Implementation and development of a new user defined element in Abaqus for analysis of composite plates using the 3rd-order theory of Reddy

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Abstract

The increasing usage of composite materials in the aerospace industry and the associated need for more accurate estimations of structural response motivated the development of a user-defined element (UEL) for static and free vibration analysis of composite laminated plates. The user element developed is based in an equivalent single-layer (ESL) theory, specifically in the third-order shear deformation theory (TSDT) of Reddy. Both $C^0$ and $C^1$ continuity interpolation functions are used. The implementation of the UEL is done in Abaqus, a commercial finite element analysis (FEA) software, which provides standard procedures for users to define their own elements and make use of its solver and interface to easily visualize and interpret solutions. As a preliminary phase, results for an isotropic plate are presented and validated through comparison with solutions obtained using a finite element available in Abaqus library. Then, numerical applications of composite laminates are shown and compared with first-order shear deformation theory (FSDT) analytical solutions available in literature. Assessment and discussion of results for thicker plates are of particular interest. Furthermore, the quadratic distribution of transverse shear stresses through thickness predicted by the UEL is understood as an important advantage of the usage of the TSDT of Reddy.

Keywords: Finite element, Equivalent single-layer, Third-order theory, Laminated plates, Abaqus UEL

1. Introduction

Composite materials have been widely used in a number of engineering applications. They are a combination of two or more materials to take the best advantage of the different desired features of each one. Composites are usually built in one of the following three types: fibrous, particulate and laminated composites. The present work focuses on laminated ones, which are made of several layers (or laminae) of different materials that can include fibrous and particulate composites. A fibre-reinforced composite consists of several high strength and high modulus fibres embedded in a matrix material.

Some of the industries that use composites are the automotive, naval, sports and recreation, rail and aerospace. In particular, fibre-reinforced composite materials are of major interest in the aerospace sector since they can address important requirements in this industry, such as the usage of light materials having high strength, good fatigue performance, high impact energy, corrosion resistance and high fracture toughness [1].

The constant demand for more efficient and affordable structures continuously motivates advances in composite materials. Specifically, structural analysis must be able to provide an increasingly accurate estimation of the structure response. Analysis of laminated composite materials involves a knowledge of anisotropic elasticity, structural theories, analytical or computational methods (such as the Finite Element Method [2, 3, 4]) to derive solutions of the governing equations and failure theories to anticipate modes of failures and determine failure loads [5]. The last is not in the scope of this work.

Modelling of composite laminates has been studied for several decades and different approaches and theories have been developed and applied. Specifically, composite laminates can generally be treated as plate elements since they have their planar dimensions at least one order of magnitude larger than their thickness. For an extended overview of the most relevant contributions in modelling of composite plates, the reader can consult [5, 6, 7, 8, 9, 10]. The present work uses equiva-
lent single-layer (ESL) theories, both in the development as well in the validation of a user-defined finite element.

The main objective of the present work is to develop and implement in Abaqus a new user-defined element (UEL) for analysis of composite laminated plates using an ESL theory, namely the third-order shear deformation theory (TSDT) of Reddy.

2. Background

Some important concepts and definitions of anisotropic elasticity and laminated composites are reviewed. ESL theories are also briefly described, especially the TSDT of Reddy.

2.1. Anisotropic Elasticity

An anisotropic material is that with directionally dependent properties. A comprehensive review on the fundamentals of anisotropic elasticity can be found in [5, 11, 12].

In view of linear elasticity and using the assumption of infinitesimal displacements, the strain components can be written in terms of the displacements as

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}; & \varepsilon_{zz} &= \frac{\partial w}{\partial z}; & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y}; & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{align*}
\]

where \(\gamma_{ij}\) are the engineering shear strains, defined as \(\gamma_{ij} = 2\varepsilon_{ij}, i \neq j\).

The equations that characterize a material and how it reacts to applied loads are called the constitutive equations. When considering elastic materials, their constitutive behaviour is only a function of the current state of deformation. In linear elasticity, the state of deformation is described by a linear relation between the states of stress and strain. This stress-strain relationship is known as the generalized Hooke’s law. The constitutive relations for different anisotropic materials, in respect to its planes of elastic symmetry, are deduced from the generalized Hooke’s law. In particular, a material that has three mutually orthogonal planes of material symmetry is called an orthotropic material. In this framework, noting that the in-plane transverse strain \(\varepsilon_{zz}\) is postulated zero, the constitutive equations for an orthotropic material can be written in the so-called material coordinate system, \((x_1, x_2, x_3)\), as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12} \\
\gamma_{13} \\
\gamma_{23}
\end{bmatrix}
\]

where \(\sigma_{ij}\) is the Young’s modulus in the \(i\)-direction, \(\nu_{ij}\) is the Poisson’s ratio, \(\nu_{23} = \frac{E_2}{E_1}\nu_{12}, \) and \(G_{ij}\) is the shear modulus in the \(x_i-x_j\)-plane.

