt{r^2 + s^2 - \sqrt{2}rs}$ and the law of sines gives

$$\frac{\sin B\widehat{A}C}{\overline{BC}} = \frac{\sin A\widehat{B}C}{\overline{AC}} \Leftrightarrow \frac{\sin \pi/2 - \alpha}{s} = \frac{\sqrt{2}/2}{\sqrt{r^2 + s^2 - \sqrt{2}rs}} \Leftrightarrow \cos\alpha = \frac{\sqrt{2}s}{2}(r^2 + s^2 - \sqrt{2}rs)^{-1/2}$$

Now, $s = r\big(1 + O(K^{-1})\big)$ yields

$$\cos\alpha = (4 - 2\sqrt{2})^{-1/2} + O\bigg(\frac{1}{K}\bigg) \Rightarrow \alpha = \frac{\pi}{8} + O\bigg(\frac{1}{K}\bigg)$$

as we wanted. The mod $\frac{\pi}{2}$ comes from the different signs r and s may have. □

The uniform comparability principle now follows easily if we assume $H(v_{k,k+1}) \geq 0$. To make up for this extra restriction, we allow an addictive constant on H. Note how this makes no difference in the statement of Proposition 5.25.

Corollary 5.29. *Let $H : \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta} \longrightarrow \mathbb{R}$ be defined according to Proposition 5.24, possibly with an addictive constant added. Then,*

$$H(v) \leq H(f) \leq Const \cdot H(v)$$

for any adjacent $f \in \operatorname{Int}\mathcal{F}_{\Omega_\delta} \setminus \{\boldsymbol{b}\}$ and $v \in \operatorname{Int}\mathcal{V}_{\Omega_\delta} \setminus \big(\{\boldsymbol{s}\} + \delta\big)$ surrounded by inner faces and vertices, as long as $H \geq 0$ at the inner vertices adjacent to v.

Proof. The first inequality follows from (45), the second one from Proposition 5.25 and the extra positivity condition. □

Regularity We now state the regularity of F, the original s-holomorphic function. We refer Theorem 3.12 of [CS12] for its rather lengthy proof.

Proposition 5.30. *Let $F : [\mathcal{C}_{\Omega_\delta} \setminus \{\boldsymbol{s}^{\rightarrow}\} \cup \mathcal{E}_{\Omega_\delta}; \boldsymbol{b}] \longrightarrow \mathbb{C}$ and $H : \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta} \longrightarrow \mathbb{R}$ be functions defined according to Proposition 5.24. Let $z_0 \in \operatorname{Int}\mathcal{E}_{\Omega_\delta}$ be at some definite distance from the boundary and the branching points: $d := \operatorname{dist}(z, \partial\Omega_\delta \cup \{\boldsymbol{b}\}) \geq Const \cdot \delta$. Set $M := \max_{z \in \mathcal{V}_{\Omega_\delta} \cup \mathcal{F}_{\Omega_\delta}} \big|H(z)\big|$. Then,*

$$\big|F(z_0)\big| \leq Const \cdot \frac{M^{1/2}}{d^{1/2}}$$

and for any adjacent z_1

$$\big|F(z_1) - F(z_0)\big| \leq Const \cdot \frac{M^{1/2}}{d^{3/2}} \delta$$

6 Auxiliary full-plane spinors

Following Section 3.2 of [CHI15], we construct two discrete spinors which will be very useful when handling the convergence of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$.

The first, $F_{[\mathbb{C}_\delta;a]}$, is a full-plane analogue of the discrete spinor observable $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$. Its role is to handle the singularity of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ near the branching points a_1 and $\mathbf{u}$: as seen in Proposition 6.4, $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ and $F_{[\mathbb{C}_\delta;a_1]}$ have the same "singularity" near a_1, hence $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]} - F_{[\mathbb{C}_\delta;a_1]}$ (seen in a neighbourhood of a_1 and extended to be 0 at $a_1^{\rightarrow}$) is s-holomorphic there, and similarly for $\mathbf{u}$. This is critical to prove the convergence of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ away from the boundary and the branching points.

With the convergence proven, we wish to relate values of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ near branching points (which, as stated in Propositions 4.17 and 4.18, are useful in computing ratios of spin correlations) with the expansion of the continuous version of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ near a_1. For this purpose we will make use of $F_{[\mathbb{C}_\delta;a_1]}$ (whose continuous counterpart is $1/\sqrt{z-a_1}$) and the second spinor $G_{[\mathbb{C}_\delta;a_1]}$, a discrete counterpart of $\sqrt{z-a_1}$. Seeing as $\widetilde{a}_1^{\rightarrow}+\delta \in \mathcal{C}^1_{\mathbb{C}_\delta}$ in Proposition 4.17, it is enough to define the real part of $G_{[\mathbb{C}_\delta;a_1]}$.

6.1 Outline

We start by stating the technical results regarding harmonic measures needed: namely, discrete Beurling estimates and convergence results. The proofs can be found in [LL04] and [CS11].

The spinor $F_{[\mathbb{C}_\delta;a]}$ is constructed in an explicit way. Using the fact the function $\Re(1/\sqrt{z-a})$ can be seen as the (properly normalized) harmonic measure of the tip point $a^{\rightarrow}+\delta = a+\frac{3\delta}{2}$ in the slit discrete plane $\mathbb{C}\setminus\{x+a : x \leq 0\}$, we define the real part of $F_{[\mathbb{C}_\delta;a]}$ on $\mathcal{C}^1_{\mathbb{C}_\delta}$. The imaginary part is then defined on $\mathcal{C}^i_{\mathbb{C}_\delta}$ by "discrete harmonic conjugation". The extension to $\mathcal{E}_{\mathbb{C}_\delta}$, $\mathcal{C}^1_{\mathbb{C}_\delta}$ and $\mathcal{C}^i_{\mathbb{C}_\delta}$ is done by means of Proposition 5.18. The convergence to the continuous counterpart follows from the convergence of harmonic measures.

Before stating the result, let us clarify what we mean when working with convergence of discrete functions or spinors. In this Section, the domain considered is the complete $\mathbb{C}$ from which we consider discretizations $\mathbb{C}_\delta$ done as described in Section 3.

Definition 6.1. A sequence of s-holomorphic functions (spinors) F_δ defined on $\mathbb{C}_\delta$ ($[\mathbb{C}_\delta;\mathbf{b}]$) is said to *converge* to a holomorphic function (spinor) f as $\delta \longrightarrow 0$ defined on $\mathbb{C}$ ($[\mathbb{C};\mathbf{b}]$) if, for every $z\in\mathbb{C}$, the values of the discrete versions at edge midpoints $F_\delta\big|_{\mathcal{E}_{\mathbb{C}_\delta}}$ at approximations of z ($\widetilde{z}$) on the lattice converge to the value of f at z ($\widetilde{z}$) as $\delta \longrightarrow 0$.

From now on, when writing $F_\delta(z)$ for any $z\in\mathbb{C}_\delta$, we imply F_δ is computed at an (adequate) approximation of z on $\mathcal{E}_{\mathbb{C}_\delta}$, and similarly for discrete spinors.

Definition 6.2. A convergence of discrete functions (spinors) F_δ to f is said to be *uniform* in a compact set $C\subset\mathbb{C}$ if the differences $|F_\delta(z)-f(z)|$ $\big(|F_\delta(\widetilde{z})-f(\widetilde{z})|\big)$ are uniformly small for $z\in C$.

With this clarification, we now state the main result describing $F_{[\mathbb{C}_\delta;a]}$.

Proposition 6.3. *For $a \in \mathcal{F}_{\mathbb{C}_\delta}$, there exists a unique s-holomorphic spinor*

$$F_{[\mathbb{C}_\delta;a]} : [\mathcal{C}_{\mathbb{C}_\delta} \cup \mathcal{E}_{\mathbb{C}_\delta} \setminus \{a^{\rightarrow}\}; a] \longrightarrow \mathbb{C}$$

such that $F_{[\mathbb{C}_\delta;a]}(\widetilde{a}^{\rightarrow} + \delta) = 1$ for a given lift of $a^{\rightarrow} + \delta$ and $F_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = o(1)$ as $z \to \infty$. Moreover,

$$\frac{1}{\vartheta(\delta)} F_{[\mathbb{C}_\delta;a]}(\widetilde{z}) \xrightarrow{\delta \to 0} f_{[\mathbb{C}_\delta;a]}(\widetilde{z}) := \frac{1}{\sqrt{z-a}}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{a\}$, with the normalizing factor given by

$$\vartheta(\delta) := F_{[\mathbb{C}_\delta;a]}\left(\widetilde{a}^{\rightarrow} + \delta + 2\delta \left\lfloor \frac{1}{2\delta} \right\rfloor\right).$$

The normalizing factor is the value $F_{[\mathbb{C}_\delta;a]}$ takes at an appropriate approximation of $a+1$ in $\mathcal{C}^1_{\mathbb{C}_\delta}$. We will prove the following estimates on this factor:

$$Const \cdot \sqrt{\delta} \leq \vartheta(\delta) \leq Const \cdot \sqrt{\delta}. \tag{51}$$

We then study the singularity of $F_{[\mathbb{C}_\delta;a]}$ at $a^{\rightarrow}$.

Proposition 6.4. *The equality*

$$\operatorname{Proj}_{i\mathbb{R}}\left[F_{[\mathbb{C}_\delta;a]}\left(\widetilde{a}^{\rightarrow} \pm \frac{i}{2}\delta\right)\right] = \mp i$$

holds, with the lift taken to be on the same sheet as the lift of $a^{\rightarrow} + \delta$ of Proposition 6.3.

The construction of $G_{[\mathbb{C}_\delta;a]}$ is done by integrating $F_{[\mathbb{C}_\delta;a]}$ using Proposition 5.13.

Proposition 6.5. *For $a \in \mathcal{F}_{\mathbb{C}_\delta}$, there exists a unique discrete harmonic spinor*

$$G_{[\mathbb{C}_\delta;a]} : [\mathcal{C}^1_{\mathbb{C}_\delta}; a] \longrightarrow \mathbb{R}$$

such that $G_{[\mathbb{C}_\delta;a]}(\widetilde{a}^{\rightarrow} + \delta) = \delta$ for a given lift of $a^{\rightarrow} + \delta$, $G_{[\mathbb{C}_\delta;a]} = 0$ on the half-line $\{a + x : x \leq 0\}$ and $G_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = O\big(|z-a|^{-\frac{1}{2}}\big)$ as $z \to \infty$. Moreover,

$$\frac{1}{\vartheta(\delta)} G_{[\mathbb{C}_\delta;a]}(\widetilde{z}) \xrightarrow{\delta \to 0} g_{[\mathbb{C}_\delta;a]}(\widetilde{z}) := \Re\sqrt{z-a}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{a\}$.

6.2 Discrete harmonic measure

We introduce the discrete harmonic measure, which will be an important ingredient to not only define $F_{[\mathbb{C}_\delta;a]}$ but also bound discrete functions.

For our purposes, these functions are defined in a subset L of a lattice with the structure of $\mathbb{Z}$ after being scaled, shifted and rotated in some way. Define ∂L as the points of such lattice that are not in L

but have at least one neighbour in L. Recall that a function $F : L \cup \partial L \longrightarrow \mathbb{R}$ is discrete harmonic at $z \in L$ if

$$\sum_{w \sim z} \big(F(w) - F(z)\big) = 0$$

Just as in the continuous setting, a discrete harmonic function verifies a *maximum principle* if L is bounded: its maximum and minimum is attained on ∂L. The argument is the same as with the super/sub-harmonic function of Proposition 5.24 (in fact, discrete harmonicity is equivalent to simultaneous super and sub-harmonicity): if there if a minimum or maximum at $z \in L$, then discrete harmonicity propagates that extreme to its neighbours, then to the neghbours of the neighbours and so on, until the boundary is reached.

Remark 6.6. The maximum principle holds the same way when L is not bounded, with the added detail that the maximum may occur at infinity.

Definition 6.7. The *discrete harmonic measure* of $A \subseteq L$ viewed from z is the probability a simple random walk on the lattice starting at z reaches A before it leaves L. It is denoted by $\mathrm{hm}_A^L(z)$.

Considering the random walk starting at $z \in L \setminus A$ and conditioning on the first step,

$$\mathrm{hm}_A^L(z) = \frac{1}{4} \sum_{w \sim z} \mathrm{hm}_A^L(w) \Leftrightarrow \sum_{w \sim z} \big(\mathrm{hm}_A^L(w) - \mathrm{hm}_A^L(z)\big) = 0$$

therefore hm_A^L is discrete harmonic on $L \setminus A$. It also vanishes on ∂L and equals 1 on A.

The harmonic measure can be used to estimate a discrete harmonic function near its zeros.

Lemma 6.8. *Let $F : L \cup \partial L \longrightarrow \mathbb{R}$ be a discrete harmonic function defined on a bounded set L and its boundary ∂L which vanishes on $A \subseteq L$. Then, for any $z \in L$,*

$$|F(z)| \leq \max_{\partial L} |F| \cdot \mathrm{hm}_{\partial L}^{(L \cup \partial L)\setminus A}(z)$$

Proof. The statement holds on A, where both sides vanish, and on ∂L, where the harmonic measure is 1. By contradiction, suppose the inequality fails for $z \in L \setminus A$, and in particular that $F(z) \neq 0$. Depending on whether $F(z) \in \mathbb{R}^{\pm}$, consider the function $\max_{\partial L} |F| \cdot \mathrm{hm}_{\partial L}^{(L \cup \partial L)\setminus A} \mp F$. This is a discrete harmonic function on $L \setminus A$ that vanishes on A, is non-negative on ∂L but is positive on z, contradicting the maximum principle. □

Such a bound is also possible for a function that is superharmonic or subharmonic.

Lemma 6.9. *Let $F : L \cup \partial L \longrightarrow \mathbb{R}$ be a function defined on a bounded set L and its boundary ∂L and $A \subseteq L$. Then,*

1. *If F is superharmonic on $L \setminus A$, then for all $z \in L$*

$$F(z) \geq \min_A F \cdot \mathrm{hm}_A^L(z) + \min_{\partial L} F \cdot \big(1 - \mathrm{hm}_A^L(z)\big)$$

2. If F is subharmonic on $L \setminus A$, then for all $z \in L$

$$F(z) \leq \max_A F \cdot \mathrm{hm}_A^L(z) + \max_{\partial L} F \cdot \big(1 - \mathrm{hm}_A^L(z)\big)$$

Remark 6.10. Lemma 6.8 is an easy corollary of Lemma 6.9 and $1 - \mathrm{hm}_A^L(z) = \mathrm{hm}_{\partial L}^{(L \cup \partial L) \setminus A}(z)$.

Proof. Let us focus on the first inequality, the second one is analogous. Note that we know

$$\mathrm{hm}_A^L(z) \geq \frac{1}{4} \sum_{w \sim z} \mathrm{hm}_A^L(w)$$

for every $z \in L$, which implies the minimum of F must occur on $A \cup \partial L$. The bound is trivially true when $z \in A$ or $z \in \partial L$, and we can check the case $z \in L \setminus A$ by adapting the argument of Lemma 6.8 using the function $F - \Big(\min_A F \cdot \mathrm{hm}_A^L + \min_{\partial L} F \cdot \big(1 - \mathrm{hm}_A^L\big)\Big)$. □

Remark 6.11. Let us provide another reasoning to see that the first inequality of Lemma 6.9 holds.

