Conformal limit of the planar Ising model with disorder lines

Extended Abstract

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1 Introduction and model definition

The Ising model is a mathematical model used in Statistical Physics. The model is defined on a graph and defines random variables associated to the vertices of the graph, which can take one of two values $\{\pm 1\}$. These represent the orientation of dipoles, and the main characteristic is that each dipole can interact with their neighbours: configurations where more neighbouring dipoles agree occur with higher probability.

Given a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, to each vertex $v \in \mathcal{V}$ we associate a variable $\sigma_v \in \{\pm 1\}$, referred to as the *spin* of the vertex. A *spin configuration* is an assignment of spins $\sigma = (\sigma_v)_{v \in \mathcal{V}} \in \{\pm 1\}^{\mathcal{V}}$ to every vertex. The *Hamiltonian* function is defined as

$$\begin{aligned} H\colon \{\pm 1\}^{\mathcal{V}} &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto -\sum_{\substack{e \in \mathcal{E} \\ e=(vu)}} \sigma_v \sigma_u \end{aligned}$$

which can be seen as the sum of "contributions" from all edges: each edge $e = (vu) \in \mathcal{E}$ contributes with -1 if $\sigma_v = \sigma_u$ or $+1$ if $\sigma_v = -\sigma_u$. The model is defined by the probability distribution $\mathbb{P}(\sigma)$ such that $\mathbb{P}(\sigma)$ is proportional to $\exp\big(-\beta H(\sigma)\big)$, where $\beta > 0$ is a fixed constant. More explicitly,

$$\mathbb{P}(\sigma) = \frac{1}{\mathcal{Z}_\beta} \exp\big(-\beta H(\sigma)\big)$$

where

$$\mathcal{Z}_\beta := \sum_{\sigma \in \{\pm 1\}^{\mathcal{V}}} \exp\big(-\beta H(\sigma)\big)$$

is called the *partition function* of the model. Another expression for the probability can be obtained by expanding the Hamiltonian:

$$\mathbb{P}(\sigma) = \frac{1}{\mathcal{Z}_\beta} \prod_{\substack{e \in \mathcal{E} \\ e=(vu)}} \exp(\beta \sigma_v \sigma_u).$$

Remark 1.1. In a physical context β is the inverse temperature of the system (assuming the units are such that the Boltzmann constant k_B equals 1) and the Hamiltonian of a configuration is interpreted as its energy. The probability distribution is therefore the Gibbs measure.

Many changes have been proposed to this model throughout the years. The most common generalization is to introduce interaction constants $(J_e)_{e\in\mathcal{E}}$, allowing some connections to be stronger than others. The Hamiltonian becomes

$$H_{(J_e)}(\sigma) := -\sum_{\substack{e\in\mathcal{E}\\ e=(vu)}} J_e\sigma_v\sigma_u$$

with the probability measure defined in the same way:

$$\mathbb{P}_{(J_e)} := \frac{1}{\mathcal{Z}_{(J_e),\beta}} \exp\big(-\beta H_{(J_e)}(\sigma)\big) = \frac{1}{\mathcal{Z}_{(J_e),\beta}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \exp(\beta J_e\sigma_v\sigma_u)$$

where the partition function is now given by

$$\mathcal{Z}_{(J_e),\beta} := \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \exp\big(-\beta H_{(J_e)}(\sigma)\big) = \sum_{\sigma\in\{\pm1\}^{\mathcal{V}}} \prod_{\substack{e\in\mathcal{E}\\ e=(vu)}} \exp(\beta J_e\sigma_v\sigma_u).$$

Some combinations of interaction constants have particular importance. An Ising model with *disorder insertions* is a model where $J_e \in \{\pm1\}$. Informally speaking, an edge with a disorder insertion behaves opposite from normal, making configurations where the neighbouring spins have opposite signs more likely. Usually these are described using dual paths, and the edges crossed by those dual lines are the ones where $J_e = -1$.

Definition 1.2. Given a planar graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ with faces $\mathcal{F}$, the set of all dual edges is labelled $\mathcal{E}^\dagger$. The *dual edge* of an edge $e\in\mathcal{E}$ is denoted by $e^\dagger$ and the *dual set* of a set $E\subseteq\mathcal{E}$ is $E^\dagger = \{e^\dagger : e\in E\}$. The *dual graph* is the graph $\mathcal{G}^\dagger = (\mathcal{F},\mathcal{E}^\dagger)$.

Definition 1.3. Given an Ising model on a planar graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ with faces $\mathcal{F}$, a *set of disorder lines* is a set $\Gamma\subseteq\mathcal{E}^\dagger$. We say the model has a set of disorder lines Γ if the Hamiltonian is defined as

$$H^\Gamma(\sigma) := -\sum_{\substack{e\in\mathcal{E}\setminus\Gamma^\dagger\\ e=(vu)}} \sigma_v\sigma_u + \sum_{\substack{e\in\Gamma^\dagger\\ e=(vu)}} \sigma_v\sigma_u.$$

The partition function, the probability distribution and expected values of the model are written as $\mathcal{Z}^\Gamma_\beta$, $\mathbb{P}^\Gamma$ and $\mathbb{E}^\Gamma$, respectively.