The correspondent \(D\) matrix for an isotropic material (one with infinite planes of elastic symmetry) can be obtained from Eq. (2) by setting \(E_1 = E_2 = E, \nu_{12} = \nu_{23} = \nu\) and \(G_{12} = G_{13} = G_{23} = G\).

2.2. Laminated Composites

Generally, a fibre-reinforced composite material is made in the form of a thin layer - a lamina or ply. There are several possibilities for the fibres distribution in a lamina, although this work focuses on unidirectional fibre-reinforced laminae. When several plies are stacked over each other, with the same or different orientation, to better meet design requirements, it’s called a laminate. The lamination scheme is the term given to the sequence of orientations of the plies in a laminate. To characterize a unidirectional fibre-reinforced lamina, one considers it as an orthotropic material that has its material symmetry planes parallel and transverse to the fibre direction, \(\theta\). In Fig. 1 the material coordinate system \((x_1, x_2, x_3)\) is represented, with the \(x_1-\) and \(x_2-\)axes in the plane of the lamina, the first oriented along the fiber direction and the second transverse to it, and the \(x_3-\)axis oriented along the thickness direction.

![Figure 1: A unidirectional fibre-reinforced lamina with material and problem coordinate systems.](image)

There is a need to establish transformation relations among stresses and strains in the principal material coordinate system to the respective quantities in the problem coordinate system since, usually, they do not coincide. Besides, a laminate has several plies with different fibre orientation and hence, different material coordinate systems.
A laminate is assembled stacking its layers with their $x_1x_2$–planes parallel to each other, constraining the $x_3$–axis of each ply to coincide with the $z$–axis of the problem coordinate system. Therefore, the coordinate transformation for each layer is an in-plane rotation of the lamination angle $\theta$ – see Fig.1. Note that $(x, y, z)$ is the problem coordinate system (subscript $d$) and $(x_1, x_2, x_3)$ is the laminar (or material) coordinate system (subscript $p$). In this framework, the stress and strain vectors in problem and material coordinates are

$$\sigma_p = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\}^T$$

$$\sigma_t = \{\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}\}^T$$

$$\varepsilon_p = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}\}^T$$

$$\varepsilon_t = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}\}^T$$

and the stress and strain transformation relations can be written as

$$\sigma_p = T\sigma_t; \quad \sigma_t = T^{-1}\sigma_p$$

$$\varepsilon_p = T^{-1}\varepsilon_t; \quad \varepsilon_t = T^T\varepsilon_p$$

where $s$ and $c$ denote sine and cosine functions, respectively.

2.3. Equivalent Single-Layer Theories

In ESL theories, the original 3-D problem is reduced to a two-dimensional one or, in the case of multi-layered plates, to a 2-D problem at a layer level. These theories are based on the understanding of the kinematics of deformation and the state of stress through the thickness ($z$–coordinate) of the plate to assume a given $z$–expansion for displacements and/or stresses. A heterogeneous plate is treated as a statically equivalent single-layer having a complex constitutive behaviour, as an extension of classical theories developed for one-layered plates. The constitutive behaviour of such plate is then a sum of through-the-thickness integrated contributions of each layer. Two commonly used ESL theories are the classical laminated plate theory (CLPT) and the first-order shear deformation theory (FSDT) - see [5]. The third-order shear deformation theory (TSDT) of Reddy [13, 14, 5] gained wide acceptance and is based on the following displacement field

$$u(x, y, z, t) = u_0(x, y, t) + z\phi_x(x, y, t)$$

$$+ z^3 \left(-\frac{4}{3h^2}\right) \left(\phi_x + \frac{\partial u_0}{\partial x}\right)$$

$$v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t)$$

$$+ z^3 \left(-\frac{4}{3h^2}\right) \left(\phi_y + \frac{\partial v_0}{\partial y}\right)$$

$$w(x, y, z, t) = w_0(x, y, t)$$

where $(u_0, v_0, w_0)$ denote the displacement components in the midplane ($z = 0$), $(\phi_x, \phi_y)$ are the rotations of a transverse normal about the $y$– and $x$–axes, respectively, and $h$ is the plate thickness. The TSDT of Reddy provides an increase in accuracy with the disadvantage of requiring more computational effort when compared to the FSDT. The most important benefit of this theory is the transverse shear stresses quadratic variation through the layer thickness that it predicts, dismissing the need for shear correction coefficients. Also, it constrains the transverse shear stresses on the top and bottom surfaces of the plate to be zero, which is kinematically true.