Use the superharmonicity to bound F at z. If any of those neighbours are in either A or ∂L, bound the value of F at those points by either $\min_A F$ or $\min_{\partial L} F$, respectively. The remaining points must belong to $L \setminus A$, thus we can bound F there with the superharmonicity. Note that we can apply these bounds as many times as we want. After n steps, our bound takes the form

$$F(z) \geq c_A(n) \cdot \min_A F + c_{\partial L}(n) \cdot \min_{\partial L} F + \sum_{w \in L \setminus A} c_w(n) \cdot F(w)$$

where $c_A(n), c_{\partial L}(n), c_w(n) \in \mathbb{R}$ are constants obtained after grouping the terms according to the bound used. The trick is to look at these constants from the point of view of a random walk starting at z: we have

$$\begin{aligned}
c_A(n) &= \mathbb{P}\big(\text{Random walk hits } A \text{ for the first time in } n \text{ steps or less and before hitting } \partial L\big) \\
c_{\partial L}(n) &= \mathbb{P}\big(\text{Random walk hits } \partial L \text{ for the first time in } n \text{ steps or less and before hitting } A\big) \\
c_w(n) &= \mathbb{P}\big(\text{Random walk reaches } w \text{ in exactly } n \text{ steps without hitting } A \text{ nor } \partial L\big) \\
\sum_{w \in L \setminus A} c_w(n) &= \mathbb{P}\big(\text{Random walk does not hit } A \text{ nor } \partial L \text{ in the first } n \text{ steps}\big)
\end{aligned}$$

which can be formally proven by induction. Taking the limit $n \longrightarrow +\infty$,

$$\begin{aligned}
c_A(n) &\xrightarrow{n \to +\infty} \mathbb{P}\big(\text{Random walk hits } A \text{ before hitting } \partial L\big) = \mathrm{hm}_A^L(z) \\
c_{\partial L}(n) &\xrightarrow{n \to +\infty} \mathbb{P}\big(\text{Random walk hits } \partial L \text{ before hitting } A\big) = 1 - \mathrm{hm}_A^L(z) \\
\sum_{w \in L \setminus A} c_w(n) &\xrightarrow{n \to +\infty} \mathbb{P}\big(\text{Random walk never hits } A \text{ nor } \partial L\big) = 0
\end{aligned}$$

With this brief introduction, we proceed to the setting relevant for defining $F_{[\mathbb{C}_\delta; a]}$. For a face a in the square grid $\mathbb{C}_\delta$, let $L_a := \{(a^\rightarrow + \delta) + x : x < 0\}$ and define the slit discrete plane $\mathbb{X}_\delta := \mathcal{C}^1_{\mathbb{C}_\delta} \setminus L_a$. We state the Beurling estimates that we will use.

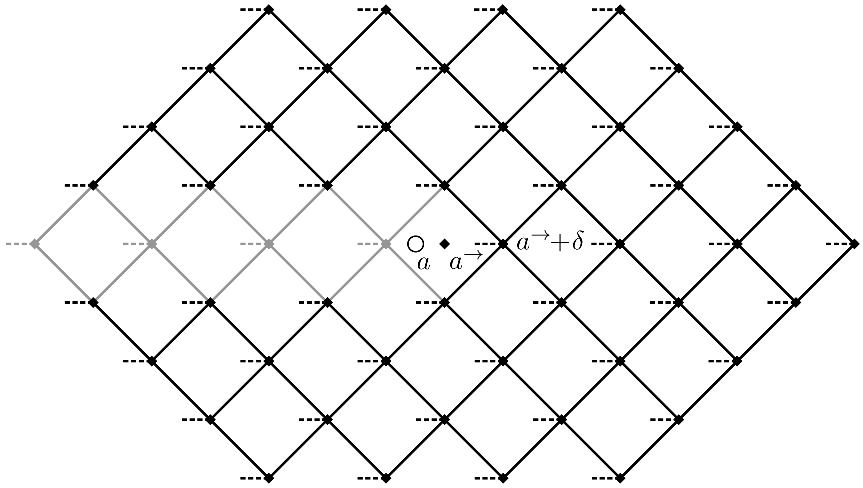

Figure 22: The slit plane $\mathbb{X}_\delta$. The corners from $\mathcal{C}^1_{\mathbb{C}_\delta}$ are connected according to its lattice structure. Grey corners belong to L_a.

Lemma 6.12 (Discrete Beurling Estimates)**.** *The following inequalities hold:*

$$\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z) \le Const \cdot \delta^{1/2}|z-a|^{-1/2} \tag{52}$$

$$\mathrm{hm}^{\mathbb{X}_\delta}_{A}(a^{\rightarrow}+\delta) \le Const \cdot \delta^{1/2}\big(\mathrm{dist}(a;A)\big)^{-1/2} \tag{53}$$

where $\mathrm{dist}(a;A) := \inf\{|a-a'| : a' \in A\}$.

Proof. These follow from reversibility arguments for random walks, refer to [LL04] for more details. ☐

Some additional estimates will be required.

Proposition 6.13. *The following inequalities hold*

1. *For every* $z \in \mathbb{X}_\delta$ *such that* $|\arg(z-a)-\pi| < \frac{\pi}{6}$*:*

$$\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z) \le Const \cdot \delta^{1/2}|\Im(z-a)||z-a|^{-3/2} \tag{54}$$

2. *For all neighbouring* $z, z' \in \mathbb{X}_\delta$*:*

$$\left|\frac{\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z') - \mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z)}{\delta}\right| \le Const \cdot \delta^{1/2}|z-a|^{-3/2} \tag{55}$$

Proof. For (54), set $r = \frac{1}{2}|z-a|$ and let $B(z,r)$ be the ball of radius r around z. Note that

$$\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z) = \mathbb{P}\Big(\substack{\text{Random walk reaches } \partial B(z,r) \\ \text{before hitting } \partial\mathbb{X}_\delta}\Big)\mathbb{P}\Big(\substack{\text{Random walk reaches } a^{\rightarrow}+\delta \\ \text{before hitting } \partial\mathbb{X}_\delta}\Big|\substack{\text{Random walk reaches } \partial B(z,r) \\ \text{before hitting } \partial\mathbb{X}_\delta}\Big).$$

The first factor is bounded by $O(|\Im(z-a)|\cdot|z-a|^{-1})$, whereas the second factor is bounded by $O\big(\delta^{\frac{1}{2}}|w-a|^{-\frac{1}{2}}\big)$ using (52) and assuming the random walk starts at $w \in \partial B(z,r)$, which yields the worst estimate when w is the point closest to a and $|w-a| = r = \frac{1}{2}|z-a|$.

Estimate (55) follows from a discrete version of Harnack's inequality, which applies because $\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}} \geq 0$ is discrete harmonic. Proposition 2.7(i) of [CS11] yields

$$\left| \frac{\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z') - \mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z)}{\delta} \right| \leq Const \cdot \frac{\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z)}{R}$$

for any $r > 0$ as long as $a^{\rightarrow} + \delta \notin B(z, r) \subset \mathbb{X}_\delta$. We take either $r = \frac{1}{2}|z - a|$ or $r = |\Im(z - a)|$, depending on whether z is far from or close to L_a, and then use respectively (52) or (54). Note how the cases $z = a^{\rightarrow} + \delta$ and $z' = a^{\rightarrow} + \delta$ are also covered. □

Finally, we state the convergence of $\mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}$.

Lemma 6.14. *We have*

$$\frac{1}{\vartheta(\delta)} \mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}(z) \xrightarrow{\delta \to 0} \left| \Re\left(\frac{1}{\sqrt{z-a}} \right) \right|$$

uniformly on compact subsets of $\mathbb{C} \setminus L_a$.

Additionally, the discrete derivatives in the left-hand side of (55) — after being normalized by $\vartheta(\delta)$ *— converge to the corresponding partial derivatives uniformly on compact subsets of* $\mathbb{C} \setminus L_a$.

Proof. The C^1-convergence is a result of Theorem 3.13 of [CS11], which states the convergence of normalized discrete Poisson kernels to their continuous versions. The only issue is that our domain is unbounded; however, if we prove the functions $F_\delta := \vartheta(\delta)^{-1} \mathrm{hm}^{\mathbb{X}_\delta}_{\{a^{\rightarrow}+\delta\}}$ are uniformly bounded on the annulus $\{z : |z - a| \geq r\}$ as $\delta \longrightarrow 0$ then this is not a problem.

Fix $r > 0$ and suppose by contradiction that we can take $F_\delta(z)$ to be as large as we want for some $\delta > 0$ and $z \in \{z : |z - a| \geq r\}$. Using the discrete harmonicity, we can build a path along which F_δ takes values equal or greater. Because of (52), this path cannot go to infinity. Hence, it must end at either $\partial\mathbb{X}_\delta$ or $a^{\rightarrow} + \delta$. But $F_\delta = 0$ at $\partial\mathbb{X}_\delta$ and $F_\delta(a^{\rightarrow} + \delta) = 1$, which limits the value of F_δ at the end of said path, therefore arriving at a contradiction. □

6.3 The spinor $F_{[\mathbb{C}_\delta;a]}$

We proceed to the construction of the full-plane spinor $F_{[\mathbb{C}_\delta;a]}$. We start with the real part $F^1_{[\mathbb{C}_\delta;a]}$, then move to the imaginary part $F^i_{[\mathbb{C}_\delta;a]}$ and then extend to all points. Recall that, under the theory of s-holomorphic functions developed in Section 5, the real and imaginary parts of $F_{[\mathbb{C}_\delta;a]}$ should be defined on $\mathcal{C}^1_{\mathbb{C}_\delta}$ and $\mathcal{C}^i_{\mathbb{C}_\delta}$, respectively.

Real part of $F_{[\mathbb{C}_\delta;a]}$. Consider $[\mathcal{C}^1_{\mathbb{C}_\delta} \setminus L_a; a]$ as the lifts $\mathbb{X}^{\pm}_\delta$ of the slit plane $\mathbb{X}_\delta$, with the upper side of the cut of $\mathbb{X}^{\pm}_\delta$ identified with the lower side of the cut of $\mathbb{X}^{\mp}_\delta$. Define

$$F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = \begin{cases} \pm \mathrm{hm}^{\mathbb{X}^{\pm}_\delta}_{\{\widetilde{a}^{\rightarrow}+\delta\}}(\widetilde{z}), & \widetilde{z} \in \mathbb{X}^{\pm}_\delta \\ 0, & \widetilde{z} \in [L_a; a] \end{cases}$$

This function is discrete harmonic on $[\mathbb{X}_\delta \setminus \{a^{\rightarrow} + \delta\}; a]$. We claim it is also discrete harmonic on $[L_a; a]$. Take any $\widetilde{z} \in [L_a; a]$ and assume without loss of generality that $\widetilde{z} + \sqrt{2}\delta\lambda$ is in the sheet $\mathbb{X}^{+}_\delta$.

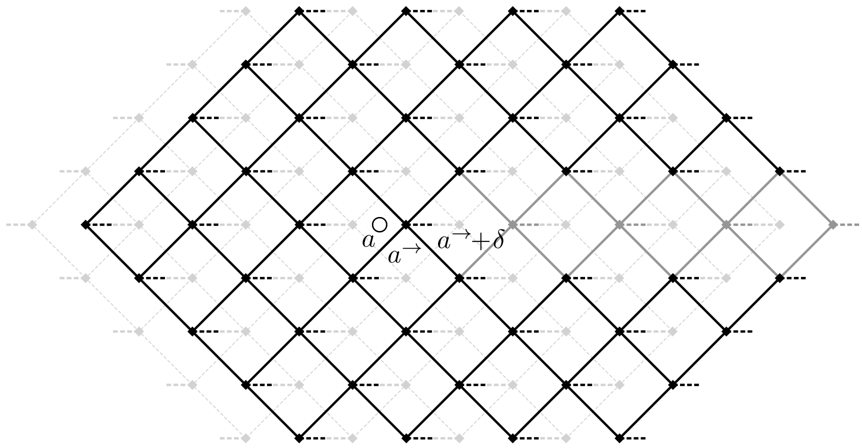

Figure 23: The slit plane $\mathbb{Y}_\delta$, with $\mathbb{X}_\delta$ in the background. The corners from $\mathcal{C}^i_{\mathbb{C}_\delta}$ are connected according to its lattice structure. Grey corners belong to R_a.

Note that $\mathrm{hm}^{\mathbb{X}^+_\delta}_{\{\widetilde{a}^{\rightarrow}+\delta\}}(\widetilde{z}+\sqrt{2}\delta\lambda) = \mathrm{hm}^{\mathbb{X}^-_\delta}_{\{\widetilde{a}^{\rightarrow}+\delta\}}(\widetilde{z}+\sqrt{2}\delta\overline{\lambda})$ by symmetry of the harmonic measure; hence, $F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z}+\sqrt{2}\delta\lambda) = -F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z}+\sqrt{2}\delta\overline{\lambda})$ because they live in different sheets. Similarly, $F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z}+\sqrt{2}\delta\lambda^3) = -F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z}+\sqrt{2}\delta\overline{\lambda}^3)$. Therefore, the discrete Laplacian at $\widetilde{z}$ equals 0.

In addition, note that it cannot be discrete harmonic at any lift $\widetilde{a}^{\rightarrow}+\delta$ of $a^{\rightarrow}+\delta$ because the point is a maximum (if $\widetilde{a}^{\rightarrow}+\delta \in \mathbb{X}^+_\delta$) or a minimum (if $\widetilde{a}^{\rightarrow}+\delta \in \mathbb{X}^-_\delta$). If it were discrete harmonic there, then $F^1_{[\mathbb{C}_\delta;a]}$ at its neighbours would be equal to it, which is contradictory with the definition.

Imaginary part of $F_{[\mathbb{C}_\delta;a]}$. Our objective is to define $F^i_{[\mathbb{C}_\delta;a]}$ on $[\mathcal{C}^i_{\mathbb{C}_\delta};a]$ so that we can follow Proposition 5.18. This is accomplished by doing a discrete version of harmonic conjugation: set $F^i_{[\mathbb{C}_\delta;a]}(\widetilde{a}^{\rightarrow}+2\delta)=0$ for one of the lifts of $a^{\rightarrow}+2\delta$ and then use (44) to extend the function to $[\mathcal{C}^i_{\mathbb{C}_\delta}\setminus\{a^{\rightarrow}\};a]$. We have to prove $F^i_{[\mathbb{C}_\delta;a]}$ is well defined, which amounts to checking sums of increments along loops γ of $[\mathcal{C}^i_{\mathbb{C}_\delta}\setminus\{a^{\rightarrow}\};a]$ going through $\widetilde{a}^{\rightarrow}+2\delta$ is 0. The strategy has the same idea to the one from the proof of Proposition 5.13, but using the discrete harmonicity of $F^1_{[\mathbb{C}_\delta;a]}$ instead.

(i) If γ is a simple loop (ie, does not intersect itself) on $[\mathcal{C}^i_{\mathbb{C}_\delta}\setminus\{a^{\rightarrow}\};a]$ and no lift of $a^{\rightarrow}+\delta$ is in the interior of γ, let $I\subset[\mathcal{C}^1_{\mathbb{C}_\delta};a]$ be the set of lifted sites in the interior of γ. Then

$$\sum_{(\widetilde{z}_1\widetilde{z}_2)\in\gamma}\left(F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z}_2)-F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z}_1)\right) = \pm i\sum_{\widetilde{z}\in I}[\Delta_\delta F^1_{[\mathbb{C}_\delta;a]}](\widetilde{z})$$

where the sign depends on the orientation of γ. To see that this is indeed true, expand the sum on the right-hand side and group the contributions of each edge; the ones from edges inside γ are cancelled out. Using the discrete harmonicity of $F^1_{[\mathbb{C}_\delta;a]}$ (note that no lift of $a^{\rightarrow}+\delta$ belongs to I), we conclude that

$$\pm i\sum_{\widetilde{z}\in I}[\Delta_\delta]F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z})=0.$$

(ii) If γ is the lift of $\omega\circ\omega$ where ω is a simple loop on $\mathcal{C}^i_{\mathbb{C}_\delta}$ with $a^{\rightarrow}+\delta$ in its interior, then the sum of the increments is 0 due to the spinor property of $F^1_{[\mathbb{C}_\delta;a]}$.

(iii) If γ_1 and γ_2 are simple loops on $[\mathcal{C}^i_{\mathbb{C}_\delta} \setminus \{a^\rightarrow\}; a]$ oriented in such a way that edges shared by γ_1 and γ_2 have opposite orientations, then the sum of the increments along $\gamma_1 \oplus \gamma_2$ is equal to the sum of the increments along γ_1 plus the sum of the increments along γ_2.