2 Scaling limits

The thesis is dedicated to a continuous version of the two-dimensional Ising model, with a spin variable assigned to each point of a domain $\Omega\subseteq\mathbb{C}$. Such a model would be described by a Statistical Field Theory. The first difficulty one faces is how to formally define such an object, and the approach followed is to consider it as a *scaling limit of discrete models.*

Given $\Omega\subseteq\mathbb{C}$, for every $\delta>0$ consider appropriate discretizations Ω_δ of Ω (Figure 1) which converge in some sense to the original domain as $\delta\longrightarrow 0$. We then consider a family of Ising models defined on

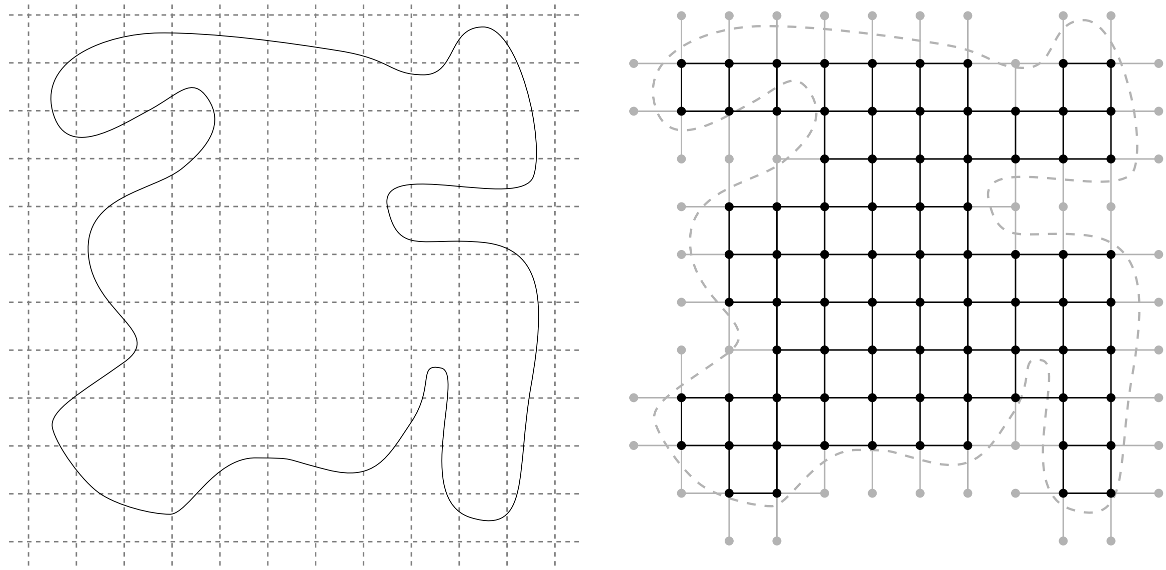

Figure 1: A domain Ω with a square grid (on the left) and an example of a discretization Ω_δ (on the right). Boundary sites are often considered for the model, and coloured grey.

every Ω_δ. The properties of the continuous version are defined by the corresponding properties of the discrete models passed to the limit. For example, given $a, b \in \Omega$, the expected value of a product of spin variables $\sigma_a \sigma_b$ for the continuous Ising model would be defined as:

$$\mathbb{E}_\Omega[\sigma_a \sigma_b] := \lim_{\delta \to 0} \mathbb{E}_{\Omega_\delta}[\sigma_a \sigma_b]$$

and note that there is an abuse of notation here: the sites on the right-hand side may not be a and b but instead appropriate approximations of these points on Ω_δ. Apart from proving that such limits exist, the well-definedness of these quantities requires proving that choices regarding discretizations are not relevant when passing to the limit. Most works in literature consider discretizations using square lattices, and our work is no different in this regard. This formulation of the continuous model has seen recent success in formally defining and proving Physics conjectures regarding conformal invariance [Smi06, Smi10, CHI15, CS12, CI13].

3 Setting

We consider a discrete domain Ω_δ which is the union of faces of a square grid with mesh size $\delta > 0$. Such faces are called *interior faces* and are denoted by $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$. Given such a domain, the set of *interior vertices* is the set $\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ of vertices of the grid that are corners of any face of $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$, and the set of *interior edges* is the set $\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$ of edges that are adjacent to any face of $\mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$.

Additionally, we define the sets of *boundary faces*, *vertices* and *edges* as being the respective elements adjacent/incident to their interior counterparts that do not belong to those sets, and are denoted by $\partial\mathcal{F}_{\Omega_\delta}$, $\partial\mathcal{V}_{\Omega_\delta}$ and $\partial\mathcal{E}_{\Omega_\delta}$. The sets of *faces*, *vertices* and *edges* are the union of the corresponding interior and boundary elements: $\mathcal{F}_{\Omega_\delta} := \mathrm{Int}\,\mathcal{F}_{\Omega_\delta} \cup \partial\mathcal{F}_{\Omega_\delta}$, $\mathcal{V}_{\Omega_\delta} := \mathrm{Int}\,\mathcal{V}_{\Omega_\delta} \cup \partial\mathcal{V}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta} := \mathrm{Int}\,\mathcal{E}_{\Omega_\delta} \cup \partial\mathcal{E}_{\Omega_\delta}$. Figure

1 shows an example of such a discretization with boundary elements coloured grey.