3. Development and Implementation

The UEL is hereby developed for static and free-vibration analysis of laminated composite plates. The implementation in Abaqus is also described.

3.1. Theoretical Model

The development of the present UEL starts with the TSDT of Reddy displacement field stated in Eq.(5). The 7 primary values are grouped to form the generalized displacement vector $d$:

$$d = \left\{u_0, v_0, w_0, \phi_x, \phi_y, \frac{\partial u_0}{\partial x}, \frac{\partial w_0}{\partial y} \right\}^T$$

Substituting the displacement field into the expressions for the infinitesimal strains in Eq.(1), one can write the strain vector containing the significant strains ($\varepsilon_{zz} = 0$) as

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial y} \\ \frac{\partial w_0}{\partial y} \\ 0 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{z\partial\phi_x}{\partial x} + z^3 c_1 \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial x \partial y} \right) \\ \frac{z\partial\phi_y}{\partial y} + z^3 c_1 \left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial^2 w_0}{\partial x \partial y} \right) \\ \phi_x + \frac{\partial u_0}{\partial x} + z^2 c_1 \left(\phi_x + \frac{\partial u_0}{\partial x} \right) \\ \phi_y + \frac{\partial u_0}{\partial y} + z^2 c_1 \left(\phi_y + \frac{\partial u_0}{\partial y} \right) \end{bmatrix}$$

(7)
where \( c_1 = -\frac{4}{3} \). In a more compact form, the strain vector \( \varepsilon \) can be expressed as

\[
\varepsilon = S\hat{\varepsilon}
\]

in a way to isolate the strains from the \( z \)-coordinate. In Eq.(8), the strain transformation matrix \( S \) is as follows

\[
S = \begin{bmatrix}
1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and the generalized strain vector \( \hat{\varepsilon} \) is defined as

\[
\hat{\varepsilon} = \begin{bmatrix} \varepsilon_m \varepsilon_b \varepsilon_s \end{bmatrix}^T
\]

where

\[
\varepsilon_m = \begin{bmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial v_0}{\partial y} & \frac{\partial w_0}{\partial x} \end{bmatrix}^T, \\
\varepsilon_b = \begin{bmatrix} \frac{\partial^2 u_0}{\partial x^2} & \frac{\partial^2 u_0}{\partial x \partial y} & \frac{\partial^2 u_0}{\partial y^2} \end{bmatrix}, \\
\varepsilon_s = \begin{bmatrix} \phi_x & \phi_y & \frac{\partial w_0}{\partial y} \end{bmatrix},
\]

is the total number of layers and \( \bar{S} \) are the bottom and top \( z \)-coordinate of layer \( k \), respectively.

3.2. Finite Element Model

The finite element model hereafter developed requires \( C^0 \) continuity functions for interpolation of \((u_0, v_0, \phi_x, \phi_y)\) and \( C^1 \) continuity functions for interpolation of \( w_0 \).

The generalized displacements \((u_0, v_0, \phi_x, \phi_y)\) are approximated over an element by

\[
u_0(x, y) = \sum_{i=1}^{n} u_0, \psi_i(x, y) \] (13a)

\[
v_0(x, y) = \sum_{i=1}^{n} v_0, \psi_i(x, y) \] (13b)

\[
\phi_x(x, y) = \sum_{i=1}^{n} \phi_x, \psi_i(x, y) \] (13c)

\[
\phi_y(x, y) = \sum_{i=1}^{n} \phi_y, \psi_i(x, y) \] (13d)

where \((u_0, v_0, \phi_x, \phi_y)\) denote the values of \((u_0, v_0, \phi_x, \phi_y)\) at the \( i \)-th node of the Lagrange elements, \( n \) is the total number of nodes in those elements and \( \psi_i \) are the Lagrange interpolation functions. A four-noded Lagrange quadrangular element is used \((-1 \leq (\xi, \eta) \leq 1)\). The associated linear interpolation functions in terms of the natural coordinates \((\xi, \eta)\) are given by [3]

\[
\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} 1 - \xi (1 - \eta) \\ 1 + \xi (1 - \eta) \\ (1 + \xi)(1 + \eta) \\ (1 - \xi)(1 + \eta) \end{bmatrix} \] (14)