(iv) If γ is a simple loop on $[\mathcal{C}^i_{\mathbb{C}_\delta} \setminus \{a^\rightarrow\}; a]$ and $\widetilde{a}^\rightarrow + \delta$ is is in the interior, separate it into a collection of loops of the types described in (i) amd (ii) and use (iii) to decompose the sum of the increments along γ. Note how the fact that $F^1_{[\mathbb{C}_\delta;a]}$ is not defined at lifts of $a^\rightarrow$ means any γ that goes around the branching point a must go around the lifts of $a^\rightarrow + \delta$.

(v) If γ is a generic loop on $[\mathcal{C}^i_{\mathbb{C}_\delta} \setminus \{a^\rightarrow\}; a]$, separate it into a collection of simple loops and use (iii) to decompose the sum of increments along γ.

Now, take a simple path γ in $\mathcal{C}^i_{\mathbb{C}_\delta} \setminus \{a^\rightarrow\}$ that runs from $a^\rightarrow + 2\delta$ to some point in L_a and let $\overline{\gamma}$ be the reflection of γ about the line $\{a + x : x \in \mathbb{R}\}$ running in the opposite direction. Then, the lift of $\gamma \circ \overline{\gamma}$ connects both lifts of $a^\rightarrow + 2\delta$ and, because of the antisymmetry of $F^1_{[\mathbb{C}_\delta;a]}$ with respect to L_a, the sum of the increments along the lift of $\gamma \circ \overline{\gamma}$ is 0. Hence, $F^i_{[\mathbb{C}_\delta;a]}$ vanishes at both lifts of $a^\rightarrow + 2\delta$. This implies that $F^i_{[\mathbb{C}_\delta;a]}$ inherits the spinor property of $F^1_{[\mathbb{C}_\delta;a]}$.

Furthermore, if $z = a^\rightarrow + k \cdot 2\delta$ for $k \in \mathbb{Z}^+$ then

$$\begin{aligned} F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z} + 2\delta) - F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z}) &= \Big[F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z} + 2\delta) - F^i_{[\mathbb{C}_\delta;a]}\big(\widetilde{z} + (1 \pm i)\delta\big)\Big] \\ &\quad + \Big[F^i_{[\mathbb{C}_\delta;a]}\big(\widetilde{z} + (1 \pm i)\delta\big) - F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z})\Big] \\ &= \mp i \Big[F^1_{[\mathbb{C}_\delta;a]}\big(\widetilde{z} + (2 \pm i)\delta\big) - 2F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z} + \delta) + F^1_{[\mathbb{C}_\delta;a]}\big(\widetilde{z} \pm i\delta\big)\Big] \end{aligned}$$

and the symmetry of $F^1_{[\mathbb{C}_\delta;a]}$ about the half-line $R_a := \{a^\rightarrow + x : x > 0\}$ yields

$$2\big(F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z} + 2\delta) - F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z})\big) = 0$$

Therefore, $F^i_{[\mathbb{C}_\delta;a]}$ vanishes on $[R_a; a]$ by induction.

Remark 6.15. Note that it impossible to define $F^i_{[\mathbb{C}_\delta;a]}$ at the lifts of $a^\rightarrow$: for instance, the increments along the smallest loop of $\mathcal{C}^i_{\mathbb{C}_\delta}$ around $a^\rightarrow + \delta$ do not add up to 0. In general, increments alongside loops going around the branching point a but without going around the lifts of $a^\rightarrow + \delta$ may not add up to 0.

However, let $\mathbb{Y}_\delta := \mathcal{C}^i_{\mathbb{C}_\delta} \setminus R_a$ be a slit discrete plane and consider $[\mathcal{C}^i_{\mathbb{C}_\delta} \setminus (\{a^\rightarrow\} \cup R_a); a]$ as the lifts $\mathbb{Y}^\pm_\delta \setminus \{\widetilde{a}^\rightarrow\}$ of $\mathbb{Y}_\delta \setminus \{a^\rightarrow\}$, choosing the signs so that $F^1_{[\mathbb{C}_\delta;a]} > 0$ on the upper half of $\mathbb{Y}^+_\delta$ and lower half of $\mathbb{Y}^-_\delta$. When restricted to only one of the sheets $\mathbb{Y}^\pm_\delta$, $F^i_{[\mathbb{C}_\delta;a]}$ can be extended to $\widetilde{a}^\rightarrow$ using the same arguments as before: the loops whose increments do not sum up to 0 cross R_a, hence they do not pose a problem here.

Remark 6.16. One can write $F^i_{[\mathbb{C}_\delta;a]}$ using a discrete harmonic measure representation. Consider $F^i_{[\mathbb{C}_\delta;a]}$ restricted to a single sheet $\mathbb{Y}^\pm_\delta$. In this situation, Remark 6.15 establishes $F^i_{[\mathbb{C}_\delta;a]}$ can be extended to the point $\widetilde{a}^\rightarrow$. With this extension, $F^i_{[\mathbb{C}_\delta;a]}$ is a bounded function that is harmonic on $\mathbb{Y}^\pm_\delta \setminus \{\widetilde{a}^\rightarrow\}$, equals $F^i_{[\mathbb{C}_\delta;a]}(\widetilde{a}^\rightarrow)$ on $\widetilde{a}^\rightarrow$ and vanishes on R_a. This implies that it must be the function $-iF^i_{[\mathbb{C}_\delta;a]}(\widetilde{a}^\rightarrow) \cdot \mathrm{hm}^{\mathbb{Y}^\pm_\delta}_{\{\widetilde{a}^\rightarrow\}}$, since it verifies the same boundary conditions.

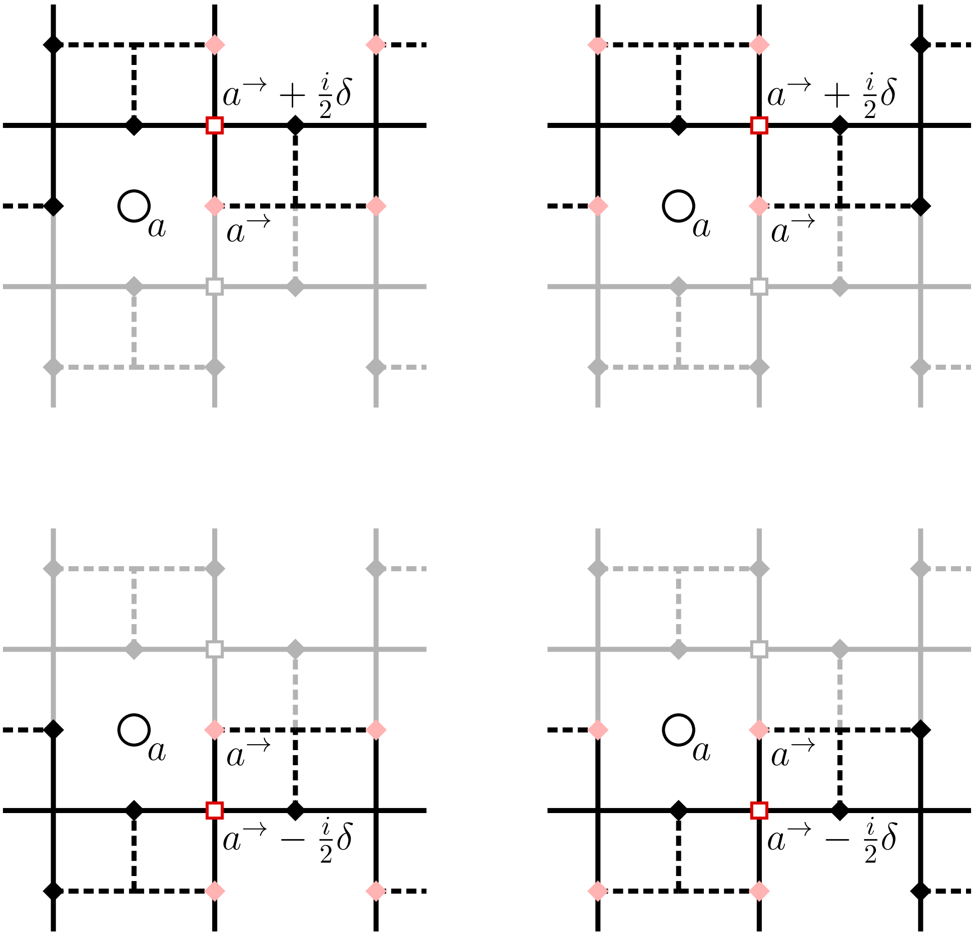

Figure 24: Image of lattice near a, with connections done according to Section 5 (namely, Figure 17). The four red corners in each image are the ones for which equation (44) is verified, depending on whether we are looking at defining $F_{[\mathbb{C}_\delta;a]}$ near $a^{\rightarrow}+\frac{i}{2}\delta$ (top two images) or near $a^{\rightarrow}-\frac{i}{2}\delta$ (bottom two images).

Knowing $F^i_{[\mathbb{C}_\delta;a]}$ restricted to $\mathbb{Y}^{\pm}_\delta$ is a multiple of $\mathrm{hm}^{\mathbb{Y}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}\}}$, we can compute the multiplicative constant. Using symmetry arguments to compare $\mathrm{hm}^{\mathbb{Y}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}\}}$ with $\mathrm{hm}^{\mathbb{X}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}+\delta\}}(\tilde{z})$ and the definition of $F^i_{[\mathbb{C}_\delta;a]}$, we get

$$\begin{aligned}\mathrm{hm}^{\mathbb{Y}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}\}}\big(\tilde{a}^{\rightarrow}+(1+i)\delta\big)-\mathrm{hm}^{\mathbb{Y}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}\}}(\tilde{a}^{\rightarrow}) &= \pm\Big[\mathrm{hm}^{\mathbb{X}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}+\delta\}}(\tilde{a}^{\rightarrow}+i\delta)-\mathrm{hm}^{\mathbb{X}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}+\delta\}}(\tilde{a}^{\rightarrow}+\delta)\Big]\\ &= \pm\Big[F^1_{[\mathbb{C}_\delta;a]}(\tilde{a}^{\rightarrow}+i\delta)-F^1_{[\mathbb{C}_\delta;a]}(\tilde{a}^{\rightarrow})\Big]\\ &= \pm i\Big[F^i_{[\mathbb{C}_\delta;a]}\big(\tilde{a}^{\rightarrow}+(1+i)\delta\big)-F^i_{[\mathbb{C}_\delta;a]}(\tilde{a}^{\rightarrow})\Big]\end{aligned}$$

Hence,

$$F^i_{[\mathbb{C}_\delta;a]}(\tilde{z})=\mp i\cdot\mathrm{hm}^{\mathbb{Y}^{\pm}_\delta}_{\{\tilde{a}^{\rightarrow}\}}(\tilde{z})\text{ for }z\in\mathbb{Y}^{\pm}_\delta$$

Full extension of $F_{[\mathbb{C}_\delta;a]}$. Consider the function equal to $F^1_{[\mathbb{C}_\delta;a]}$ on $[\mathcal{C}^1_{\mathbb{C}_\delta};a]$ and equal to $F^i_{[\mathbb{C}_\delta;a]}$ on $[\mathcal{C}^i_{\mathbb{C}_\delta}\setminus\{a^{\rightarrow}\};a]$, then extend it to $[\mathcal{C}^{\lambda}_{\mathbb{C}_\delta}\cup\mathcal{C}^{\bar{\lambda}}_{\mathbb{C}_\delta}\cup\mathcal{E}_{\mathbb{C}_\delta};a]$ using Proposition 5.18. The objective is to extend it to $[\mathcal{C}_{\mathbb{C}_\delta}\cup\mathcal{E}_{\mathbb{C}_\delta}\setminus\{a^{\rightarrow}\};a]$, which is not guaranteed by the statement: since $F^i_{[\mathbb{C}_\delta;a]}$ is not defined at the lifts of $a^{\rightarrow}$, $F_{[\mathbb{C}_\delta;a]}$ is not defined at the lifts of the edge midpoints $a^{\rightarrow}\pm\frac{i}{2}\delta$ and corners $a^{\rightarrow}+\frac{1\pm i}{2}\delta$ and $a^{\rightarrow}+\frac{-1\pm i}{2}\delta$. Some additional care is thus required.

Let $\tilde{z}$ be a lift of $a^{\rightarrow}+\frac{i}{2}\delta$ and consider the sheet $\mathbb{Y}^{\pm}_\delta$ it belongs to. As explored in Remark 6.15, one can define $F^i_{[\mathbb{C}_\delta;a]}$ at $\widetilde{a}^{\rightarrow}$ by harmonic conjugation of $F^1_{[\mathbb{C}_\delta;a]}$. This implies that equation (44) is satisfied for

$$\widetilde{c}_{NW} = \widetilde{z} + \left(-1 + \frac{i}{2}\right)\delta \qquad \widetilde{c}_{SW} = \widetilde{z} + \left(-1 - \frac{i}{2}\right)\delta \qquad \widetilde{c}_{SE} = \widetilde{z} - \frac{i}{2}\delta \qquad \widetilde{c}_{NE} = \widetilde{z} + \frac{i}{2}$$

and

$$\widetilde{c}_{NW} = \widetilde{z} + \frac{i}{2}\delta \qquad \widetilde{c}_{SW} = \widetilde{z} - \frac{i}{2}\delta \qquad \widetilde{c}_{SE} = \widetilde{z} + \left(1 - \frac{i}{2}\right)\delta \qquad \widetilde{c}_{NE} = \widetilde{z} + \left(1 + \frac{i}{2}\right)\delta$$

(Figure 24, top two images) and now Proposition 5.18 allows us to define $F_{[\mathbb{C}_\delta;a]}$ at the edge midpoint $\widetilde{a}^{\rightarrow} + \frac{i}{2}\delta$ and corners $\widetilde{a}^{\rightarrow} + \frac{1+i}{2}\delta$ and $\widetilde{a}^{\rightarrow} + \frac{-1+i}{2}\delta$. The same argument applies to the lifted $\widetilde{z} - i\delta = \widetilde{a}^{\rightarrow} - \frac{i}{2}\delta$ near it (Figure 24, bottom two images), with the issue that the values given to $F_{[\mathbb{C}_\delta;a]}(\widetilde{a})$ do not agree because $\widetilde{z} \in \mathbb{Y}^{\pm}_{\delta} \Rightarrow \widetilde{z} - i\delta \in \mathbb{Y}^{\mp}_{\delta}$. However, if we remove $\widetilde{a}$ from the domain of $F_{[\mathbb{C}_\delta;a]}$, the projections of $\widetilde{a}^{\rightarrow} \pm \frac{i}{2}\delta$ to the remaining corners still match. We thus define $F_{[\mathbb{C}_\delta;a]}$ at $\widetilde{a}^{\rightarrow} \pm \frac{i}{2}\delta$, $\widetilde{a}^{\rightarrow} + \frac{1\pm i}{2}\delta$ and $\widetilde{a}^{\rightarrow} + \frac{-1\pm i}{2}\delta$ this way, leave $F_{[\mathbb{C}_\delta;a]}$ at $\widetilde{a}^{\rightarrow}$ undefined and the resulting function is s-holomorphic in its domain.