The domain Ω_δ is any polygonal domain resulting from the union of square grid faces, and to simplify arguments we will assume that Ω_δ is simply connected and any edges connecting vertices of $\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}$ belong to $\mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$.

4 Disorder variables and Duality

Section 2 of the thesis is dedicated to introducing Kramers-Wannier duality: a symmetry relating Ising models defined on dual graphs. Historically, it was an important step to find the critical temperature of the model. From this point of view, disorder lines are found to be dual objects of spin variables [KC71]. The most common formulation of this fact is achieved by using disorder variables.

Definition 4.1. Given an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with faces $\mathcal{F}$ and a set of disorder lines $\Gamma \subseteq \mathcal{E}^\dagger$, the *degree* of a face $a \in \mathcal{F}$ in Γ is the cardinality $|\{e \in \Gamma : a \text{ is an endpoint of } e\}|$. We write $\Gamma \equiv \Gamma[a_1, \dots, a_{2m}]$ to emphasize the faces $a_1, \dots, a_{2m} \in \mathcal{F}$ that have an odd degree on Γ, which must exist in an even number.

Remark 4.2. When employing the notation $\Gamma[a_1, \dots, a_{2n}]$ (and more generally, when speaking about disorder lines), we allow for some of the endpoints a_k to be repeated. The faces that are endpoints of an odd number of elements of Γ are the ones that appear an odd number of times on the list $a_1, \dots, a_{2n}$ (note that the list must still have even length). We will make an abuse of language when referring to $a_1, \dots, a_{2n}$ as the vertices with odd degree in $\Gamma[a_1, \dots, a_{2n}]$, even if some of the a_k repeat. This "cancellation" will be a recurring pattern in the sequel, and will be left implicit.

Definition 4.3. Given an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with faces $\mathcal{F}$, a *disorder variable* is a random variable of the form

$$\left(\prod_{j=1}^{2n} \mu_{a_j} \right)_\Gamma := \prod_{\substack{e \in \Gamma^\dagger \\ e=(vu)}} \exp(-2\beta J_e \sigma_v \sigma_u)$$

where $a_1, \dots, a_{2m} \in \mathcal{F}$ and $\Gamma \equiv \Gamma[a_1, \dots, a_{2n}] \subseteq \mathcal{E}^\dagger$ is a set of disorder lines.

The following statement from [KC71] relates disorder lines and disorder variables

Proposition 4.4. *Consider an Ising model on a planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with parameter β and let $\Gamma \equiv \Gamma[a_1, \dots, a_{2n}] \subseteq \mathcal{E}^\dagger$ be a set of disorder lines. Then,*

$$\mathbb{P}^\Gamma(\sigma) = \frac{\left(\prod_{j=1}^{2n} \mu_{a_j} \right)_\Gamma}{\mathbb{E}\left[\left(\prod_{j=1}^{2n} \mu_{a_j} \right)_\Gamma \right]} \cdot \mathbb{P}(\sigma) \qquad \text{and} \qquad \mathbb{E}^\Gamma[\mathbb{X}] = \frac{\mathcal{Z}_\beta}{\mathcal{Z}_\beta^\Gamma} \cdot \mathbb{E}\left[\mathbb{X} \cdot \left(\prod_{j=1}^{2n} \mu_{a_j} \right)_\Gamma \right]$$

where $\mathbb{X} \equiv \mathbb{X}(\sigma)$ is any random variable depending on the spin variables.

A classic duality result is the following:

Theorem 4.5. *Consider an Ising model on* $\mathcal{G}_{\Omega_\delta} = (\mathrm{Int}\,\mathcal{V}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}_{\Omega_\delta})$ *with parameter* β*, together with another Ising model on* $\mathcal{G}^\dagger_{\Omega_\delta} = (\mathcal{F}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta})$ *with parameter* $\beta^\dagger$ *and* $+$ *boundary conditions: that is, all the spins of* $\partial\mathcal{F}_{\Omega_\delta}$ *are conditioned to be* $+1$*. Let* $\Theta \equiv \Theta[v_1, \ldots, v_{2m}] \subseteq \mathrm{Int}\,\mathcal{E}_{\Omega_\delta}$ *and* $\Gamma \equiv \Gamma[a_1, \ldots, a_{2n}] \subseteq \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta}$*. If* $\tanh\beta = \exp(-2\beta^\dagger)$*, then*

$$\mathbb{E}_{\mathcal{G}_{\Omega_\delta}}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_\Gamma\right] = (-1)^{|\Theta\cap\Gamma^\dagger|}\cdot\mathbb{E}_{\mathcal{G}^\dagger_{\Omega_\delta}}\left[\prod_{j=1}^{2n}\sigma_{a_j}\left(\prod_{k=1}^{2m}\mu_{v_k}\right)_\Theta\right]. \tag{1}$$