The generalized transverse displacement \( w_0 \) is approximated over an element by

\[
w_0(x, y) = \sum_{j=1}^{n} \Delta_{0j} \varphi_j(x, y) \] (15)

where \( \Delta_{0j} \) denote the values of \( w_0 \) and its derivatives with respect to \( x \) and \( y \) at the \( j \)-th node of the finite element, \( n \) is the element’s number of nodes and \( \varphi_j \) are the Hermite interpolation functions. A nonconforming quadrangular element with three degrees of freedom \((w_0, \frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y})\) per node, \( 2 \times 2 \) dimensions and \((x_0, y_0) = (0, 0)\) (centre global coordinates) is used. The associated Hermite functions in terms of the natural coordinates are as follows

\[
\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \varphi_1, \varphi_4, \varphi_7, \varphi_{10} \\ \varphi_2, \varphi_5, \varphi_8, \varphi_{11} \\ \varphi_3, \varphi_6, \varphi_9, \varphi_{12} \end{bmatrix} \] (16a)
where

\[
\begin{align*}
\varphi_1 &= (1 - \xi)(1 - \eta)(2 - \xi - \eta - \xi^2 - \eta^2) \\
\varphi_2 &= (\xi + 1)(1 - \eta)(1 - \xi^2) \\
\varphi_3 &= (\eta + 1)(1 - \xi)(1 - \eta) \\
\varphi_4 &= (1 + \xi)(1 - \eta)(2 + \xi - \eta - \xi^2 - \eta^2) \\
\varphi_5 &= (\xi - 1)(1 - \eta)(1 + \xi^2) \\
\varphi_6 &= (\eta - 1)(1 + \xi)(1 - \eta^2) \\
\varphi_7 &= (1 + \xi)(1 + \eta)(2 + \xi - \eta - \xi^2 - \eta^2) \\
\varphi_8 &= (\xi - 1)(1 + \eta)(1 + \xi^2) \\
\varphi_9 &= (\eta - 1)(1 + \xi)(1 + \eta^2) \\
\varphi_{10} &= (1 - \xi)(1 + \eta)(2 - \xi - \eta - \xi^2 - \eta^2) \\
\varphi_{11} &= (\xi + 1)(1 + \eta)(1 - \xi^2) \\
\varphi_{12} &= (\eta + 1)(1 - \xi)(1 + \eta^2) \\
\end{align*}
\]

\[
(16b)
\]

In equation (16), \(g_1, g_2\) and \(g_3\) include the interpolation functions for \(w_0, \partial w_0 / \partial x\) and \(\partial w_0 / \partial y\), respectively, for each node in the element.

The generalized displacement vector for a node \(i\) is defined as

\[
d_i = \begin{bmatrix} w_{0i} \\ \partial w_{0i} / \partial x \\ \partial w_{0i} / \partial y \end{bmatrix}
\]

\[
(17)
\]

Using the approximations in Eqs.(13) and (15), the corresponding interpolation functions in Eqs. (14) and (16), and the expressions in Eq.(10), the generalized strain vector is expressed in terms of the nodal displacements as

\[
\dot{\epsilon} = \sum_{i=1}^{n} B_i d_i = [B_1, \ldots, B_n] \left\{ \begin{array}{c} d_1 \\ \vdots \\ d_n \end{array} \right\} = Bd
\]

\[
(18)
\]

where \(B\) and \(B_i\) are the generalized strain matrices for the element and a node \(i\), respectively, \(d\) is the generalized displacement vector for the element and \(n = 4\) is the total number of nodes in the element. From equation (18) one can deduce

\[
B_i =
\begin{bmatrix}
\partial \varphi_1 / \partial x & \partial \varphi_1 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_2 / \partial x & \partial \varphi_2 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_3 / \partial x & \partial \varphi_3 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_4 / \partial x & \partial \varphi_4 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_5 / \partial x & \partial \varphi_5 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_6 / \partial x & \partial \varphi_6 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_7 / \partial x & \partial \varphi_7 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_8 / \partial x & \partial \varphi_8 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_9 / \partial x & \partial \varphi_9 / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_{10} / \partial x & \partial \varphi_{10} / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_{11} / \partial x & \partial \varphi_{11} / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial \varphi_{12} / \partial x & \partial \varphi_{12} / \partial y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
(19)
\]

All integrals in the finite element model are evaluated numerically using the Gauss-Legendre quadrature. Gauss quadrature requires the integrals to be expressed over a square region \(\Omega\) of dimensions \(2 \times 2\) and the coordinate system \((\xi, \eta)\) be such that \(-1 \leq \xi, \eta \leq 1\) [5]. The transformation between global \((x, y)\) and natural \((\xi, \eta)\) coordinates is accomplished by a coordinate transformation of the form

\[
x = \sum_{i=1}^{n} \dot{\psi}_i (\xi, \eta) x_i \\
y = \sum_{i=1}^{n} \dot{\psi}_i (\xi, \eta) y_i
\]

\[
(20)
\]

where \(\dot{\psi}_i\) are the finite element interpolation functions of the master element \(\Omega\). The same Lagrange interpolation functions as in Eq.(13) are used. Relations between \(\dot{\psi}\) derivatives in \((x, y)\) and \((\xi, \eta)\) coordinate systems are established as

\[
\begin{bmatrix}
\partial \dot{\psi}_i \\
\partial \xi \\
\partial \eta \\
\end{bmatrix} = J \begin{bmatrix}
\partial \psi_i \\
\partial \xi \\
\partial \eta \\
\end{bmatrix}
\]

\[
(21a)
\]