Proof of Proposition 6.3. The convergence of $F^1_{[\mathbb{C}_\delta;a]}$ and its discrete derivatives is given by Lemma 6.14:

$$\frac{1}{\vartheta(\delta)} F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z}) \xrightarrow{\delta\to 0} \Re\left(\frac{1}{\sqrt{z-a}}\right)$$

uniformly on compact subsets of $[\mathbb{C} \setminus L_a; a]$, which we can extend to L_a. For $F^i_{[\mathbb{C}_\delta;a]}$, Remark 6.16 yields

$$\frac{1}{\vartheta(\delta)} F^i_{[\mathbb{C}_\delta;a]}(\widetilde{z}) \xrightarrow{\delta\to 0} \Im\left(\frac{1}{\sqrt{z-a}}\right)$$

uniformly on compact subsets of $[\mathbb{C} \setminus R_a; a]$, which again can also be extended to R_a, together with the convergence of the discrete derivatives of $F^i_{[\mathbb{C}_\delta;a]}$. Since for $z \in \mathcal{E}_{\mathbb{C}_\delta}$

$$F_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = F_{[\mathbb{C}_\delta;a]}\left(\widetilde{z} + \frac{i}{2}\delta\right) + F_{[\mathbb{C}_\delta;a]}\left(\widetilde{z} - \frac{i}{2}\delta\right)$$

and the two terms are values of $F^1_{[\mathbb{C}_\delta;a]}$ and $F^i_{[\mathbb{C}_\delta;a]}$, the C^1-convergence follows. □

Proof of Proposition 6.4. This is a direct consequence of the considerations done before. On each sheet $\mathbb{Y}^{\pm}_{\delta}$,

$$\mathrm{Proj}_{i\mathbb{R}}\left[F_{[\mathbb{C}_\delta;a]}\left(a + \frac{1+i}{2}\delta\right)\right] = \mathrm{Proj}_{i\mathbb{R}}\left[F_{[\mathbb{C}_\delta;a]}\left(a + \frac{1-i}{2}\delta\right)\right] = F^i_{[\mathbb{C}_\delta;\widetilde{a}]}(a^{\rightarrow}) = \mp i$$

by Remark 6.16, and $\mathbb{X}^{+}_{\delta}$ coincides with $\mathbb{Y}^{+}_{\delta}$ on the upper half and with $\mathbb{Y}^{-}_{\delta}$ on the lower half. □

6.4 The spinor $G_{[\mathbb{C}_\delta;a]}$

To define $G_{[\mathbb{C}_\delta;a]}$ as a discrete version of $\Re\sqrt{z-a}$, we start with $F_{[\mathbb{C}_\delta;a]}$, integrate it by following Proposition 5.13 and compute the real part divided by 2 (because the primitive of $1/\sqrt{z-a}$ is $2\sqrt{z-a}$). Since we want a spinor defined on $[\mathcal{C}^1_{\mathbb{C}_\delta}; a]$, we use the primitive on the vertices $H|_{\mathcal{V}_{\mathbb{C}_\delta}}$ and for each corner

$z \in \mathcal{C}^1_{\mathbb{C}_\delta}$ define $G_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = \frac{1}{2}H|_{\mathcal{V}_{\mathbb{C}_\delta}}\big(f(\widetilde{z})\big)$[15]. This function is discrete harmonic everywhere — including at $a^{\to}+\delta$ — because $H|_{\mathcal{V}_{\mathbb{C}_\delta}}$ is discrete harmonic and the change of lattices does not change the neighbours taken in the computation of the discrete Laplacians.

Regarding the singularity of $F_{[\mathbb{C}_\delta;a]}$ at $a^{\to}$, we claim this is not an issue. Recalling the proof of Proposition 5.13, the well definition of $H|_{\mathcal{V}_{\mathbb{C}_\delta}}$ hinges on the discrete holomorphicity of $F_{[\mathbb{C}_\delta;a]}$ around the vertices and the singularity of $F_{[\mathbb{C}_\delta;a]}$ at $a^{\to}$ only affects the discrete holomorphicity of $F_{[\mathbb{C}_\delta;a]}$ at the vertex $a+\delta$ (Proposition 5.17), which is not needed — see Remark 5.14. Furthermore, the discrete harmonicity of $H|_{\mathcal{V}_{\mathbb{C}_\delta}}$ uses the discrete holomorphicity at vertices, so the argument follows all the same.

Let us find an expression for $G_{[\mathbb{C}_\delta;a]}$. After fixing a lifted corner $\widetilde{c}$, for $z \in \mathcal{C}^1_{\mathbb{C}_\delta}$ we have

$$G_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = G_{[\mathbb{C}_\delta;a]}(\widetilde{c}) + \Re \int_{f(\widetilde{c})}^{f(\widetilde{z})} F_{[\mathbb{C}_\delta;a]}(\widetilde{w})\, d\widetilde{w}$$

Now, we would like to choose paths that make the integral simple to compute. Consider for instance four corners $\widetilde{c}_{NW}, \widetilde{c}_{SE} \in [\mathcal{C}^1_{\mathbb{C}_\delta};a]$ and $\widetilde{c}_{SW}, \widetilde{c}_{NE} \in [\mathcal{C}^i_{\mathbb{C}_\delta};a]$ that form a square of side δ. Let $\widetilde{w}_W = \frac{1}{2}(\widetilde{c}_{NW}+\widetilde{c}_{SW})$ and $\widetilde{w}_E = \frac{1}{2}(\widetilde{c}_{NE}+\widetilde{c}_{SE})$, and let $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in [\mathcal{F}_{\mathbb{C}_\delta};a]$ be such that $\frac{1}{2}(\widetilde{f}_1+\widetilde{f}_2) = \widetilde{w}_W$ and $\frac{1}{2}(\widetilde{f}_2+\widetilde{f}_3) = \widetilde{w}_E$. Along the path $\gamma = \widetilde{f}_1 \sim \widetilde{f}_2 \sim \widetilde{f}_3$, we have

$$\begin{aligned}
\int_\gamma F_{[\mathbb{C}_\delta;a]}(\widetilde{w})\, d\widetilde{w} &= F_{[\mathbb{C}_\delta;a]}(\widetilde{z}_W)\big(\widetilde{f}_2 - \widetilde{f}_1\big) + F_{[\mathbb{C}_\delta;a]}(\widetilde{z}_E)\big(\widetilde{f}_3 - \widetilde{f}_2\big) \\
&= \Big(F_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NW}) + F_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SW})\Big)\cdot(1+i)\delta + \big(F_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SE}) + F_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NE})\big)\cdot(1-i)\delta \\
&= \delta\Big[F^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NW}) + iF^i_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SW}) + F^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SE}) - iF^i_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NE})\Big] \\
&\quad + \delta\Big[iF^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NW}) + F^i_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SW}) - iF^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SE}) + F^i_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NE})\Big)
\end{aligned}$$

and taking the real part yields

$$\begin{aligned}
\Re\int_\gamma F_{[\mathbb{C}_\delta;a]}(\widetilde{w})\, d\widetilde{w} &= \delta\Big[F^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NW}) + iF^i_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SW}) + F^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SE}) - iF^i_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{NE})\Big] \\
&= 2\delta F^1_{[\mathbb{C}_\delta;a]}(\widetilde{c}_{SE})
\end{aligned}$$

using (44). Thus, if our path is composed of multiple paths like γ, the integral admits a simple expression. Since $G_{[\mathbb{C}_\delta;a]}$ is a discrete version of $\Re\sqrt{z-a}$, intuitively one should be able to take as a fixed face $\widetilde{f}$ the points $f = u+iv$ with $u \to -\infty$, where $G_{[\mathbb{C}_\delta;a]} \equiv 0$. In fact, one can see that

$$\Re\int_{\widetilde{u+iv}}^{\widetilde{u+iv'}} F_{[\mathbb{C}_\delta;a]}(\widetilde{w})\, d\widetilde{w} \xrightarrow{u\to-\infty} 0$$

using the above argument rotated by $\frac{\pi}{2}$ and the bound (54). Hence, an expression for $G_{[\mathbb{C}_\delta;a]}$ is

$$G_{[\mathbb{C}_\delta;a]}(\widetilde{z}) = \delta\sum_{k=0}^{\infty} F^1_{[\mathbb{C}_\delta;a]}(\widetilde{z} - k\cdot 2\delta) \tag{56}$$

[15]Recall that $f(z)$ is the face containing z.

Remark 6.17. The relevant properties of $G_{[\mathbb{C}_\delta;a]}$ can be deduced from the expression (56). Considering the two sheets $\mathbb{X}_\delta^\pm$ individually, for $z \in \mathbb{X}_\delta^\pm$ we set

$$G_{[\mathbb{C}_\delta;a]}(z) := \pm\delta \sum_{k=0}^{\infty} \mathrm{hm}_{\{a^\to+\delta\}}^{\mathbb{X}_\delta}(z - k \cdot 2\delta)$$

and estimate (54) guarantees the sum convergence. In addition, $G_{[\mathbb{C}_\delta;a]} = 0$ in L_a (again confirming the additive constant is correct), which ensures the function is well-defined in $[\mathcal{C}^1_{\mathbb{C}_\delta};a]$, as there is no ambiguity for the points of R_a. The discrete harmonicity outside R_a follows directly from the discrete harmonicity of $F^1_{[\mathbb{C}_\delta;a]}$, and for the points of R_a one can use estimate (55); for more details, see Section 3.2.3 of [CHI15].

Proof of Proposition 6.5. Similarly to Lemma 6.14, we use Theorem 3.13 of [CS11]. The function vanishes on R_a, is bounded when $z \longrightarrow \infty$ by (52) and (54) and also near 0 due to the maximum principle. Hence,

$$\frac{1}{\nu(\delta)} G_{[\mathbb{C}_\delta;a]}(\widetilde{z}) \xrightarrow{\delta\to 0} \Re\sqrt{z-a}$$

uniformly on compact subsets of $\mathbb{C} \setminus L_a$, with a multiplicative normalization $\nu(\delta)$ which we choose to fix at $a+1$. Since we also have convergence to the directional derivatives,

$$\frac{1}{\nu(\delta)} \cdot \frac{1}{2} F_{[\mathbb{C}_\delta;a]}(\widetilde{z}) \xrightarrow{\delta\to 0} \partial_x \Re\sqrt{z-a} = \frac{1}{2}\Re\left(\frac{1}{\sqrt{z-a}}\right)$$

implying $\nu(\delta)/\vartheta(\delta) \xrightarrow{\delta\to 0} 1$. Finally, we can extend the convergence to near L_a using the convergence of $F^1_{[\mathbb{C}_\delta;a]}$ and uniform bounds on the tails in the sum (56), which are provided by (54). □

Proof of estimate (51). Since $\vartheta(\delta)$ is $\mathrm{hm}_{\{a^\to+\delta\}}^{\mathbb{X}_\delta}$ at an approximation of $a+1$, the upper bound follows readily from (52).

For the lower bound, consider the functions $\vartheta(\delta)^{-1} G_{[\mathbb{C}_\delta;a]}$ in the unitary ball $B(a,1)$ around a. The previous proof shows they are uniformly bounded by a constant on $\partial B(a,1)$. Hence,

$$\frac{1}{\vartheta(\delta)}\left|G_{[\mathbb{C}_\delta;a]}(\widetilde{z})\right| \leq Const \cdot \mathrm{hm}_{\partial B(a,1)\setminus L_a}^{(B(a,1)\cup\partial B(a,1))\setminus L_a}(z) = Const \cdot \mathrm{hm}_{\partial B(a,1)\setminus L_a}^{\mathbb{X}_\delta}(z)$$

for $z \in B(a,1)$ by Lemma 6.8. Knowing this, the bound follows from (53):

$$\frac{1}{\vartheta(\delta)}\delta = \frac{1}{\vartheta(\delta)} G_{[\mathbb{C}_\delta;a]}(\widetilde{a}^\to + \delta) \leq Const \cdot \mathrm{hm}_{\partial B(a,1)\setminus L_a}^{\mathbb{X}_\delta}(a^\to + \delta) \leq Const \cdot \sqrt{\delta}.$$

□

7 Convergence of the discrete observables

We are now ready to prove the convergence of the discrete observables of Section 4, using the theory developed Section 5 and the tools of Section 6. The first step is to define the continuous counterparts, then we prove convergence and finally we relate the values of the discrete spinors near branching points (from which we can compute ratios of spin correlations, according to Propositions 4.17 and 4.18) with the expansion of the continuous spinors at those branching points.

7.1 The continuous spinors

Let us try to define the continuous version $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ of the spinor observable $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ from Definitions 4.11 and 4.12. Following Proposition 4.21, $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ should be holomorphic in $[\Omega;\mathbf{a};\mathbf{u}]$, should have $\mathbf{a},\mathbf{u}$ as branching points with multiplicative monodromy -1 and should verify $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}$ on the boundary. To complete the boundary value problem, some information regarding the behaviour of $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ near the branching points is needed.

Let us start with the branching points $b = a_2, \dots, a_n$ where there are no discrete singularities. Around these points, the discrete primitive $H = \Re \int (F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]})^2$ defined in Proposition 5.24 is bounded from below: this is because H is superhamonic everywhere around b, therefore it cannot blow up to negative values there. Imposing this for the limit, together with $(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$ being holomorphic in a punctured neighbourhood of b, we conclude $h = \Re \int (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$ should behave like $C \log|z-b| + C'$ for some $C \leq 0, C' \in \mathbb{C}$:

1. If b were a removable singularity of $(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$, then for $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ to branch around b we would have $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}(b) = 0$, therefore h is constant around b.

2. If b were a pole of order n of $(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$, then $n > 1$ would imply that h is not bounded from below. For $n = 1$ we have $\int (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2 = C\log(z-b) + C'$ for some complex constants C, C', which is defined around the logarithmic branching point b. Now, the real part of $\log(z-b)$ is $\log|z-b|$, which is well-defined in the punctured plane, whereas the imaginary part is $\arg(z-b)$, which increases (or decreases) by 2π when going around b once. For $\Re \int (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$ to be defined in a punctured plane, we must have $C \in \mathbb{R}$, and the boundedness from below implies $C < 0$.

3. If b were an essential singularity of $(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$, then it would also be an essential singularity of $\int (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$. This would imply the images of the punctured neighbourhood previously fixed would form a dense set in $\mathbb{C}$ (Casorati–Weierstrass theorem). This is contradictory with h being bounded from below.

Therefore, $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ behaves like $\sqrt{C}(z-b)^{-1/2}$, or $C(z-b)^{-1/2}$ with $C \in i\mathbb{R}$.

For the other branching points $b = a_1, \mathbf{u}$, Lemmas 4.23 and 4.25 describe the "discrete singularities" of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$. Proposition 6.3 defines a spinor $F_{[\mathbb{C}_\delta;b]}$ which, according to Proposition 6.4, has the same type of "singularity". For the case $b = a_1$ this is conveyed by simply stating that $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]} - F_{[\mathbb{C}_\delta;b]}$ can be extended to be s-holomorphic at $b + \delta/2$, since the projections with opposing signs of each spinor cancel each other out. Seeing as $F_{[\mathbb{C}_\delta;b]}$ is a discretization of $1/\sqrt{z-b}$. This means $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ does not blow up faster than $\pm 1/\sqrt{z-b}$ at $b = a_1$.

For $b = \mathbf{u}$, the same idea works but Lemma 4.25 states there is an additional multiplicative constant, which is the ratio of two other spinors. Note that said spinors are simple, in the sense that they have two less endpoints in their disorder lines. This establishes a recursive method for accurately defining these spinors: one starts by studying the spinors with no disorder lines (for which the case $b = \mathbf{u}$ is vacuous), then uses those objects to define spinors with one disorder line, which are used in the definition of spinors with two disorder lines and so on. Although more involved, this situation does not require additional technical tools to handle when compared to the case $b = a_1$.

Signs of square roots: choosing c_k and η_{c_k}. To accurately define $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ near a_1 and $\mathbf{u}$ it is necessary to state unambiguously which square root we are referring to when writing $\sqrt{z-a_1}$ and $\sqrt{z-u_j}$, both for the continuous and the discrete versions. The continuous square root is set to agree with the limit of the discrete one, and the choice of the discrete square root's sign is related to the computation of signs given by Proposition 4.8, which depend on the choices of which corner c_k near v_k should we take and the sign of the η_{c_k}.

These choices are mostly free[16], and we will set them now before proceeding. Our objective is to choose the square roots so that $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ behaves like $1/\sqrt{z-b}$ near $b = a_1, \mathbf{u}$. This way, the signs do not interfere in future arguments. For other branching points, the square root chosen has no relevance: the constant $C \in i\mathbb{R}$ can be either in the upper or lower half plane.

When computing signs τ_0 as stated in Proposition 4.8 for the continuous case, we use the expression

$$\tau^0 = (-1)^{\Theta\cdot\Gamma} \cdot (-1)^{\Theta^0\cdot\Gamma} \cdot (-1)^{\Theta^0\cdot\Theta^0} \cdot \operatorname{sign}\big(s^0\big) \cdot \prod_{k=1}^{m} -i\eta_{c_{2k-1}}\overline{\eta}_{c_{2k}} \exp\left(-\frac{i}{2}\operatorname{wind}(\gamma_k^0)\right)$$

in similar conditions for Γ, Θ and Θ^0 with some extra details:

1. The Θ^0 used should be formed by paths that leave and enter the midpoints from the left. Formally, if $\gamma_k : [0,1] \longrightarrow \Omega$ is such a path that is then lifted to $[\Omega;\mathbf{a};\mathbf{u}]$, we require $\gamma'(0) = 1$ and $\gamma'(1) = -1$. This is to ensure convergence and is a consequence of the endpoints of the γ_k being $c_k = u_k + \frac{\delta}{2}$.
2. If the paths from Γ, Θ or Θ^0 overlap near the endpoints, we change Γ or Θ slightly. This is to make sure the intersection numbers $\Theta \cdot \Gamma$ and $\Theta^0 \cdot \Gamma$ are defined unambiguously. Note that Θ is a free choice and duality arguments show that such changes on Γ do not affect computations.