From this result follows a well-known fact about disorder lines: changing Γ while keeping the endpoints fixed only affects the factor $(-1)^{|\Theta\cap\Gamma^\dagger|}$ on the right-hand side of (1), therefore the expected value

$$\mathbb{E}_{\mathcal{G}_{\Omega_\delta}}\left[\prod_{k=1}^{2m}\sigma_{v_k}\left(\prod_{j=1}^{2n}\mu_{a_j}\right)_\Gamma\right]$$

is independent of the choice of Γ up to a sign.

5 Spinor observables

The bulk of the thesis focuses on spinor observables $F^\Gamma_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$. The definition given in the thesis is a "combinatorial definition", which expresses the function as a combinatorial sum. It is rather lengthy and requires quite a bit of work to check that it is well defined, but it is essential to prove s-holomorphicity (more details below). Another possibility is provided by an "analytical definition", which gives insight as to why these objects are interesting to study. The latter will be the one followed in the present text.

Figure 2 shows an example of a discretization using a lattice of mesh size $\sqrt{2}\delta$ and rotated by 45°. The sites $\mathcal{V}_{\Omega_\delta}$, $\mathcal{E}_{\Omega_\delta}$ and $\mathcal{F}_{\Omega_\delta}$ are displayed, and note how the edges are identified with their midpoints and faces with their centers. Around each vertex v we add four neighbour *corners* $v \pm \frac{\delta}{2}$ and $v \pm \frac{\delta}{2}i$. The collection of all corners is denoted by $\mathcal{C}_{\Omega_\delta}$ and is partitioned into $\mathcal{C}_{\Omega_\delta} = \mathcal{C}^1_{\Omega_\delta} \cup \mathcal{C}^i_{\Omega_\delta} \cup \mathcal{C}^\lambda_{\Omega_\delta} \cup \mathcal{C}^{\overline{\lambda}}_{\Omega_\delta}$ depending on the position of each corner relative to the neighbouring vertex, as shown in Figure 2. Depending on where a corner c is placed in this partition, we associate to it a complex number η_c amongst $1, i, \lambda := e^{i\frac{\pi}{4}}$ and $\overline{\lambda} = e^{-i\frac{\pi}{4}}$. For a corner c, $v(c)$ denotes the neighbouring vertex and $f(c)$ refers to the face where it is inserted.

We consider the Ising model on the graph $\mathcal{G}^\dagger_{\Omega_\delta} = (\mathcal{F}_{\Omega_\delta}, \mathrm{Int}\,\mathcal{E}^\dagger_{\Omega_\delta})$ at the critical temperature $\beta = \beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$ with boundary conditions $+$: that is, all the spins of $\partial\mathcal{F}_{\Omega_\delta}$ are set as $+1$. Our objective is to study such a model with disorder lines $\Gamma \equiv \Gamma[v_1, \ldots, v_{2m}]$, but we often consider the same model with other disorder lines. Let $c_1, \ldots, c_{2m} \in \mathrm{Int}\,\mathcal{C}_{\Omega_\delta}$ be pairwise disjoint corners adjacent to $v_1, \ldots, v_{2m}$ and set $u_k = f(c_k)$. Consider some additional faces $a_1, \ldots, a_n \in \mathrm{Int}\,\mathcal{F}_{\Omega_\delta}$, not necessarily distinct from $u_1, \ldots, u_{2m}$, and let Θ be a collection of edge-disjoint paths in $\mathcal{G}^\dagger_{\Omega_\delta}$ linking $u_1, \ldots, u_{2m}, a_1, \ldots, a_n$ and possibly $a_{\mathrm{out}} \in \partial\mathcal{F}_{\Omega_\delta}$. We write $\mathbf{a} \equiv a_1, \ldots, a_n$, $\mathbf{c} \equiv c_1, \ldots, c_{2m}$, $\mathbf{u} \equiv u_1, \ldots, u_{2m}$ and $\mathbf{v} \equiv v_1, \ldots, v_{2m}$.

The spinor observables are functions defined on the canonical double cover of $\Omega_\delta \setminus \{\mathbf{a}, \mathbf{u}\}$ branching around each point $\mathbf{a}, \mathbf{u}$, which is denoted by $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$. There is a natural 2-to-1 correspondence between $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ and $\Omega_\delta \setminus \{\mathbf{a}, \mathbf{u}\}$; the two representatives of $z \in \Omega_\delta$ in $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ are called the lifts of z, whereas