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & 0 \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & 0 \\
0 & 0 & 1 \\
\end{bmatrix} = \left[ \sum_{i=1}^{n} \frac{\partial \psi_i}{\partial \xi} x_i \right] \left[ \frac{\partial \psi_i}{\partial \eta} y_i \right]
\]

\[
(21b)
\]

where \(n = 4\) and \(J\) is the Jacobian matrix of the transformation of the derivatives of \(\dot{\psi}\) in the natural and global coordinate system. Since second order derivatives of Hermite functions appear in a number of terms of the generalized strain matrix, there is also a need to relate second derivatives of \(\dot{\psi}\) in the natural and global coordinate system - Eq.(22).

\[
\begin{bmatrix}
\frac{\partial^2 \dot{\psi}_i}{\partial \xi^2} \\
\frac{\partial^2 \dot{\psi}_i}{\partial \xi \partial \eta} \\
\frac{\partial^2 \dot{\psi}_i}{\partial \eta^2} \\
\end{bmatrix} = J^A \begin{bmatrix}
\frac{\partial^2 \psi_i}{\partial \xi^2} \\
\frac{\partial^2 \psi_i}{\partial \xi \partial \eta} \\
\frac{\partial^2 \psi_i}{\partial \eta^2} \\
\end{bmatrix}
\]

\[
(22a)
\]

\[
J^A = \begin{bmatrix}
\frac{\partial^2 x}{\partial \xi^2} & \frac{\partial^2 y}{\partial \xi^2} & 2 \frac{\partial^2 y}{\partial \xi \partial \eta} \\
\frac{\partial^2 x}{\partial \xi \partial \eta} & \frac{\partial^2 y}{\partial \xi \partial \eta} & \frac{\partial^2 y}{\partial \eta^2} \\
\frac{\partial^2 x}{\partial \eta^2} & \frac{\partial^2 y}{\partial \eta^2} & \frac{\partial^2 y}{\partial \xi \partial \eta} \\
\end{bmatrix}
\]

\[
(22b)
\]

\(J^A\) is the Jacobian matrix of the transformation of the second derivatives of \(\dot{\psi}\) in the natural and global coordinate system.

The element stiffness and mass matrices \((K\) and \(M\), respectively) are deduced using the well-known Lagrange equations [15]. The equations of motion for static and dynamic analysis (in the absence of damping) are

\[
\text{static: } Kd = f
\]

\[
\text{dynamic: } M\ddot{d} + Kd = f
\]

\[
(23)
\]
where
\[ K = \int_A B^T \dot{D} B dA \]
\[ M = \int_A \rho F^T P F dA \] (24)

Matrix \( P \) is determined according to
\[ P = \int_{-h/2}^{h/2} Z^T Z dz \] (25a)
\[ Z = \begin{bmatrix} 1 & 0 & 0 & z + z^3 c_1 & 0 & z^3 c_1 & 0 \\ 0 & 1 & 0 & 0 & z + z^3 c_1 & 0 & z^3 c_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \] (25b)
where \( Z \) is obtained from the displacement field in Eq.(5) that can be written as
\[ u = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = Z \begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_x \\ \phi_y \\ \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = Zd \] (25c)

\( F \) in Eq.(24) is a matrix defined as
\[ d = F d^e = [F_1, \ldots, F_n] d^e \] (26)
where \( n = 4 \) is the number of nodes in the element. Combining equation (26) with the approximations in Eqs.(13) and (15), for \( i = 1, \ldots, n \) one can deduce
\[ F_i = \begin{bmatrix} \psi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{i1} & 0 & 0 & g_{i2} & g_{i3} \\ 0 & 0 & 0 & \psi_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_i & 0 & 0 \\ 0 & 0 & \frac{\partial g_{i1}}{\partial x} & 0 & 0 & \frac{\partial g_{i2}}{\partial x} & \frac{\partial g_{i3}}{\partial x} \\ 0 & 0 & \frac{\partial g_{i1}}{\partial y} & 0 & 0 & \frac{\partial g_{i2}}{\partial y} & \frac{\partial g_{i3}}{\partial y} \end{bmatrix} \] (27)
Using the Gauss-Legendre quadrature to numerically evaluate the integrals in Eq.(24), the stiffness and mass matrices are computed as follows
\[ K_{ij} = \sum_{i=1}^{q} \sum_{j=1}^{q} B^T(\xi, \eta_j) \dot{D} B(\xi, \eta_j) | J | (\xi, \eta_j) W_i W_j \]
\[ M_{ij} = \sum_{i=1}^{q} \sum_{j=1}^{q} \rho F^T(\xi, \eta_j) P F(\xi, \eta_j) | J | (\xi, \eta_j) W_i W_j \] (28)
where 2×2 and 4×4 Gauss integration points are used, respectively, for \( K_{ij} \) and \( M_{ij} \). Reduced integration is included for shear terms. Gauss points coordinates and respective weights are summarized in Tab.1.