In all discretizations, we take v_k to be $u_k + \delta$ so that Lemma 4.25 holds. The choice of the η_c for corners c is not important, hence we simply pick for consistency: η_c is set to be either $\overline{\lambda}, 1, \lambda$ or i depending on whether $c \in \mathcal{C}^{\overline{\lambda}}_{\Omega_\delta}, c \in \mathcal{C}^{1}_{\Omega_\delta}, c \in \mathcal{C}^{\lambda}_{\Omega_\delta}$ or $c \in \mathcal{C}^{i}_{\Omega_\delta}$.

Next, there is a choice of which lift $\widetilde{a}_1^{\rightarrow}$ of $a_1^{\rightarrow}$ to take. This is essentially a choice between the two branches around a_1, and the actual choice is irrelevant so long as they all agree for different δ.

For simplicity, we will assume the $\pm$ sign in Proposition 4.17 is $+$. However, there is no guarantee that this is true. In case it is not, the simplest tweak is to universally consider $-F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ instead of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$

[16] It is possible to make the signs of η_{c_k} flip as $\delta \longrightarrow 0$ so that the scaling limit does not exist.

and reverse all of the choices described after this. Note that the $\pm$ sign in Proposition 4.17 is consistent for different δ when it is small enough.

To define unambiguously $\sqrt{z-a_1}$, take the square root such that, for lifts of $a_1+\varepsilon$ in the same sheet of $\widetilde{a}_1^{\rightarrow}$, the square root is positive if $\tau^0=-1$ and negative otherwise. When using $F_{[\mathbb{C}_\delta;a_1]}$ from Proposition 6.3, the lift of $a_1^{\rightarrow}+\delta$ should be on the sheet of $\widetilde{a}_1^{\rightarrow}$ if $\tau^0=-1$ and the other sheet otherwise. This way, Lemma 4.23 and Proposition 6.4 imply $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}-F_{[\mathbb{C}_\delta;a_1]}$, defined in a neighbourhood of a_1 and extended to be 0 at $a_1^{\rightarrow}$, is s-holomorphic. From now on, when writing $\sqrt{z-a_1}$, the sign is defined in this way.

For the other branching points, $\sqrt{z-u_j}$ is determined in an identical way. Following Lemma 4.25, we take the square root that is positive at lifts of $u_j+\varepsilon$ such that

$$(-1)^{j+1}\widetilde{\tau}^0\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\varepsilon,\widetilde{c}_j+\varepsilon)=1$$

and we proceed the same way to fix the sign of $F_{[\mathbb{C}_\delta;u_j]}$.

With the square roots rigorously defined, we define $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ as a solution of a boundary value problem whose conditions have the intuition explained earlier. We state the problem and prove unicity of solutions, but not existence. The latter will follow when we prove that successions of discrete spinors converge to a continuous spinor which solves the problem. For now, we simply assume the existence when needed.

Definition 7.1. Let $\Omega\subset\mathbb{C}$ be a bounded and simply connected domain with smooth boundary, some $\mathbf{a},\mathbf{u}\in\Omega$ and a set of disorder lines Γ linking $\mathbf{u}$. For every $k\in\{1,\dots,2m\}$, fix another index $j_k\in\{1,\dots,2m\}\setminus\{k\}$ as well as a set Γ_k of disorder lines linking $[\mathbf{u}]_{k,j_k}$.

Define $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ as a solution (if it exists) of the boundary value problem for holomorphic spinors $f:[\Omega;\mathbf{a};\mathbf{u}]\to\mathbb{C}$ branching around each $\mathbf{a}$ and $\mathbf{u}$ and comprised of the following conditions:

$$f(\widetilde{z})\sqrt{\nu_{\mathrm{out}}(z)}\in\mathbb{R},\ \text{for } z\in\partial\Omega \tag{57}$$

$$\lim_{z\to a_1}\sqrt{z-a_1}\cdot f(\widetilde{z})=1 \tag{58}$$

$$\lim_{z\to a_k}\sqrt{z-a_k}\cdot f(\widetilde{z})\in i\mathbb{R},\ \text{for } k=2,\dots,n \tag{59}$$

$$\lim_{z\to u_k}\sqrt{z-u_k}\cdot f(\widetilde{z})=\lim_{z\to u_{j_k}}\frac{f^{\Gamma_k}_{\left[\Omega;\mathbf{a},u_k,u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z})}{f^{\Gamma_k}_{\left[\Omega;u_k,\mathbf{a},u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z})},\ \text{for } k=1,\dots,2m \tag{60}$$

where both lifts of z in (60) are on the same sheet of $\left[\Omega;\mathbf{a},u_j,u_{k_j};[\mathbf{u}]_{k,j}\right]=\left[\Omega;u_j,\mathbf{a},u_{k_j};[\mathbf{u}]_{k,j}\right]$.

Remark 7.2. The existence of $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ will be proven under "general conditions", which we will describe now. We assume that

$$\mathbb{E}^{\Gamma,+}_{\Omega}[\sigma_{a_1}\cdots\sigma_{a_n}]\neq 0$$

otherwise the normalizing factors $\mathcal{Z}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\cdots\sigma_{a_n}]$ in Definitions 4.11 and 4.12 do not blow up. In addition, and we assume that, for every $k\in\{1,\cdots,2m\}$ it is possible to choose $j_k\in\{1,\cdots,2m\}\setminus\{k\}$

such that

$$\lim_{z\to u_{j_k}} f^{\Gamma_k}_{\left[\Omega;u_k,\mathbf{a},u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z}) \neq 0$$

so that condition (60) can be well defined. These conditions do not appear to be restrictive .

Remark 7.3. For a fixed Ω the boundary value problem (57 – 60) for all possible values of $\mathbf{a}$, $\mathbf{u}$ and Γ is handled by induction on m. This means that the proof of the existence and well-definedness of $f^{\Gamma_k}_{\left[\Omega;\mathbf{a},u_k,u_{j_k};[\mathbf{u}]_{k,j_k}\right]}$ and $f^{\Gamma_k}_{\left[\Omega;u_k,\mathbf{a},u_{j_k};[\mathbf{u}]_{k,j_k}\right]}$ is done before considering $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$. Hence, such facts will be assumed on the arguments that follow. For instance, note that condition (59) for $f^{\Gamma_k}_{\left[\Omega;\mathbf{a},u_k,u_{j_k};[\mathbf{u}]_{k,j_k}\right]}$ and $f^{\Gamma_k}_{\left[\Omega;u_k,\mathbf{a},u_{j_k};[\mathbf{u}]_{k,j_k}\right]}$ implies the right-hand side limit of (60) is real.

Lemma 7.4. *The boundary value problem (57 – 60) has at most one solution.*

Proof. Let f_1 and f_2 be two solutions and consider the spinor $f_1 - f_2$. It satisfies (57) and (59), while $\lim_{z\to a_1}\sqrt{z-a_1}\cdot(f_1-f_2)(\widetilde{z}) = \lim_{z\to u_k}\sqrt{z-u_k}\cdot(f_1-f_2)(\widetilde{z}) = 0$. Now, take the function $g=(f_1-f_2)^2$, defined in the original domain Ω and holomorphic on $\Omega\setminus\{\mathbf{a},\mathbf{u}\}$. The Cauchy residue theorem yields

$$-i\oint_{\partial\Omega} g(z)dz = 2\pi\sum_{k=2}^{n}\lim_{z\to a_k}(z-a_k)\cdot g(z) \leq 0$$

Additionally, $(f_1-f_2)(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R} \Rightarrow g(z)\cdot\big(i\nu_{\text{out}}(z)\big) \in i\mathbb{R}_0^+ \Rightarrow -i\oint_{\partial\Omega} g(z)dz \geq 0$. Both inequalities together yield $\sum_{k=2}^{n}\lim_{z\to a_k}(z-a_k)\cdot g(z) = 0$, and seeing as each residue is in $\mathbb{R}_0^-$ we conclude they are all 0, implying $g=0$ and $f_1=f_2$. □

Before moving on to the existence of $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$, let us prove the conformal covariance of the boundary value problem. For a function φ, we write $\varphi(\mathbf{a}) \equiv \varphi(a_1),\ldots,\varphi(a_n)$, $\varphi(\mathbf{u}) \equiv \varphi(u_1),\ldots,\varphi(u_{2m})$ and $\varphi(\Gamma) \equiv \{\varphi(\gamma) : \gamma\in\Gamma\}$.

Proposition 7.5. *Given a conformal mapping $\varphi:\Omega\to\Omega'$ and assuming $f^{\varphi(\Gamma)}_{[\Omega';\varphi(\boldsymbol{a});\varphi(\boldsymbol{u})]}$ exists, consider the induced mapping $\varphi:[\Omega;\boldsymbol{a};\boldsymbol{u}]\to[\Omega';\varphi(\boldsymbol{a});\varphi(\boldsymbol{u})]$. Then, $f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}$ exists and is given by*

$$f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}(\widetilde{z}) = \varphi'(z)^{1/2}\cdot f^{\varphi(\Gamma)}_{[\Omega';\varphi(\boldsymbol{a});\varphi(\boldsymbol{u})]}\big(\varphi(\widetilde{z})\big)^{17} \tag{61}$$

Remark 7.6. There are two choices for the induced mapping $\varphi:[\Omega;\mathbf{a};\mathbf{u}]\to[\Omega';\mathbf{a};\mathbf{u}]$, we pick the one that sends the chosen branch of $[\Omega;\mathbf{a};\mathbf{u}]$ to the right of a_1 to the chosen branch of $[\Omega';\varphi(\mathbf{a});\varphi(\mathbf{u})]$ to the right of $\varphi(a_1)$. In addition, the square root of the derivative is chosen so that

$$\varphi'(z)^{1/2} = \frac{\sqrt{\varphi(z)-\varphi(a_1)}}{\sqrt{z-a_1}}$$

as $z\longrightarrow a_1$.

Remark 7.7. The definition of $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ is extended to any simply connected domain $\Omega\subseteq\mathbb{C}$ using this covariance property. Note how this rule is coherent across compositions of conformal mappings.

[17] We consider $\varphi:[\Omega;\mathbf{a};\mathbf{u}]\to[\Omega';\mathbf{a};\mathbf{u}]$ defined as one would expect to make sense of $\varphi(\widetilde{z})$.

Proof. The result is done by induction on m and checking (61) verifies all constrains. The base case $m = 0$ occurs when (60) is vacuous, therefore the proof is correct as long as the induction hypothesis is only used to prove (60).

Denote the outer normals to the domains as $\nu_{\text{out}}^{\Omega}$ and $\nu_{\text{out}}^{\Omega'}$. For $z \in \partial\Omega$, we know $\varphi(z) \in \partial\Omega'$ and thus $f^{\varphi(\Gamma)}_{[\Omega';\varphi(\mathbf{a}),\varphi(\mathbf{u})]}\big(\varphi(\widetilde{z})\big)\sqrt{\nu_{\text{out}}^{\Omega'}\big(\varphi(z)\big)} \in \mathbb{R}$. Therefore, we have to check

$$\left(\varphi'(z)\cdot\frac{\nu_{\text{out}}^{\Omega}(z)}{\nu_{\text{out}}^{\Omega'}\big(\varphi(z)\big)}\right)^{1/2} \in \mathbb{R} \Leftrightarrow \varphi'(z)\cdot\frac{\nu_{\text{out}}^{\Omega}(z)}{\nu_{\text{out}}^{\Omega'}\big(\varphi(z)\big)} \in \mathbb{R}_0^+$$

Take a parametrization g of $\partial\Omega$ around z in the counterclockwise direction with speed 1 and such that $g(0) = z$. Then, $\varphi \circ g$ parametrizes a neighbourhood of $\partial\Omega'$ around $\varphi(z)$ in the same direction. Thus,

$$\|(\varphi\circ g)'(0)\|\cdot(-i)\nu_{\text{out}}^{\Omega'}\big(\varphi(z)\big) = (\varphi\circ g)'(0) = \varphi'\big(g(0)\big)\cdot g'(0) = \varphi'(z)\cdot(-i)\nu_{\text{out}}^{\Omega}(z).$$

For (58),

$$\begin{aligned}
&\lim_{z\to a_1}\sqrt{z-a_1}\cdot\big(\varphi'(z)\big)^{1/2}\cdot f^{\varphi(\Gamma)}_{[\Omega';\varphi(\mathbf{a}),\varphi(\mathbf{u})]}\big(\varphi(\widetilde{z})\big) = \\
&\qquad = \lim_{z\to a_1}\left(\varphi'(z)\frac{z-a_1}{\varphi(z)-\varphi(a_1)}\right)^{1/2}\cdot\sqrt{\varphi(z)-\varphi(a_1)}\,f^{\varphi(\Gamma)}_{[\Omega';\varphi(\mathbf{a}),\varphi(\mathbf{u})]}\big(\varphi(\widetilde{z})\big) \\
&\qquad = 1
\end{aligned}$$

and (59) follows from the same argument. The proof of (60) starts out the same way but requires using the statement for spinors with one less disorder line:

$$\begin{aligned}
\lim_{z\to u_k}\sqrt{z-u_k}\cdot\big(\varphi'(z)\big)^{1/2}\cdot f^{\varphi(\Gamma)}_{[\Omega';\varphi(\mathbf{a}),\varphi(\mathbf{u})]}\big(\varphi(\widetilde{z})\big) &= \lim_{x\to\varphi(u_{j_k})}\frac{f^{\varphi(\Gamma_k)}_{\left[\Omega';\varphi(\mathbf{a},u_k,u_{j_k});\varphi([\mathbf{u}]_{k,j_k})\right]}\big(\varphi(\widetilde{z})\big)}{f^{\Gamma_k}_{\left[\Omega;\varphi(u_k,\mathbf{a},u_{j_k});\varphi([\mathbf{u}]_{k,j_k})\right]}\big(\varphi(\widetilde{z})\big)} \\
&= \frac{\big(\varphi'(v_k)\big)^{1/2}}{\big(\varphi'(v_k)\big)^{1/2}}\cdot\lim_{z\to u_{j_k}}\frac{f^{\Gamma_k}_{\left[\Omega;\mathbf{a},u_k,u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z})}{f^{\Gamma_k}_{\left[\Omega;u_k,\mathbf{a},u_{k_j};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z})} \\
&= \lim_{z\to u_{j_k}}\frac{f^{\Gamma_k}_{\left[\Omega;\mathbf{a},u_k,u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z})}{f^{\Gamma_k}_{\left[\Omega;u_k,\mathbf{a},u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{z})}
\end{aligned}$$

□

7.2 Integrating f^2

As stated at the beginning of Section 5, to prove the convergence we consider the primitive of $(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]})^2$ so that the boundary condition can pass to the scaling limit. To match this in the continuous setting, we describe the problem (57 – 60) using primitives of the squares of the spinors.