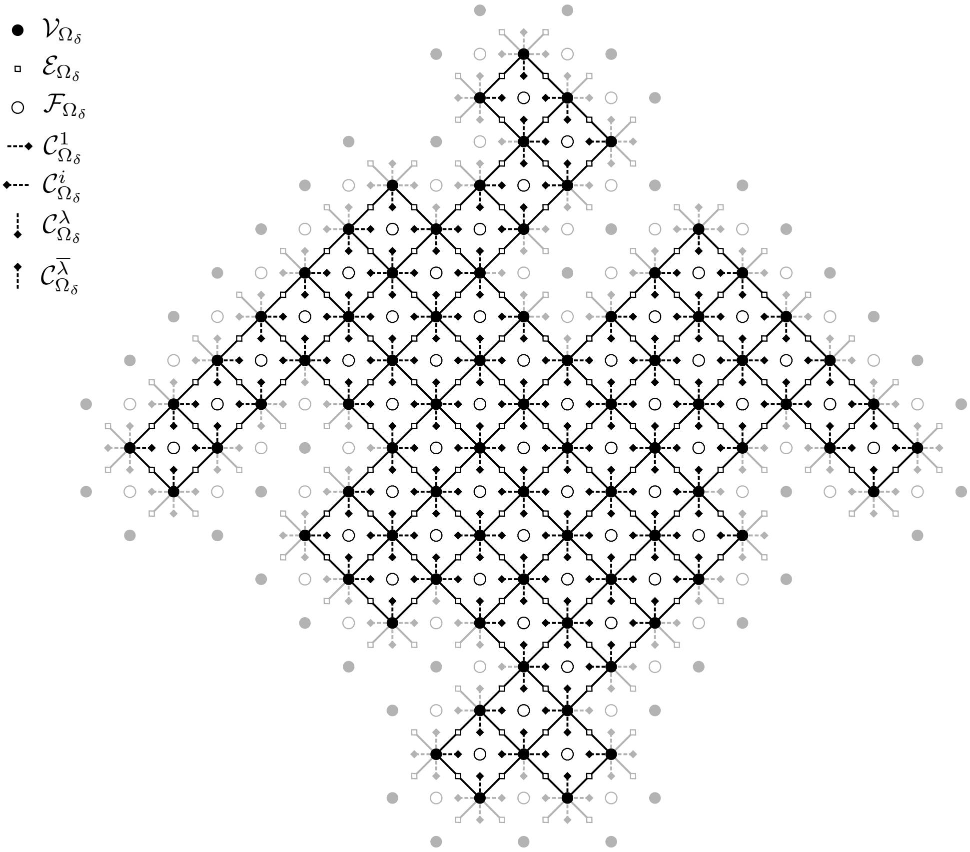

Figure 2: Example of sites for a discretization Ω_δ, with boundary elements coloured grey.

the representative of $\widetilde{z} \in [\Omega_\delta; \mathbf{a}; \mathbf{u}]$ in Ω_δ is the *projection* of $\widetilde{z}$. We write $\widetilde{z}$ when referring to an element of $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ whose projection is z, and $[A; \mathbf{a}; \mathbf{u}]$ for the set of lifts of elements of $A \subseteq \Omega_\delta$ in $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$. Finally, for each $\widetilde{z} \in [\Omega_\delta; \mathbf{a}; \mathbf{u}]$ we define $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\widetilde{z})$ as follows:

1. Fix forever a lift $\widetilde{a_1^{\rightarrow}} \in [\Omega_\delta; \mathbf{a}; \mathbf{u}]$ of the corner $a_1^{\rightarrow} := a_1 + \frac{\delta}{2}$.

2. Take any smooth path π running from $a_1^{\rightarrow}$ to z such that π intersects Θ an even number of times and π does not go through any of the points $\mathbf{a}$ and $\mathbf{u}$.

3. Lift π to the double cover $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ starting from $\widetilde{a_1^{\rightarrow}}$. Such a path must end in one of the two lifts of z.

4. $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\widetilde{z}) = +1$ if the lifted path ends at $\widetilde{z}$ and $\mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\widetilde{z}) = -1$ otherwise.

Finally, we assume $\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}] \neq 0$.

Definition 5.1. Given a lifted corner $\widetilde{z} \in [\mathcal{C}_{\Omega_\delta} \setminus \{\mathbf{c}, a_1^{\rightarrow}\}; \mathbf{a}; \mathbf{u}]$ with associated $\eta_c \in \{1, i, \lambda, \overline{\lambda}\}$, define

$$F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{z}) := \frac{\mathcal{Z}^{\widetilde{\Gamma},+}_{\Omega_\delta}}{\mathcal{Z}^{\Gamma,+}_{\Omega_\delta} \cdot \mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} \cdot \mathrm{sheet}_{\mathbf{a},\mathbf{u}}(\widetilde{z})\eta_z \cdot \tau^0 \mathbb{E}^{\widetilde{\Gamma},+}_{\Omega_\delta}[\sigma_{f(z)}\sigma_{a_2}\cdots\sigma_{a_n}]$$

where $\widetilde{\Gamma} \equiv \widetilde{\Gamma}[\mathbf{v}, a_1 + \delta, v(z)]$ is another set of disorder lines linking additional vertices and $\tau^0 \in \{\pm 1\}$ is a normalizing sign depending on the choice of $\widetilde{\Gamma}$: by duality arguments, $\mathbb{E}^{\widetilde{\Gamma},+}_{\Omega_\delta}[\sigma_{f(z)}\sigma_{a_2}\cdots\sigma_{a_n}]$ is independent of the choice of $\widetilde{\Gamma}$ up to a sign, and τ^0 corrects this.