<table>
<thead>
<tr>
<th>( \xi_i, \eta_j )</th>
<th>( W_i, W_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2×2</td>
<td>± \sqrt{1/3}</td>
</tr>
<tr>
<td>4×4</td>
<td>1</td>
</tr>
<tr>
<td>± 0.3399810435</td>
<td>0.6521451548</td>
</tr>
<tr>
<td>± 0.8611363116</td>
<td>0.3478548451</td>
</tr>
</tbody>
</table>

Table 1: 2×2 and 4×4 Gauss points coordinates and weights [2].

3.3. Implementation in Abaqus
The objective in implementing the UEL in Abaqus is to use its solver to determine solutions for static and dynamic analysis. To do so, a UEL subroutine and input file must be written following standard Abaqus procedures [16] and then these files must be correctly linked with the software.

The user subroutine must include all element definitions and perform its calculations according to the desired analysis. It is coded using Fortran 90 programming language and must begin with a default header provided in Abaqus documentation. For a better understanding on the input and output variables provided by and back to Abaqus and how the subroutine is coded see Fig.2. It should be noted that the schematic inside the UEL subroutine block is quite simplified. Even for static analysis, Abaqus requires the definition of the mass matrix. Thus, in this cases, it is simply defined as an identity matrix. For free vibrations, one can substitute the AMATRX (mass) block in figure 2 by the flowchart in figure 3.

The input and output variables are defined by default (in view of this work) as:

- **COORDS** - matrix containing the \((x, y)\) coordinates of the element nodes;
- **U (dU)** - vector containing the degrees of freedom values of each node in the current (previous) time increment;
- **AMATRIX** - element mass and stiffness matrices;
- **RHS** - internal force vector for the element.

The subroutine is called for an element at a current time increment two times. In the first one, **COORDS** is the only input that is not zero and the only output that is defined within the subroutine is **AMATRIX** (both the mass and stiffness matrices are passed to Abaqus). In the second time, **U** and
\[ \textbf{dU} \] are already computed by Abaqus solver, using AMATRX, that are passed to the subroutine as inputs. The subroutine can then perform the calculations of the SVARS (stresses and strains at 2×2 Gauss points) and RHS vectors, if a static analysis is being performed.

The input file contains information that complements the subroutine needed for Abaqus to build the user-defined element. Furthermore, it defines the mesh, the boundary conditions, the applied loads and the desired analysis to be performed. It can be edited in Notepad (text editor) and should be saved with the .inp extension.

Several files are generated by Abaqus when linking the subroutine and input files with the software. The file with .odb extension is an output database and can be opened in Abaqus/CAE. The visualization module is started and the user can view the model and its analysis results, such as displacement distribution, reaction force distribution and, in the case of free vibration, the mode shapes and corresponding eigenvalues. The file with .log extension is also important since the user can print variables in it using the subroutine. For example, the strains and stresses at an element integration point are obtained by printing them in the .log extension file. Then, to obtain the value at a specific point of the plate other than a Gauss point, a direct local extrapolation is used [3].

**4. Results**

A number of finite element results of square plates using the UEL previously developed are compared with finite element results using a conventional Abaqus shell element and with analytical solutions using the FSDT.

**4.1. Static Analysis of an Isotropic Plate**

The UEL results are validated through comparison with finite model results obtained using the S4R element - a four-noded reduced integration quadrilateral shell element available in Abaqus library. An isotropic plate with \( a \times a \times h \) dimensions is used, considering a side-to-thickness ratio of \( a/h = 20 \).

Two load cases are studied: application of a concentrated force of magnitude \( F \) in the positive \( z \)-direction at the node with coordinates \((a, a)\) and a maximum imposed displacement of magnitude \( d_{\text{imp}} \) in the positive \( z \)-direction at the node with coordinates \((a, a)\). The boundary conditions are the same for both load cases: all degree of freedom fixed zero at \( x = 0 \) and \( y = 0 \). A 25×25 mesh (625 elements) is used.