Proposition 7.8. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain for which $f \equiv f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}$ is defined, either because it solves the boundary value problem (57 – 60) or by extension using the covariance property (61).*

Let h and h^b be the harmonic functions

$$h := \Re \int \big(f(z)\big)^2 dz \qquad\qquad h^b := \Re \int \big(f(z) - c^b \cdot f_{[\mathbb{C}_\delta;b]}(z)\big)^2 dz$$

for $b = a_1, \boldsymbol{u}$ with

$$c^{a_1} := 1 \qquad\qquad c^{v_k} := \lim_{z\to u_{j_k}} \frac{f^{\Gamma_j}_{\left[\Omega;\boldsymbol{a},u_k,u_{j_k};[\boldsymbol{u}]_{k,j_k}\right]}(\widetilde{z})}{f^{\Gamma_j}_{\left[\Omega;u_k,\boldsymbol{a},u_{j_k};[\boldsymbol{u}]_{k,j_k}\right]}(\widetilde{z})} \text{ for } k = 1,\ldots,2m$$

Then, the following holds:

(i) h is a single-valued function in $\Omega_\delta \setminus \{\boldsymbol{a},\boldsymbol{u}\}$, it is continuous up to $\partial\Omega$ and it satisfies the Dirichlet boundary condition $h \equiv Const$ on $\partial\Omega$ (since h is defined up to an additive constant, we assume $h \equiv 0$ on $\partial\Omega$).

(ii) For every $z \in \partial\Omega$ there exists a neighbourhood of z where $h \leq 0$.

(iii) The function h is bounded from below in a neighbourhood of each $a_2,\ldots,a_n$.

(iv) For every $b = a_1, \boldsymbol{u}$, the function h^b is single-valued and bounded in a neighbourhood of b.

Furthermore, if h and every h^b satisfy (i – iv), then f solves the problem (57 – 60).

Proof. Squaring and integrating both sides of (61), we conclude h is conformally invariant. Similarly, all h^b are also conformally invariant. Since the properties (i – iv) are preserved under conformal mappings, we can assume that $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ is defined as the solution of the problem (57 – 60), and in particular that Ω is bounded and has a smooth boundary.

If $\tau(z)$ is the normalized vector tangent to $\partial\Omega$ in the direct orientation at $z \in \partial\Omega$,

$$f(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R} \Leftrightarrow \begin{cases} f^2(z)\tau(z) = f^2(z)\big[i\nu_{\text{out}}(z)\big] \in i\mathbb{R} \\ f^2(z)\nu_{\text{out}}(z) \in \mathbb{R}_0^+ \end{cases} \Leftrightarrow \begin{cases} \partial_{\tau(z)} h = 0 \\ \partial_{\nu_{\text{out}}(z)} h \geq 0 \end{cases}$$

therefore (57) is equivalent to (i) and (ii). In addition, squaring and integration the asymptotics (58 – 60) near the branching points gives

$$\begin{aligned} &\text{When } z \to b, h(z) &&= -C_b \log|z-b| + O(1), && b = a_2,\ldots,a_n \\ &\text{When } z \to b,\ h^b(z) = O(1), && && b = a_1, \mathbf{u} \end{aligned}$$

where $C_b \geq 0$ are some constants, which imply (iii) and (iv) respectively. Likewise, (iii) and the fact f^2 is holomorphic in a punctured neighbourhood of $b = a_2,\ldots,a_n$ implies (59) by analysing the singularity b, and (iv) implies (58) and (60) by building the harmonic conjugate — which must exist, $\Im \int (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]} - c^b \cdot f_{[\mathbb{C}_\delta;b]})^2$ is a valid candidate —, differentiating and taking the square root. ☐

7.3 Convergence of discrete spinors

We are now finally able to prove the convergence of the discrete spinor observables to their continuous versions. When dealing with such convergences, the definitions are natural generalizations of the ones from Section 6, with the added detail of having discrete domains converge to a continuous one. The formal details are handled by following [CS11]. Note that the discretization of domains and functions has a real parameter $\delta > 0$, but the convergence will be studied alongside a sequence $\delta_n \to 0$.

Definition 7.9. A family of discrete domains Ω_δ *converges* to a continuous domain Ω as $\delta \to 0$ if they converge in the Carathéodory sense along any subsequence Ω_{δ_n} such that $\delta_n \to 0$:

1. For any compact set $K \subset \Omega$, we have $K \subset \Omega_{\delta_n}$ for all δ_n small enough.

2. For any connected open set U, if $U \subset \Omega_{\delta_n}$ for infinitely many δ_n then $U \subset \Omega$.

We will also require $\Omega \subseteq \bigcup_{n=1}^{+\infty} \bigcap_{m=n}^{+\infty} \Omega_{\delta_m}$, to simplify arguments.

Definition 7.10 (Generalization of Definition 6.1)**.** Given a family of domains Ω_δ converging to Ω, a family of functions (spinors) F_δ defined in Ω_δ ($[\Omega_\delta; \mathbf{b}]$) is said to *converge* to a function (spinor) f defined in Ω (a double cover of Ω) as $\delta \to 0$ if, for every $z \in \Omega$ and $\delta_n \to 0$, the values of the discrete versions at edge midpoints $F_{\delta_n}\big|_{\mathcal{E}_{\mathbb{C}_\delta}}$ at approximations of z ($\widetilde{z}$) on the lattice converge to the value of f at z ($\widetilde{z}$) as $\delta_n \to 0$.

As before, when writing $F_\delta(z)$ for any $z \in \Omega$, we imply F_δ is computed at approximations of z on Ω_δ, and similarly for discrete spinors.

Definition 7.11 (Generalization of Definition 6.2)**.** A convergence of discrete functions (spinors) F_δ to f is said to be *uniform* in a compact set $K \subseteq \Omega$ if the differences $|F_\delta(z) - f(z)|$ $\big(|F_\delta(\widetilde{z}) - f(\widetilde{z})|\big)$ are uniformly small for $z \in K$.

With these definitions in hand, we state the main result.

Theorem 7.12. *Given a bounded, simply connected domain $\Omega \subset \mathbb{C}$, let Ω_δ be a family of discrete, simply connected domains that converges to Ω as $\delta \to 0$. Then, under general conditions (Remark 7.2), for any $\varepsilon > 0$,*

$$\frac{1}{\vartheta(\delta)} F^{\Gamma}_{[\Omega_\delta; \boldsymbol{a}; \boldsymbol{u}]}(\widetilde{z}) \xrightarrow{\delta \to 0} f^{\Gamma}_{[\Omega; \boldsymbol{a}; \boldsymbol{u}]}(\widetilde{z})$$

uniformly on compact sets of distance at least ε from the branching points.

As announced, for this proof it is vital to consider the primitive of the square of the spinors. Define

$$H_\delta := \int \Re\left(\frac{1}{\vartheta(\delta)} F_\delta^2(z)\right) dz$$

as described in Proposition 5.24 (with $\mathbf{b} = \mathbf{a}, \mathbf{u}$ and $\mathbf{s} = a_1, \mathbf{u}$), where we write $F_\delta = F^{\Gamma}_{[\Omega_\delta; \mathbf{a}; \mathbf{u}]}$ for simplicity. Recall the results proven regarding F_δ and H_δ at the end of Section 5: namely, the minimum and maximum principle (49), Corollary 5.29 and Proposition 5.30.

The first step is to prove a sufficient condition for the existence of a subsequence of F_δ that converges. Set

$$\Omega_\delta(\varepsilon) := \Omega_\delta \cap \left\{z : \min_{b=\mathbf{a},\mathbf{u}} |z-b| > \varepsilon\right\}$$

$$\Omega(\varepsilon) := \Omega \cap \left\{z : \min_{b=\mathbf{a},\mathbf{u}} |z-b| > \varepsilon\right\}.$$

Lemma 7.13. *For any $\varepsilon > 0$, the functions H_δ are uniformly bounded on $\Omega(\varepsilon)$ by some constant $C(\varepsilon)$ for δ small enough. More precisely,*

$$\max_{\Omega(\varepsilon)\cap\Omega_\delta} |H_\delta| < C(\varepsilon) < +\infty$$

for all $\delta < \delta(\varepsilon)$.

For a better exposition, we prove the convergence assuming this Lemma.

Proof of Theorem 7.12. The proof is done by induction on m.

Fix any $\varepsilon > 0$. Consider first the functions $\vartheta(\delta)^{-1}F_\delta$, and extend them in a "mostly analytic" way so that they are defined in the whole $[\Omega_\delta(\varepsilon); \mathbf{a}; \mathbf{u}]$: for example, make it so that they are linear along line segments connecting a face and a vertex adjacent, then extend them analytically inside the squares that are left. If Lemma 7.13 holds, then Proposition 5.30 asserts that the functions are uniformly bounded and uniformly equicontinuous on δ, even after being extended. Hence, the Arzelà-Ascoli theorem yields uniform convergence on compact subsets of $\Omega(\varepsilon)$.

We follow an analogous strategy for H_δ: extend them to $\Omega_\delta(\varepsilon)$ (now in a "mostly harmonic" way) and use Arzelà-Ascoli, which is possible because Lemma 7.13 directly imposes an uniform bound and (45) yields uniform equicontinuity. Proposition 5.24(iii) relates H_δ with a discrete version of $\Re\int F_\delta^2$. Hence, we can say that along a subsequence of δ

$$F_\delta \xrightarrow{\delta\to 0} \widetilde{f} \quad \text{and} \quad H_\delta \xrightarrow{\delta\to 0} \widetilde{h} = \Re \int \widetilde{f}^2$$

uniformly on compact subsets $K \subseteq \Omega_\delta(\varepsilon)$, where $\widetilde{f} : [\Omega_\delta(\varepsilon); \mathbf{a}; \mathbf{u}] \longrightarrow \mathbb{C}$ and $\widetilde{h} : \Omega_\delta(\varepsilon) \longrightarrow \mathbb{R}$. Finally, by doing a "diagonal process" in which we decrease ε alongside δ, we can say the convergence happens on compact subsets of $\Omega \setminus \{\mathbf{a}, \mathbf{u}\}$.

We claim that if we take any subsequence of the F_δ and H_δ that converges to some $\widetilde{f}$ and $\widetilde{h}$, then $\widetilde{f} = f$. This is demonstrated by using the unicity of the boundary value problem (57 – 60), and it amounts to checking $\widetilde{h}$ satisfies all four conditions (i – iv) from Proposition 7.8. Note that such a statement yields the convergence along the full sequence due to a compactness trick: if that were false, then there would be a subsequence of F_δ whose elements would be at distance[18] of f of at least $\varepsilon_0 > 0$, but since the Arzelà-Ascoli theorem implies the set of functions is compact there would be a subsequence of this subsequence that converges; since this is a subsequence of the full sequence, the limit is f, which is contradictory.

The s-holomorphicity and spinor nature of the F_δ implies $\widetilde{f}$ is a holomorphic spinor on $[\Omega; \mathbf{a}; \mathbf{u}]$, thus $\widetilde{h}$ is harmonic. To prove (i) it remains to check the Dirichlet boundary condition passes to the limit. This

[18]The metric here is induced by the uniform norm.

is not immediate because the discrete Dirichlet boundary condition holds on $\partial\Omega$, which may not coincide with $\partial\Omega_\delta$. It is therefore required to estimate the values of H_δ at points $z \in \partial\Omega$, which approach $\partial\Omega_\delta$ as $\delta \longrightarrow 0$ because $\partial\Omega_\delta$ is the limit of the $\partial\Omega_\delta$. For that, we turn to Lemma 6.9: taking $A = \partial\Omega_\delta$, either $L = \mathcal{V}_{\Omega_\delta} \cap \Omega_\delta(\varepsilon)$ or $L = \mathcal{F}_{\Omega_\delta} \cap \Omega_\delta(\varepsilon)$ and using the uniform bound from Lemma 7.13, we arrive at

$$H_\delta(z) \leq C(\varepsilon)\Big[1 - \mathrm{hm}_{\partial\Omega_\delta}^{\mathcal{V}_{\Omega_\delta}\cap\Omega_\delta(\varepsilon)}(z)\Big]$$
$$H_\delta(z) \geq -C(\varepsilon)\Big[1 - \mathrm{hm}_{\partial\Omega_\delta}^{\mathcal{F}_{\Omega_\delta}\cap\Omega_\delta(\varepsilon)}(z)\Big]$$

with the small detail that the jumps in $\mathcal{V}_{\Omega_\delta}$ do not occur with equal probability for neighbours of boundary vertices because the Laplacian was modified to (48) as a result of the boundary modification trick, therefore the measure in the first inequality is not exactly the harmonic measure. As $\delta \longrightarrow 0$ the points z approach $\partial\Omega_\delta$, therefore $\mathrm{hm}_{\partial\Omega_\delta}^{\mathcal{V}_{\Omega_\delta}\cap\Omega_\delta(\varepsilon)}(z), \mathrm{hm}_{\partial\Omega_\delta}^{\mathcal{F}_{\Omega_\delta}\cap\Omega_\delta(\varepsilon)}(z) \longrightarrow 1$ uniformly in δ (and note how this still holds even with the modified Laplacian). Hence, $\widetilde{h} = 0$ on $\partial\Omega$, concluding the proof of (i).

Statement (ii) follows from Remark 6.3 of [CS12], whereas statement (iii) is a result of the superharmonicity of H_δ near $a_2, \ldots, a_n$:

$$\min_{\{z:|z-a_k|\leq\varepsilon\}} H_\delta \geq \min_{\{z:\varepsilon<|z-a_k|<2\varepsilon\}} H_\delta \geq -C(\varepsilon)$$

which holds in the limit $\delta \longrightarrow 0$.

For (iv), fix any $b = a_1, \mathbf{u}$ and we consider the functions

$$H_\delta^b := \Re \int \frac{1}{\vartheta(\delta)} \big(F_\delta(z) - C_\delta^b \cdot F_{[\mathbb{C}_\delta;b]}(z)\big)^2 dz$$

defined near b, say in the disk $D_r := \{z : |z - b| < r\}$ for $r > 0$ small enough, where the constants C_δ^b are given by

$$C_\delta^{a_1} := 1 \qquad C_\delta^{u_k} := \frac{F^{(\Gamma_\delta)_j}_{\left[\Omega_\delta;\mathbf{a},u_k,u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{c}_k)}{F^{(\Gamma_\delta)_j}_{\left[\Omega_\delta;u_k,\mathbf{a},u_{j_k};[\mathbf{u}]_{k,j_k}\right]}(\widetilde{c}_k)} \text{ for } k = 1, \ldots, 2m$$

where $(\Gamma_\delta)_j$ are appropriate discretizations of the disorder lines Γ_j. As discussed, we extend $F_\delta - C_\delta^b \cdot F_{[\mathbb{C}_\delta;b]}$ to be 0 at $b + \frac{\delta}{2}$. In addition, the projections from nearby edge midpoints now match, so the argument for statement (v) of Proposition 5.21 proves that H_δ^b is subharmonic at b. Since on compact subsets of $D_r \setminus \{b\}$ we have the uniform convergences $\vartheta(\delta)^{-1}F_\delta \longrightarrow \widetilde{f}$, $\vartheta(\delta)^{-1}F_{[\mathbb{C}_\delta;b]} \longrightarrow f_{[\mathbb{C}_\delta;b]}$ (by Proposition 6.3) and $C_\delta^b \longrightarrow c^b$ (trivial when $b = a_1$ and by induction hypothesis when $b = \mathbf{u}$), we have

$$H_\delta^b \xrightarrow{\delta\to 0} \widetilde{h}^b := \Re \int \big(\widetilde{f}(z) - c^b \cdot f_{[\mathbb{C}_\delta;b]}(z)\big)^2 dz$$

everywhere on $D_r \setminus \{b\}$. The subharmonicity and superharmonicity on D_r (including at b) imply the H_δ^b are uniformly bounded, hence the limit $\widetilde{h}_\delta^b$ is bounded too. □

We now prove the functions H_δ remain uniformly bounded as δ goes to 0.

Proof of Lemma 7.13. Fix $\varepsilon > 0$ and, proceeding by contradiction, suppose that

$$M_\delta = M_\delta(\varepsilon) := \max_{\Omega(\varepsilon)\cap\Omega_\delta} |H_\delta| \xrightarrow{\delta\to 0} +\infty$$

along some subsequence of δ. Then, the re-normalized functions $M_\delta^{-1}H_\delta$ are uniformly bounded on $\Omega_\delta(\varepsilon)$. The convergence arguments from the proof of Theorem 7.12 apply identically to the renormalized functions $M_\delta^{-1/2}\vartheta(\delta)^{-1}F_\delta$ and $M_\delta^{-1}H_\delta$, yielding an uniform convergence along some subsequence to some functions $\widetilde{f}$ and $\widetilde{h} = \Re\int \widetilde{f}^2$ on $[\Omega_\delta(\varepsilon); \mathbf{a}; \mathbf{u}]$ and $\Omega_\delta(\varepsilon)$, respectively.