Definition 5.2. Given a lifted edge $\widetilde{z} \in [\mathcal{E}_{\Omega_\delta}; \mathbf{a}; \mathbf{u}]$, define

$$F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e}) := F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}\left(\widetilde{z} + \frac{\delta}{2}i\right) + F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}\left(\widetilde{z} - \frac{\delta}{2}i\right)$$

where $\widetilde{z} \pm \frac{\delta}{2}i$ is the lift of $z \pm \frac{\delta}{2}i$ located on the same sheet of $[\Omega_\delta; \mathbf{a}; \mathbf{u}]$ as $\widetilde{z}$.

Remark 5.3. In the setting of [KC71, CCK17], the spinor can be written using fermions: on the corners, the spinor is essentially given by the operator

$$\frac{1}{\left\langle \prod_{k=1}^{n} \sigma(a_k) \prod_{k=1}^{2m} \Big[\sigma(u_k)\psi(c_k)\Big] \right\rangle} \left\langle \prod_{k=1}^{n} \sigma(a_k) \prod_{k=1}^{2m} \Big[\sigma(u_k)\psi(c_k)\Big] \cdot \psi(a_1^{\rightarrow})\psi(z) \right\rangle. \tag{2}$$

For the definition of $\psi(z)$ see Definition 4.1 of the thesis.

In the particular case $v(z) = a_1 + \delta$, the second collection of disorder lines $\widetilde{\Gamma}$ should link the vertices $\mathbf{v}, a_1 + \delta, a_1 + \delta$. Therefore, we can take $\widetilde{\Gamma} = \Gamma$. This leads to the following result that is vital in extracting the information from the spinors.

Proposition 5.4. *The equalities*

$$F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}(a_1^{\rightarrow} + \delta) = \tau^0 \frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} \qquad F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}\left(\widetilde{a}_1^{\rightarrow} + \frac{1+i}{2}\delta\right) = \tau^0 \frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+(1+i)\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]}$$

hold, where $\widetilde{a}_1^{\rightarrow} + \delta$ *and* $\widetilde{a}_1^{\rightarrow} + \frac{1+i}{2}\delta$ *are the lifts of* $a_1^{\rightarrow} + \delta$ *and* $a_1^{\rightarrow} + \frac{1+i}{2}\delta$ *located on the same sheet of* $[\Omega_\delta; \boldsymbol{a}; \boldsymbol{u}]$ *as* $\widetilde{a}_1^{\rightarrow}$.

6 Strategy for the convergence proof

The main difficulty faced in the study of a conformally invariant scaling limit of discrete models is the rigorous proof of the passage to the scaling limit. In fact, although heuristic arguments from Physics have suggested many observables for which a passage to the limit is expected, both for the Ising model and other 2D models, the technical difficulties faced in this step are so substantial that only recently it has been possible to arrive at a completely rigorous proof of convergence for a specific family of observables. We describe an outline of this proof, for more in depth insights, as well as remarks regarding generalizations to other models, see [Smi06].

The converging functions have to be carefully chosen. Since we wish to prove a conformal invariance property, we require a conformally covariant object in the continuous setting together with an adequate discretization. Note that it is unreasonable to expect the covariance property to hold at the lattice level. In addition, the discretization must possess properties that allow one to work with it and prove estimates. Some important properties in the continuum are already well understood at the lattice level: namely, analyticity and harmonicity. In fact, defining the function in the continuum using either of these properties and a boundary condition (like Dirichlet, Neumann or Robin) means that the corresponding discretizations have only to verify a local condition and a boundary condition.

6.1 S-holomorphicity

Restricted to $[\mathcal{E}_{\Omega_\delta};\mathbf{a};\mathbf{u}]$, the spinor observable $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ verifies a discretized version of the Cauchy-Riemann equations. For $e_{NW}, e_{SW}, e_{SE}, e_{NE} \in \mathcal{E}_{\Omega_\delta}$ that are vertices of a square of side δ starting in the upper left corner and going counter-clockwise,

$$\frac{F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e}_{NW}) - F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e}_{SE})}{\widetilde{e}_{NW} - \widetilde{e}_{SE}} = \frac{F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e}_{NE}) - F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{e}_{SW})}{\widetilde{e}_{NE} - \widetilde{e}_{SW}}$$

for both lifts of the edge midpoints, as long as all four are on the same sheet of $[\Omega_\delta;\mathbf{a};\mathbf{u}]$. This is not true for all combinations of $e_{NW}, e_{SW}, e_{SE}, e_{NE}$: namely, the ones surrounding one of the branching points $\mathbf{a}, \mathbf{u}$ (which would make the statement "all four are on the same sheet of $[\Omega_\delta;\mathbf{a};\mathbf{u}]$" not clear), but also the squares to the right of a_1 and $\mathbf{u}$. These are seen as "discrete singularities" and will require additional care.