For the first load case, the results used for validation of the UEL are the transverse displacement distribution and the membrane stresses at the centre of the plate for \( z = h/2 \). Regarding the second load case, \( z \)-direction reaction force distribution and the same stresses are used.

The results for both load cases are consistent and considered enough to validate the UEL development and implementation for an isotropic plate.

**4.2. Static Analysis of Laminated Plates**

Finite element static analyses results of square laminated plates \((a \times a \times h)\) obtained using the UEL developed in this work are compared to FSDT analytical results. These results are derived from Navier solutions that are fully developed in chapter 7 of [5], where the corresponding results are also presented.
Symmetric and antisymmetric cross-ply laminates and antisymmetric angle-ply ones are considered, taking into account several side-to-thickness ratios for each laminate. Specifically, (0/90/0), (0/90/90/0), (0/90), (0/90), (−45/45/45) and (−45/45) laminates with \( h_k = h/L \) are studied.

All laminates are under a uniform distributed load of intensity \( q_0 \) applied on the top surface and the layers are all made of the same material with the following properties:

\[
E_1 = 25E_2; \quad G_{12} = G_{13} = 0.5E_2; \quad \nu_{12} = 0.25.
\]

The FSDT shear correction coefficient is equal to 5/6. Two types of boundary conditions are used, depending on the laminate. For cross-ply laminates, simply supported boundary conditions of type-1 (SS-1) are used. For angle-ply laminates, the simply supported boundary conditions used are of type-2 (SS-2). A 33×33 mesh (1089 elements) is used.

Maximum transverse displacement and stresses (both membrane and transverse shear stresses) values are used for the validation of the UEL. Overall, convergence of results is observed for thinner plates, as expected since the TSDT of Reddy can better predict results for thicker plates than the FSDT, due to its higher order. Significant differences are distinguished for transverse shear stresses, which are justified by the different distribution through layer thickness predicted by each modelling theory (constant and quadratic for transverse shear stresses derived from FSDT and TSDT of Reddy constitutive equations, respectively).

As an example of the comparison between UEL results and FSDT analytical solutions, Tabs.2 and 3 are presented. The locations of the maximum stresses are as follows

\[
\sigma_{xx}(a/2,a/2,−h/2) = −\sigma_{yy}(a/2,a/2,h/2) \quad \sigma_{xy}(a,a,−h/2) = \sigma_{yz}(a/2,0,z_2)
\]

where \( z_1 = z_2 = 0 \) for values computed using the UEL and FSDT constitutive equations; for equilibrium derived stresses (superscript ‘∗’), \( z_1 = −h/4 \) and \( z_2 = h/4 \).

As it can be observed, the values converge for higher side-to-thickness ratios. Observing the transverse shear stress results in Tab.3, the UEL predicts values between the FSDT constitutive and equilibrium equations derived ones. Although the location of maximum \( \sigma_{xz} \) is not the same for values derived from constitutive and equilibrium equations, it is not conclusive which location is more accurate. For example, the UEL and FSDT constitutive relations predict maximum stress in the outer layers of the (0/90/0) laminate, while the equilibrium equations predict the maximum stress to be at the plate midplane. It turns out, resorting to exact solutions presented in [17], that the constitutive relations yield, qualitatively, the correct stress variation.

The transverse shear stresses quadratic variation through layer thickness obtained using the UEL is explicitly shown in Fig.4 for \( \sigma_{xz} \) of the (0/90) antisymmetric cross-ply laminate. The zero transverse shear stresses condition in top and bottom surfaces of the plate can also be observed.

![Figure 4: (0/90) laminate distribution of transverse shear stress \( \sigma_{xz}(0,a/2,z) \) through thickness \( z/h \) (UEL, \( a/h = 10 \)).](image-url)

As an example of the comparison between UEL results and FSDT analytical solutions, Tabs.2 and 3 are presented. The locations of the maximum stresses are as follows

<table>
<thead>
<tr>
<th>( a/h )</th>
<th>( \bar{w} \times 10^2 )</th>
<th>( \bar{\sigma}_{xx} )</th>
<th>( \bar{\sigma}_{xy} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>UEL 10</td>
<td>1.2384</td>
<td>0.3621</td>
<td>0.4953</td>
</tr>
<tr>
<td>FSDT 10</td>
<td>1.2791</td>
<td>0.3476</td>
<td>0.4274</td>
</tr>
<tr>
<td>UEL 20</td>
<td>1.0784</td>
<td>0.3529</td>
<td>0.4569</td>
</tr>
<tr>
<td>FSDT 20</td>
<td>1.0907</td>
<td>0.3496</td>
<td>0.4357</td>
</tr>
<tr>
<td>UEL 100</td>
<td>1.0267</td>
<td>0.3501</td>
<td>0.4440</td>
</tr>
<tr>
<td>FSDT 100</td>
<td>1.0305</td>
<td>0.3504</td>
<td>0.4417</td>
</tr>
</tbody>
</table>

Table 2: (−45/45) laminate dimensionless maximum transverse deflections and membrane stresses.