We would like to do a similar diagonal procedure and use the rest of the argument from the proof, but it requires having the limit functions defined on $[\Omega \setminus \{\mathbf{a}, \mathbf{u}\}; \mathbf{a}; \mathbf{u}]$ and $\Omega \setminus \{\mathbf{a}, \mathbf{u}\}$ and, as it is presented, the domains only have points whose distance to all branching points is at least ε. If we take $\varepsilon' < \varepsilon$, we might not have $M_\delta(\varepsilon') = O\big(M_\delta(\varepsilon)\big)$ as $\delta \longrightarrow 0$ because the values $|H_\delta|$ takes on $\Omega_{\varepsilon'} \setminus \Omega_\varepsilon$ may blow up faster as $\delta \longrightarrow 0$ than the ones on Ω_ε, so there may not be convergence on $\Omega_\delta(\varepsilon')$ of $M_\delta(\varepsilon)H_\delta$.

To proceed, we require a key observation: that $\widetilde{h}$ cannot be identically 0. The proof is rather long, so we leave it to Lemma 7.14. Assuming this result, then we know that $M_\delta(\varepsilon') \leq C(\varepsilon', \varepsilon)M_\delta(\varepsilon)$ for a constant independent of δ, hence $M_\delta^{-1/2}\vartheta(\delta)^{-1}F_\delta$ and $M_\delta^{-1}H_\delta$ also converge on $\Omega_\delta(\varepsilon')$. Doing the same diagonal procedure, the remaining argument from the proof of Theorem 7.12 follows identically: $\widetilde{h}$ is harmonic on $\Omega \setminus \{\mathbf{a}, \mathbf{u}\}$, satisfies the Dirichlet boundary condition, has a non-negative outer normal derivative and is bounded from below near $a_2, \ldots, a_n$. In addition to the last part, take $b = a_1, \mathbf{u}$ and note that

$$\begin{aligned}
\widetilde{h}^b &= \lim_{\delta\to 0} \Re \int \frac{1}{M_\delta \cdot \vartheta(\delta)} \big(F_\delta(z) - C_\delta^b \cdot F_{[\mathbb{C}_\delta;b]}(z)\big)^2 dz \\
&= \lim_{\delta\to 0} \Re \int \frac{1}{M_\delta \cdot \vartheta(\delta)} F_\delta^2(z) dz \\
&= \widetilde{h}
\end{aligned}$$

because $\vartheta(\delta)^{-1}F_{[\mathbb{C}_\delta;b]} \longrightarrow f_{[\mathbb{C}_\delta;b]}$, which is not $+\infty$, while $M_\delta \longrightarrow +\infty$. Thus, $\widetilde{h}$ is bounded from below near all $\mathbf{a}, \mathbf{u}$. Therefore, we can extend $\widetilde{h}$ harmonically to $\{\mathbf{a}, \mathbf{u}\}$ and apply the maximum principle to conclude the minimum has to occur on the boundary, where it is 0. But $\widetilde{h}$ has a non-negative outward normal derivative. Therefore, it must be 0 on some open set. Since $\widetilde{h}$ is defined in the simply connected domain Ω, $\widetilde{h} = 0$ and contradicting Lemma 7.14. □

Lemma 7.14. *In the setting of the previous Lemma, no subsequence of $M_\delta(\varepsilon)H_\delta$ converges to an identically zero function.*

Proof. Suppose by contradiction that, along a subsequence, $M_\delta^{-1}H_\delta \longrightarrow 0$ (uniformly) on compact subsets of $\Omega_\delta(\varepsilon)$. Let $z_\delta^{\max} \in \Omega_\delta(\varepsilon)$ be a point where the maximum of $|H_\delta||_{\Omega_\delta(\varepsilon)}$ occurs. One can assume two properties (possibly after passing to a subsequence of δ):

1. Either all $z_\delta^{\max} \in \mathcal{F}_{\Omega_\delta}$ or all $z_\delta^{\max} \in \mathcal{V}_{\Omega_\delta}$, and $H|_{\mathcal{F}_{\Omega_\delta}} \geq H|_{\mathcal{V}_{\Omega_\delta}}$ at adjacent points coupled with the boundary condition implies $M_\delta = H_\delta(z_\delta^{\max})$ or $M_\delta = -H_\delta(z_\delta^{\max})$ respectively.

2. The sequence of $z_\delta^{\max}$ converges to a correspondent maximum $z^{\max}$ of $|\widetilde{h}|$.

Let us pin down where $z_\delta^{\max}$ is. Because of the sub and superharmonicity, we can take it to be either on the boundary or close to one of the branching points (but still at distance ε or greater). Note that (45) implies $H|_{\mathcal{V}_{\Omega_\delta}}(z) \leq 0$ and $H|_{\mathcal{F}_{\Omega_\delta}}(z) \geq 0$ for points adjacent to the boundary. Hence, we can make it so that $z_\delta^{\max}$ must be in one of the discrete annuli

$$A_\delta^b(\varepsilon) := \{z : \varepsilon \leq |z-b| \leq \varepsilon + 5\delta\}$$

where $b \in \{\mathbf{a}, \mathbf{u}\}$, and we can assume all $z_\delta^{\max}$ are on the annuli around the same b using the pigeonhole principle and passing to a subsequence of δ.

Suppose $z_\delta^{\max} \in A_\delta^b(\varepsilon)$ for some $b \in \{a_2, \ldots, a_n\}$. Then, the case $z_\delta^{\max} \in \mathcal{V}_{\Omega_\delta}$ is contradictory with the superharmonicity of H_δ everywhere near b: denote

$$m_\delta := \min_{|z-b|\leq 2\varepsilon} H_\delta(z) = \min_{|z-b|\leq 2\varepsilon} H_\delta|_{\mathcal{F}_{\Omega_\delta}}(z)$$

and note that the values of $H_\delta|_{\mathcal{V}_{\Omega_\delta}}$ on $\{z : |z-b| \leq 2\varepsilon\}$ are bounded from below by the values of $H|_{\mathcal{V}_{\Omega_\delta}}$ on, say, $A_\delta^b(2\varepsilon)$, where we assumed $M_\delta^{-1} H_\delta \longrightarrow 0$ uniformly. Hence,

$$-M_\delta^{-1} m_\delta \leq -M_\delta^{-1} H_\delta(z'_\delta) \xrightarrow{\delta \to 0} 0 \Rightarrow M_\delta^{-1} m_\delta \xrightarrow{\delta \to 0} 0$$

(for some $z'_\delta \in A_\delta^b$), therefore $1 = -M_\delta^{-1} H_\delta(z_\delta^{\max}) \leq -M_\delta^{-1} m_\delta \longrightarrow 0$ which is impossible.

Thus, if $z_\delta^{\max} \in A_\delta^b(\varepsilon)$ for $b \in \{a_2, \ldots, a_n\}$ then one must have $z_\delta^{\max} \in \mathcal{F}_{\Omega_\delta}$. The argument from above does not translate to faces because the subharmonicity fails at b. To get a contradiction, we use said argument, still applied to vertices, together with the uniform comparability between $H|_{\mathcal{F}_{\Omega_\delta}} - m_\delta$ and $H|_{\mathcal{V}_{\Omega_\delta}} - m_\delta$ on $\{z : |z-b| \leq 2\varepsilon\}$, as stated in Corollary 5.29.

Using subharmonicity, we can build a path $\gamma_{\mathcal{F}_{\Omega_\delta}} = z_\delta^{\max} = z_1 \sim z_2 \sim \cdots$ of adjacent faces such that $M_\delta = H(z_1) \leq H(z_2) \leq \cdots$, which ends at either b (where the subharmonicity fails) or $A_\delta^b(2\varepsilon)$. Let $\gamma_{\mathcal{V}_{\Omega_\delta}}$ be all vertices adjacent to faces of $\gamma_{\mathcal{F}_{\Omega_\delta}}$. Recall m_δ as was defined previously, and that we proved $M_\delta^{-1} m_\delta \longrightarrow 0$. When δ is small enough, one can thus take $|m_\delta| \leq Const \cdot M_\delta$ for some $Const > 0$ independent of δ. Thus, if $z \in \gamma_{\mathcal{V}_{\Omega_\delta}}$ and $w \in \gamma_{\mathcal{F}_{\Omega_\delta}}$ is adjacent,

$$H(z) = \big[H(z) - m_\delta\big] + m_\delta \geq Const \cdot \big[H(w) - m_\delta\big] + m_\delta \geq Const \cdot M_\delta$$

with $Const > 0$ independent of δ. For a generic $z \in \mathcal{V}_{\Omega_\delta} \cap \{z : |z-b| \leq 2\varepsilon\}$, we find a bound using Lemma 6.9:

$$H_\delta(z) \geq \mathrm{hm}_{\gamma_{\mathcal{V}_{\Omega_\delta}}}^{\mathcal{V}_{\Omega_\delta} \cap \{z:|z-b|\leq 2\varepsilon\}}(z) \cdot \big[Const \cdot M_\delta\big] + \Big[1 - \mathrm{hm}_{\gamma_{\mathcal{V}_{\Omega_\delta}}}^{\mathcal{V}_{\Omega_\delta} \cap \{z:|z-b|\leq 2\varepsilon\}}(z)\Big] \cdot m_\delta$$

Now, for each δ we choose some vertex $w_\delta \in \{z : \varepsilon \leq |z-b| \leq 2\varepsilon\}$ close enough to $\gamma_{\mathcal{V}_{\Omega_\delta}}$ so that

$\mathrm{hm}_{\gamma\mathcal{V}_{\Omega_\delta}}^{\mathcal{V}_{\Omega_\delta}\cap\{z:|z-b|\leq 2\varepsilon\}}(w_\delta) \geq \frac{1}{4}$. We find

$$\frac{H_\delta(w_\delta)}{M_\delta} \geq \frac{1}{4}\cdot Const + \left[1 - \mathrm{hm}_{\gamma\mathcal{V}_{\Omega_\delta}}^{\mathcal{V}_{\Omega_\delta}\cap\{z:|z-b|\leq 2\varepsilon\}}(z)\right]\cdot\frac{m_\delta}{M_\delta} = \frac{1}{4}\cdot Const + o(1)$$

as $\delta \longrightarrow 0$ (recall $M_\delta^{-1}m_\delta \longrightarrow 0$), which is contradictory with the limit $M_\delta^{-1}H_\delta(w_\delta)$ being 0.

Finally, we take care of the case $z_\delta^{\max} \in A_\delta^b$ for some $b \in \{a_1, \mathbf{u}\}$. We consider the functions

$$\frac{1}{M_\delta}H_\delta^b = \frac{1}{M_\delta}\Re\int\frac{1}{\vartheta(\delta)}\big(F_\delta(z) - C_\delta^b\cdot F_{[\mathbb{C}_\delta;b]}(z)\big)^2dz$$

defined on $\{z : |z-b| \leq 2\varepsilon\}\cup A_\delta^b(2\varepsilon)$. Note they must also be identically 0 in the limit on $\{z : \varepsilon \leq |z-b|\}$, because $\vartheta(\delta)^{-1}F_{[\mathbb{C}_\delta;b]}(z) \longrightarrow f_{[\mathbb{C}_\delta;b]}(z)$ is negligible when compared to $M_\delta \longrightarrow +\infty$. In addition, one can extend $F_\delta - F_{[\mathbb{C}_\delta;b]}$ so that the super and subharmonicity holds everywhere (on vertices and faces, respectively) everywhere around b. The same argument for the case $b \in \{a_2,\ldots,a_n\}, z_\delta^{\max} \in \mathcal{V}_{\Omega_\delta}$ now leads to a contradiction, whether $z_\delta^{\max}$ is a vertex or a face: the function H_δ^b on $\{z : |z-b| \leq 2\varepsilon\}$ is bounded both below and above by their values on A_δ^b where $M_\delta H_\delta^b \longrightarrow 0$, hence the contradiction $1 = M_\delta^{-1}|H_\delta(z_\delta^{\max})| \longrightarrow 0$ arises. □

7.4 Studying the series near a_1

With the convergence of discrete spinors to continuous ones proven, we are able to formally prove conformal invariance results. We will showcase this by using Proposition 4.17, which shows that information regarding the expected value of products of spinors can be found by studying the values $F_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}^\Gamma$ takes near a_1. Our strategy is to define the corresponding quantity for $f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$, which is found in the series expansion of this spinor around a_1, and then prove the passage to the scaling limit using Theorem 7.12.

Definition 7.15. If $f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$ exists, define $\mathcal{A}_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$ as the following coefficient in the expansion of $f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$ near the branching point a_1:

$$f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma(\widetilde{z}) = \frac{1}{\sqrt{z-a_1}} + 2\mathcal{A}_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma\sqrt{z-a_1} + O\big(|z-a_1|^{3/2}\big) \tag{62}$$

We will also use the notation $\mathcal{A}_\Omega^\Gamma(\mathbf{a};\mathbf{u}) \equiv \mathcal{A}_\Omega^\Gamma(a_1,\ldots,a_n;u_1,\ldots,u_{2m}) = \mathcal{A}_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$.

Remark 7.16. To prove that an expansion such as (62) exists, recall that $f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$ has multiplicative monodromy -1 around a_1. In particular, after going around two loops in a neighbourhood of a_1, the function has the same value. This implies $f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$ has a Puiseux series of the form

$$f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma(\widetilde{z}) = \sum_{k=-\infty}^{+\infty} b_k(z-a_1)^{k/2}$$

for some complex coefficients b_k. Due to having opposite signs on different sheets, the coefficient for all integer exponents must be 0. In addition, (58) implies $b_{-1} = 1$ and $b_k = 0$ for $k < -1$.

A covariance rule for $\mathcal{A}_\Omega^\Gamma$ is deduced from the definition and the analogue (61) for $f_{[\Omega;\mathbf{a};\mathbf{u}]}^\Gamma$.