Additionally, $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ verifies the following boundary condition:

$$\text{For } z \in [\partial\mathcal{E}_{\Omega_\delta};\mathbf{a};\mathbf{u}], \quad F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}(\widetilde{z}) \cdot \sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}$$

where $\nu_{\text{out}}(z)$ is a discrete analogue of the outer normal to the boundary at z. Together with the spinor nature of this function, this should allow us to identify the continuous version $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]} : [\Omega;\mathbf{a};\mathbf{u}] \longrightarrow \mathbb{C}$ of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$:

1. $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ is holomorphic on $[\Omega;\mathbf{a};\mathbf{u}]$[1].
2. For every $\widetilde{z} \in [\partial\Omega;\mathbf{a};\mathbf{u}]$, $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R}$.
3. $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ has multiplicative monodromy -1 around each branching point $b_1, \ldots, b_r$.

(We require additional knowledge regarding the behaviour around the branching points, which we are ignoring for now).

This reasoning faces a difficulty: the boundary condition is not robust enough to pass to the limit. Since Ω_δ is always a square grid rotated by an angle of $\frac{\pi}{4}$, the discrete version of $\nu_{\text{out}}(z)$ can only take the values of $e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}$ and $e^{\frac{7\pi i}{4}}$, hence we may not have $\nu^{\Omega_\delta}_{\text{out}}(z) \xrightarrow{\delta\to 0} \nu^{\Omega}_{\text{out}}(z)$.

A solution to this problem is to integrate the square of $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$. Note that

$$f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}(\widetilde{z})\sqrt{\nu_{\text{out}}(z)} \in \mathbb{R} \Leftrightarrow \left(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}\right)^2(z) \cdot i\nu_{\text{out}}(z) \in i\mathbb{R}^+_0$$

and $i\nu_{\text{out}}(\widetilde{z})$ is now tangent to Ω at z. Consider an antiderivative h of $(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$, which verifies

$$h(v) - h(u) = \int_\gamma \left(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}\right)^2(z)\, dz$$

for any path γ running from u to v. If $\gamma \subseteq \partial\Omega$ then the integrand must be imaginary. Hence, h verifies the boundary condition $\Re(h) \equiv Cte$, which passes to the limit naturally and is generally more pleasant to deal with.

This strategy runs into a technical difficulty: it requires some definition of “discrete primitive” of $(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]})^2$. This is problematic because, under usual definitions of discrete holomorphicity, there is no guarantee that the square of a discrete holomorphic function is discrete holomorphic, and so there may not be a well-defined primitive.

The solution to this problem is provided by a rather astounding observation first made in [Smi06]. By requiring that $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ is s-holomorphic, a stronger version of the usual discrete holomorphicity, it is possible to provide a suitable definition of $\Re\int\big(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}\big)^2$. In addition, this function shares many of the properties of discrete harmonic functions, as one would expect. The function $\Re\int\big(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}\big)^2$ plays an important role in proving the convergence of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$.

Definition 6.1. To each corner $c \in \mathcal{C}_{\Omega_\delta}$ we associate the line $l(c) := \eta_c\mathbb{R}$ — seen as a subset of $\mathbb{C}$ — and denote by $\mathrm{Proj}_{l(c)}[w]$ the *projection* of a complex number w onto the line $l(c)$, which can be written as

$$\mathrm{Proj}_{l(c)}[w] = \Re\big(w\overline{\eta}_c\big)\eta_c = \frac{1}{2}\big(w + \eta_c^2\overline{w}\big).$$

A function $F : [C\cup\mathcal{E}_{\Omega_\delta};\mathbf{a};\mathbf{u}] \longrightarrow \mathbb{C}$ defined on the lifts of sets $C \subseteq \mathcal{C}_{\Omega_\delta}$ and $\mathcal{E}_{\Omega_\delta}$ is *strongly holomorphic* at $c \in C$, or *s-holomorphic* for short, if for both $e \in \mathcal{E}_{\Omega_\delta}$ adjacent to c (that is, such that $|c-e| = \frac{\delta}{2}$)

$$F(\widetilde{c}) = \mathrm{Proj}_{l(c)}\big[F(\widetilde{e})\big]$$

for both lifts of c, with the lift of e taken to be on the same sheet of $[\Omega_\delta;\mathbf{a};\mathbf{u}]$. Moreover, F is s-holomorphic on C if it is s-holomorphic at each $c \in C$.

Remark 6.2. The function $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ is s-holomorphic on $[\mathcal{C}_{\Omega_\delta}\setminus\{a_1^{\rightarrow},\mathbf{c}\}\cup\mathcal{E}_{\Omega_\delta};\mathbf{a};\mathbf{u}]$. The projections to the corners $c = a_1^{\rightarrow},\mathbf{c}$ have opposite signs. Recalling Remark 5.3, note that the s-holomorphicity breaks down when $\psi(z)$ cancels out with another ψ on the numerator in (2).