<table>
<thead>
<tr>
<th>( a/h )</th>
<th>( \bar{\sigma}_{xz} )</th>
<th>( \bar{\sigma}_{xz}^∗ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>UEL 10</td>
<td>0.4289</td>
<td>——</td>
</tr>
<tr>
<td>FSDT 10</td>
<td>0.5072</td>
<td>0.4328</td>
</tr>
<tr>
<td>UEL 20</td>
<td>0.4324</td>
<td>——</td>
</tr>
<tr>
<td>FSDT 20</td>
<td>0.5065</td>
<td>0.4205</td>
</tr>
<tr>
<td>UEL 100</td>
<td>0.4299</td>
<td>——</td>
</tr>
<tr>
<td>FSDT 100</td>
<td>0.5068</td>
<td>0.4189</td>
</tr>
</tbody>
</table>

Table 3: (−45/45) laminate dimensionless maximum shear stress.
(0/90/90/0) laminate is shown in Fig. 5.

4.3. Free Vibration Analysis of Laminated Plates

Numerical results for free vibration of square laminated plates \((a \times a \times h)\) dimensions) computed using the UEL developed in the present work are again compared with results derived from Navier solutions using the FSDT, which are presented in chapter 7 of [5].

The same boundary conditions as for the static analysis of laminated plates are used, i.e. SS-1 for cross-ply laminates and SS-2 for angle-ply laminates. The FSDT shear correction coefficient used is again \(K = 5/6\). All results are obtained including rotary inertia. A 33\times33 mesh is also used for the free vibration analysis numerical applications.

Symmetric and antisymmetric cross-ply laminates and antisymmetric angle-ply ones are again considered, taking into account several side-to-thickness ratios, Young’s modulus ratios and different material properties. Specifically, \((0/90/0/\cdots)\) 3-, 5-, 7- and 9-ply laminates, with total thickness of all 0° and all 90° layers equal to \(h/2\), are studied. \((0/90/0), (0/90), (0/90)_4, (-45/45)\) and \((-45/45)_4\) laminates with \(h_k = h/L\) are also analysed. Convergence of results is again observed for thinner plates. The UEL results are consistent with the analytical FSDT solutions. For the thicker plates, \(a/h = 5\), the frequency predicted by the UEL is higher than that obtained with the FSDT. This is more evident for the \(L = 2\) laminate.

The first 4 modes of vibration for transverse displacement of a \((0/90/0)\) laminate studied, obtained using the UEL, are shown in figures 6.

\[
\bar{\omega} = \omega \left( \frac{a^2}{h} \right) \sqrt{\frac{\rho}{E_2}} \quad (31)
\]

5. Conclusions

The present finite element model uses \(C^0\) and \(C^1\) continuity functions for interpolation of the generalized displacements. The use of \(C^1\) functions adds complexity to the model but it is needed to ensure continuity of the transverse deflection and its...
derivatives between elements, in order to satisfy the zero transverse shear stresses condition on the top and bottom plate surfaces.

The main advantages of the implementation in Abaqus are considered to be the use of its robust solver, which is extremely fast, and the use of its graphical interface, allowing an easy and user-friendly visualization and interpretation of results.

The user-defined element developed is considered to be more accurate than one that uses the FSDT, especially for thicker plates. However, in view of ESL theories limitations, the awareness of that the present model can be incapable of accurately describing the state of stress and strain near geometric and material discontinuities and near regions of extreme loading should be kept in mind.

The major achievements of the present work are the development and implementation of a new user-defined element for analysis of composite laminated plates in Abaqus. Overall, validation of the UEL developed was successfully accomplished for static and free vibration analysis. The introduction of this finite element in Abaqus software is considered a benefit in the analysis of composite laminates. Furthermore, it can happen that this work contributes to motivate others to implement more complex and accurate finite elements in Abaqus.

5.1. Future Work
It would be advantageous to take all results directly from Abaqus interface, rather than print stress and strain values in the .log extension file. However, it was really difficult to find any information regarding this issue in Abaqus documentation.

Another suggestion for future work is to extend the UEL to a shell structure, in which is included the flat plate element as a special case, since shells are also used in a number of engineering structures, including in aerospace applications.

References