Proposition 7.17. *If $\varphi : \Omega \to \Omega'$ is a conformal mapping and $f^{\varphi(\Gamma)}_{[\Omega;\varphi(\boldsymbol{a}),\varphi(\boldsymbol{u})]}$ exists, then $\mathcal{A}^{\Gamma}_{\Omega}$ exists and is given by*

$$\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u}) = \varphi'(a_1)\cdot\mathcal{A}^{\varphi(\Gamma)}_{\Omega'}\big(\varphi(\boldsymbol{a});\varphi(\boldsymbol{u})\big) + \frac{1}{8}\frac{\varphi''(a_1)}{\varphi'(a_1)} \tag{63}$$

Proof. Write $\mathcal{A}_\varphi = \mathcal{A}^{\varphi(\Gamma)}_{\Omega'}\big(\varphi(\mathbf{a});\varphi(\mathbf{u})\big)$ for simplicity. Starting with (61),

$$\begin{aligned}
f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}(z) &= \big(\varphi'(z)\big)^{1/2}\times f^{\varphi(\Gamma)}_{[\Omega;\varphi(\mathbf{a}),\varphi(\mathbf{u})]}\big(\varphi(\widetilde{z})\big)\\
&= \left[\frac{\varphi'(z)}{\varphi(z)-\varphi(a_1)}\right]^{1/2}\times\Big[1+2\mathcal{A}_\varphi\cdot\big(\varphi(z)-\varphi(a_1)\big)+O\big(|\varphi(z)-\varphi(a_1)|^2\big)\Big]\\
&= \left[\frac{\varphi'(a_1)+\varphi''(a_1)\cdot(z-a_1)+O\big(|z-a_1|^2\big)}{\varphi'(a_1)\cdot(z-a_1)+\frac{\varphi''(a_1)}{2}\cdot(z-a_1)^2+O\big(|z-a_1|^3\big)}\right]^{1/2}\times\\
&\quad\times\Big[1+2\varphi'(a_1)\mathcal{A}_\varphi\cdot(z-a_1)+O\big(|z-a_1|^2\big)\Big] \qquad (64)\\
&= (z-a_1)^{-1/2}\left[\frac{1+\frac{\varphi''(a_1)}{\varphi'(a_1)}\cdot(z-a_1)+O\big(|z-a_1|^2\big)}{1+\frac{\varphi''(a_1)}{2\varphi'(a_1)}\cdot(z-a_1)+O\big(|z-a_1|^2\big)}\right]^{1/2}\times\\
&\quad\times\Big[1+2\varphi'(a_1)\mathcal{A}_\varphi\cdot(z-a_1)+O\big(|z-a_1|^2\big)\Big]\\
&= (z-a_1)^{-1/2}\left[1+\left(\frac{\varphi'(a_1)}{\varphi''(a_1)}-\frac{\varphi'(a_1)}{2\varphi''(a_1)}\right)\cdot(z-a_1)+O\big(|z-a_1|^2\big)\right]^{1/2}\times\\
&\quad\times\Big[1+2\varphi'(a_1)\mathcal{A}_\varphi\cdot(z-a_1)+O\big(|z-a_1|^2\big)\Big] \qquad (65)\\
&= (z-a_1)^{-1/2}\left[1+\frac{1}{2}\cdot\frac{\varphi'(a_1)}{2\varphi''(a_1)}\cdot(z-a_1)+O\big(|z-a_1|^2\big)\right]\times\\
&\quad\times\Big[1+2\varphi'(a_1)\mathcal{A}_\varphi\cdot(z-a_1)+O\big(|z-a_1|^2\big)\Big] \qquad (66)\\
&= (z-a_1)^{-1/2}\left[1+2\left(\varphi'(a_1)\mathcal{A}_\varphi+\frac{1}{8}\frac{\varphi'(a_1)}{\varphi''(a_1)}\right)\cdot(z-a_1)+O\big(|z-a_1|^2\big)\right]
\end{aligned}$$

where (64) uses the expansion of φ and φ' around a_1, (65) follows from $(1+z)^{-1} = 1-z+O\big(|z|^2\big)$ and (66) is a result of $(1+z)^{1/2} = 1+\frac{1}{2}z+O\big(|z|^2\big)$. ☐

In the continuous case, the series of $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}(z)-1/\sqrt{z-a_1} = (f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}-f_{[\mathbb{C}_\delta;a_1]})(z)$ near a_1 has leading term $2\mathcal{A}^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}\sqrt{z-a_1}$, hence one would expect a similar heuristic to be true in the discrete case. We can write such a heuristic as

$$\big(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}-F_{[\mathbb{C}_\delta;a_1]}\big)\left(\widetilde{a}_1+\frac{3}{2}\delta\right)\approx 2\Re\Big(\mathcal{A}^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}\Big)\cdot G_{[\mathbb{C}_\delta;a_1]}\left(\widetilde{a}_1+\frac{3}{2}\delta\right)$$

where we include the real part because $a_1+\frac{3}{2}\delta\in\mathcal{C}^1_{\Omega_\delta}$, therefore the spinors take real values at that point. Note how the normalizing term $\vartheta(\delta)$ is absent because it is the same for all 3 functions. In addition, this expression can be simplified using $F_{[\mathbb{C}_\delta;a_1]}\big(\widetilde{a}_1+\frac{3}{2}\delta\big)=1$ (Proposition 6.3) and $G_{[\mathbb{C}_\delta;a_1]}\big(\widetilde{a}_1+\frac{3}{2}\delta\big)=\delta$ (Proposition 6.5) for the adequate lift of $\widetilde{a}_1$. This observation is formally stated as follows:

Theorem 7.18. *Given a bounded, simply connected domain $\Omega\subset\mathbb{C}$, let Ω_δ be a family of discrete, simply* *connected domains that converges to Ω as $\delta \to 0$. Then, under general conditions (Remark 7.2),*

$$F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}\left(\widetilde{a}_1 + \frac{3}{2}\delta\right) - 1 - 2\Re\left(\mathcal{A}^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}\right) \cdot \delta = o(\delta)$$

as $\delta \longrightarrow 0$ for the lift of $a_1 + \frac{3}{2}\delta$ used in the definition of $F_{[\mathbb{C}_\delta;a_1]}$ and $G_{[\mathbb{C}_\delta;a_1]}$.

Proof. For shortness, we write $F_\delta = F^{\Gamma}_{[\Omega_\delta;a_1]}$ and $\mathcal{A} = \mathcal{A}^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$. Let $\mathcal{R}$ be the reflection with respect to the line $\{z : \Im(z-a) = 0\}$, let U_δ be a small neighbourhood of a_1 in $\Omega_\delta \cap \mathcal{R}(\Omega_\delta)$. Recalling Section 6, denote $L_{a_1} = \{a_1 + \frac{3}{2}\delta + x : x < 0\}$ and consider $[(U_\delta \cap \mathcal{C}^1_{\Omega_\delta}) \setminus L_{a_1}; a_1]$ as two sheets $U^{\pm}_\delta$, defined so that the lift of $a_1 + \frac{3}{2}\delta$ specified in the statement belongs to U^{+}_δ. Note that these sheets are the result of a discretization of $U_\delta \setminus L_{a_1}$. We will make abuses of language and give the same name to sets in U_δ and $U^{\pm}_\delta$ that have a trivial correspondence: for example, the half-line L_{a_1} as seen in $U^{\pm}_\delta$ is still called L_{a_1}.

Consider the spinor

$$F^{(\mathcal{R})}_\delta := F^{\Gamma}_{[\mathcal{R}(\Omega);\mathcal{R}(\mathbf{a});\mathcal{R}(\mathbf{u})]} \equiv F^{\Gamma}_{[\mathcal{R}(\Omega_\delta);\mathcal{R}(a_1),\ldots\mathcal{R}(a_n);\mathcal{R}(u_1),\ldots,\mathcal{R}(u_{2m})]}$$

together with a matching of the sheets between the domains of F_δ and $F^{(\mathcal{R})}_\delta$ done in a way where the lift of $a_1 + \frac{3}{2}\delta$ chosen for the definition of $F_{[\mathbb{C}_\delta;a_1]}$ (or $G_{[\mathbb{C}_\delta;a_1]}$) are in matching sheets. We can take it one step further, and say these functions are defined in the same domain: given $\widetilde{z}$, let γ be a path whose lift to $[\Omega_\delta;\mathbf{a};\mathbf{u}]$ connects $\widetilde{a}_1^{\rightarrow}$ and $\widetilde{z}$, consider $\mathcal{R}(\gamma)$, lift it to the double cover starting at $\widetilde{a}_1^{\rightarrow}$ and let the other end be $\widetilde{z}^{(\mathcal{R})}$; then, $F^{(\mathcal{R})}_\delta(\widetilde{z}) = F_\delta\big(\widetilde{z}^{(\mathcal{R})}\big)$.

Define the real-valued function $S_\delta : U^{+}_\delta \longrightarrow \mathbb{R}$ as

$$S_\delta(z) := \frac{1}{\vartheta(\delta)}\left[\frac{1}{2}\big(F_\delta + F^{(\mathcal{R})}_\delta\big)(\widetilde{z}) - F_{[\mathbb{C}_\delta;a_1]}(\widetilde{z}) - 2\Re(\mathcal{A}) \cdot G_{[\mathbb{C}_\delta;a_1]}(\widetilde{z})\right]$$

(we drop the lifts of z in $S_\delta(z)$ for ease of notation, recall that z is lifted to the sheet corresponding to U^{+}_δ when computing the spinors) and note that, when valued at $\widetilde{a}_1 + \frac{3}{2}\delta$, becomes the quantity we are interested in (after normalization):

$$S_\delta\left(a_1 + \frac{3}{2}\delta\right) = \frac{1}{\vartheta(\delta)}\left[F^{\Gamma}_{[\Omega_\delta;a_1]}\left(\widetilde{a}_1 + \frac{3}{2}\delta\right) - 1 - 2\Re\left(\mathcal{A}^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}\right) \cdot \delta\right]$$

seeing as $F^{(\mathcal{R})}_\delta\big(\widetilde{a}_1 + \frac{3}{2}\delta\big) = F_\delta\big(\widetilde{a}_1 + \frac{3}{2}\delta\big)$ because of symmetry arguments — either consider Definition 4.11 or use Proposition 4.17 and symmetry of the Ising model.

We claim S_δ is discrete harmonic and can be extended to L_{a_1}. It is clear that this is a discrete harmonic function on its domain except at $a_1 + \frac{3}{2}\delta$, since all 4 spinors are as well. It is also discrete harmonic at $a_1 + \frac{3}{2}\delta$ because the differences $\frac{1}{2}\big(F_\delta - F_{[\mathbb{C}_\delta;a_1]}\big)$ and $\frac{1}{2}\big(F^{(\mathcal{R})}_\delta - F_{[\mathbb{C}_\delta;a_1]}\big)$ correct the discrete singularity, seeing as $F_{[\mathbb{C}_\delta;a_1]} \circ \mathcal{R} = F_{[\mathbb{C}_\delta;a_1]}$. If we wanted to extend S_δ to L_{a_1} in a discrete harmonic way, we could use the definition of $S_\delta(z)$ with the lift of z taken to be on the sheet depending on whether we approach the point from above or from below. However, note that $F_{[\mathbb{C}_\delta;a_1]}$ and $G_{[\mathbb{C}_\delta;a_1]}$ both vanish on L_{a_1} and $F^{(\mathcal{R})}_\delta = -F_\delta$ there because they are computed at different points with the same projection.

Therefore, we can set $S_\delta = 0$ on L_{a_1} and still have a discrete harmonic function.

Given these properties, a bound for $S_\delta\big(a_1 + \frac{3}{2}\delta\big)$ follows from Lemma 6.8 and the bound (53) for harmonic measure: if we take the disk $D(r) = \{z : |z - a_1| < r\}$ for r small enough that $D(r) \subset U_\delta$, then

$$|S_\delta(z)| \leq \max_{A_\delta(r)} |S_\delta| \cdot \mathrm{hm}_{A_\delta(r)}^{\left(D(r)\cup A_\delta(r)\right)\setminus L_{a_1}}(z) \leq \max_{A_\delta(r)} |S_\delta| \cdot \mathrm{hm}_{A_\delta(r)}^{\mathbb{X}_\delta}(z)$$

for all $z \in D(r)$, where $A_\delta(r) = \{z : r \leq |z - b| \leq r + 5\delta\}$. Using (53), we end with

$$\left| S_\delta\left(a_1 + \frac{3}{2}\delta\right) \right| \leq Const \cdot \delta^{1/2} r^{-1/2} \max_{A_\delta(r)} |S_\delta|$$

$$\Rightarrow \left| F_{[\Omega_\delta;a_1]}^{\Gamma}\left(\widetilde{a}_1 + \frac{3}{2}\delta\right) - 1 - 2\Re\left(\mathcal{A}_{[\Omega;\mathbf{a};\mathbf{u}]}^{\Gamma}\right) \cdot \delta \right| \leq Const \cdot \vartheta(\delta)\delta^{1/2} r^{-1/2} \max_{A_\delta(r)} |S_\delta|.$$

All that is left is to bound the last factor uniformly, which we can do by using the convergence of the discrete spinors. Theorem 7.12 and Propositions 6.3 and 6.5 yield

$$S_\delta \xrightarrow{\delta \to 0} s := \Re\left(\frac{1}{2}\big(f + f^{(\mathcal{R})}\big) - f_{[\mathbb{C}_\delta;a_1]}\right) - 2\Re(\mathcal{A}) \cdot g_{[\mathbb{C}_\delta;a_1]}$$

uniformly on $A_0(r)$, where $f = f_{[\Omega;a_1]}^{\Gamma}$ and $f^{(\mathcal{R})} = f_{[\mathcal{R}(\Omega);\mathcal{R}(\mathbf{a});\mathcal{R}(\mathbf{u})]}$. Note that the $\Re$ is introduced because S_δ is computed at corners of $\mathcal{C}_{\Omega_\delta}^1$, where the spinors' values are the real part of those at edge midpoints. By definition of $\mathcal{A}$,

$$\big(f - f_{[\mathbb{C}_\delta;a_1]}\big)(z) = 2\mathcal{A}\sqrt{z - a} + O\big(|z - a|^{3/2}\big)$$

as $z \longrightarrow a$. In addition, $f^{(\mathcal{R})}(z) = \overline{f\big(\mathcal{R}(z)\big)}$ because the right-hand side solves the corresponding boundary value problem, and therefore

$$\big(f^{(\mathcal{R})} - f_{[\mathbb{C}_\delta;a_1]}\big)(z) = 2\overline{\mathcal{A}}\sqrt{z - a} + O\big(|z - a|^{3/2}\big)$$

as $z \longrightarrow a$. Putting everything together,

$$s(z) = O\big(|z - a|^{3/2}\big)$$

as $z \longrightarrow a$. Hence,

$$r^{-1/2} \max_{A_\delta(r)} |S_\delta| \xrightarrow{\delta \to 0} r^{-1/2} \cdot O\big(r^{-3/2}\big) = O(r)$$

and one arrives at

$$\left| S_\delta\left(a_1 + \frac{3}{2}\delta\right) \right| \leq Const \cdot \vartheta(\delta)\delta^{1/2} r$$

and seeing as $r > 0$ can be chosen arbitrarily small and $\vartheta(\delta) = O\big(\delta^{1/2}\big)$ by (51), this finishes the proof. □

The previous result can be reformulated more clearly.

Theorem 7.19. *Given a bounded, simply connected domain $\Omega \subset \mathbb{C}$, let Ω_δ be a family of discrete, simply* *connected domains that converges to Ω as $\delta \to 0$. Then, under general conditions (Remark 7.2),*

$$\lim_{\delta\to 0}\frac{1}{2\delta}\left(\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]}-1\right)=\Re\Big(\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\Big)$$
$$\lim_{\delta\to 0}\frac{1}{2\delta}\left(\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2i\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]}-1\right)=-\Im\Big(\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\Big)$$

Proof. The first equality is just Theorem 7.18 rewritten and with the spinor replaced using Proposition 4.17. The second equality can be easily derived from the first using the covariance rule of $\mathcal{A}^{\Gamma}_{\Omega}$, by considering the 90° clockwise rotation around a_1, given by the conformal map $\varphi(z) = -iz + (1-i)a_1$.

We have

$$\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2i\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]}=\frac{\mathbb{E}^{\varphi(\Gamma),+}_{\varphi(\Omega_\delta)}[\sigma_{\varphi(a_1+2i\delta)}\sigma_{\varphi(a_2)}\cdots\sigma_{\varphi(a_n)}]}{\mathbb{E}^{\varphi(\Gamma),+}_{\varphi(\Omega_\delta)}[\sigma_{\varphi(a_1)}\sigma_{\varphi(a_2)}\cdots\sigma_{\varphi(a_n)}]}$$

seeing as the only difference between the Ising models on both sides is that the graph they are defined on is rotated. Since $\varphi(a_1 + 2i\delta) = a_1 + 2\delta$, the first expression yields

$$\lim_{\delta\to 0}\frac{1}{2\delta}\left(\frac{\mathbb{E}^{\varphi(\Gamma),+}_{\varphi(\Omega_\delta)}[\sigma_{\varphi(a_1+2i\delta)}\sigma_{\varphi(a_2)}\cdots\sigma_{\varphi(a_n)}]}{\mathbb{E}^{\varphi(\Gamma),+}_{\varphi(\Omega_\delta)}[\sigma_{\varphi(a_1)}\sigma_{\varphi(a_2)}\cdots\sigma_{\varphi(a_n)}]}-1\right)=\Re\Big(\mathcal{A}^{\varphi(\Gamma)}_{\varphi(\Omega)}(\varphi(\mathbf{a});\varphi(\mathbf{u})\Big)$$

and now we can use (63) to return to the original domain. Using $\varphi'(a_1) = -i$ and $\varphi''(a_1) = 0$, we arrive at

$$\Re\Big(\mathcal{A}^{\varphi(\Gamma)}_{\varphi(\Omega)}(\varphi(\mathbf{a});\varphi(\mathbf{u})\Big)=\Re\Big(i\cdot\mathcal{A}^{\Gamma}_{\Omega}(\mathbf{a};\mathbf{u})\Big)=-\Im\Big(\mathcal{A}^{\Gamma}_{\Omega}(\mathbf{a};\mathbf{u})\Big).$$

□

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