6.2 Behaviour around the branching points

To completely define $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ we need to translate the behaviour of $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ around the branching points to the continuous setting. Around the branching points $b = a_2,\dots,a_n$ (where there are no discrete singularities), the discrete primitive $H = \Re\int(F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]})^2$ turns out to be bounded from below. Imposing this for the limit, together with $(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$ being holomorphic in a punctured neighbourhood of b, it follows that $h = \Re\int(f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]})^2$ should behave like $C_1\log|z-b|+C_2$ for some $C_1 \in \mathbb{R}_0^-, C_2 \in \mathbb{C}$. Therefore, $f^{\Gamma}_{[\Omega;\mathbf{a};\mathbf{u}]}$ behaves like $\sqrt{C}(z-b)^{-1/2}$, or $C(z-b)^{-1/2}$ with $C \in i\mathbb{R}$.

For the other branching points $b = a_1,\mathbf{u}$, we define a spinor $F_{[\mathbb{C}_\delta;b]}$ which has the same type of singularity. For the case $b = a_1$ this is conveyed by simply stating that $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]} - F_{[\mathbb{C}_\delta;b]}$ can be extended to be s-holomorphic at $b+\delta/2$. Using results regarding the convergence of normalized discrete Poisson kernels from [CS11], $F_{[\mathbb{C}_\delta;b]}$ is proven to be a discretization of $1/\sqrt{z-b}$. This means $F^{\Gamma}_{[\Omega_\delta;\mathbf{a};\mathbf{u}]}$ does not blow up faster than $\pm1/\sqrt{z-b}$ at $b = a_1$.

For $b = \mathbf{u}$, the same idea works but $F_{[\mathbb{C}_\delta;b]}$ needs to be multiplied by an additional multiplicative constant, which is the ratio of two other spinors. These are simpler in the sense that they have two less endpoints in their disorder lines. We thus require a recursive method in order to accurately define these spinors: one starts by studying the spinors with no disorder lines (for which the case $b = \mathbf{u}$ is vacuous), then uses them to define spinors with one disorder line, which are used in the definition of spinors with two disorder lines and so on. Although more involved, this situation does not require additional technical tools to handle when compared to the case $b = a_1$.

6.3 Convergence result and implications

Using the strategy described, we are able to prove the convergence result.

Theorem 6.3. *Given a bounded, simply connected domain $\Omega \subset \mathbb{C}$, let Ω_δ be a family of discrete, simply connected domains that converges to Ω as $\delta \to 0$. Then, for any $\varepsilon > 0$,*

$$\frac{1}{\vartheta(\delta)} F^{\Gamma}_{[\Omega_\delta;\boldsymbol{a};\boldsymbol{u}]}(\tilde{z}) \xrightarrow{\delta \to 0} f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}(\tilde{z})$$

uniformly on compact sets at distance at least ε from the branching points.

The normalizing constant $\vartheta(\delta)$ is the probability that a simple random walk on $\mathbb{C}_\delta$ — the square grid rotated by 45° with mesh size equal to $\sqrt{2}\delta$ — starting from an approximation of 1 on the lattice hits 0 before hitting the half-line $\{z < 0\}$. We prove that $\vartheta(\delta)$ is bounded both above and below by $\sqrt{\delta}$ asymptotically.

We then use the previous theorem to find a conformal invariance result. Proposition 5.4 states that we can extract information about the model by studying its behaviour near a_1. Passing to the scaling limit, we arrive at the result below.

Theorem 6.4. *Define $\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})$ as the following coefficient in the expansion of $f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}$ near the first branching point a_1:*

$$f^{\Gamma}_{[\Omega;\boldsymbol{a};\boldsymbol{u}]}(z) = \frac{1}{\sqrt{z-a_1}} + 2\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\sqrt{z-a_1} + O\big(|z-a_1|^{3/2}\big).$$

This coefficient verifies the conformal covariance rule

$$\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u}) = \varphi'(a_1) \cdot \mathcal{A}^{\varphi(\Gamma)}_{\Omega'}\big(\varphi(\boldsymbol{a});\varphi(\boldsymbol{u})\big) + \frac{1}{8}\frac{\varphi''(a_1)}{\varphi'(a_1)}$$

for any conformal mapping $\varphi : \Omega \to \Omega'$. In addition,

$$\lim_{\delta \to 0} \frac{1}{2\delta}\left(\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} - 1\right) = \Re\Big(\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\Big)$$
$$\lim_{\delta \to 0} \frac{1}{2\delta}\left(\frac{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1+2i\delta}\sigma_{a_2}\cdots\sigma_{a_n}]}{\mathbb{E}^{\Gamma,+}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]} - 1\right) = -\Im\Big(\mathcal{A}^{\Gamma}_{\Omega}(\boldsymbol{a};\boldsymbol{u})\Big)$$

with the respective values computed on the Ising models at the critical temperature, with + boundary conditions and with disorder lines Γ on graphs defined on Ω_δ.

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