Curved structures simply suported on inner or outer edges

João Luís dos Santos Raposo

Instituto Superior Técnico, Lisbon, Portugal

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Abstract: Because of its geometry, this type of structure requires only a simple alignment of supports along one of its edges so that equilibrium is guaranteed. Consequently, it becomes possible to construct cantilever structures without using fixed supports. Using the equations of equilibrium, compatibility and the constitutive relations a first approach to characterize the exact solution of the problema is presented. Afterwards and through numerical models of finite elements, the behavior in service and the ultimate limit state were evaluated, in order to demonstrate the viability of this type of solutions. For this, a practical study case was analyzed and a parametric analysis was carried out.

Afterwards, a curved structure with ‘‘S’’ geometry was studied in order to understand whether the structural behavior was similar to the base system and evaluate benefits or disadvantages with respect to the previous solution.

The study showed that the design of this type of structures is essentially conditioned by serviceability conditions. It has been found, however, that for médium radius less than 6m, the solution leads to current dimensions and amounts of reinforcement.

The results for solutions with geometry in ‘‘S’’ allowed to conclude that this type of geometry leads to more rigid solutions and therefore to a consequent better behavior in service.

Keywords: Curved structures, structural behavior, parametric analysis, service limit state, ultimate limit state, structural concrete.

I. INTRODUCTION

A. SCOPE OF THE INVESTIGATION

With the evolution of science and engineering, over the years very impressive structures have emerged concerning aesthetics. An example is the type of structure studied in this dissertation, which have a circular shape in their horizontal plane and so are particulary interesting due to their distinct structural functioning. This type of solution only needs an alignment of simple supports along one of its edges to be statically equilibrated. The designer can take advantage of its geometry and suspend it only along one of the edges as shown in figure 1.

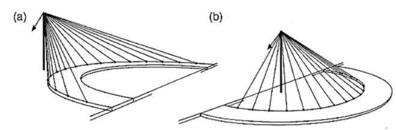

Figure 1 - Suspension of the curved deck: (a) outer edge (b) inner edge (Strasky 2011).

II. MOTIVATION AND OBJECTIVES

The purpose of this work is to study the behavior of curved structures supported by the inner or the outer edge. The first objective is to investigate the distributions of stresses that lead to statically equilibrated solutions in this type of structures.

After clarifying this question, a practical case study is analyzed, in order to evaluate the structural response in respect to the safety verifications.

Subsequently it is intended to perform a parametric analysis, varying some geometrical data, in order to perceive the influence of each one in the structural behavior and consequently in design. After that it is important to define ranges of practical applicability for this type of solution, highlighting in which situations its design is feasible.

It is intended to draw conclusions about the advantages and disadvantages in the adoption of curved structures supported along the inner or outer edges.

It was also interesting to analyze the structural behavior of a structure with “S” geometry, in order to understand the behavior differences comparatively to the one previously analyzed.

I. THEORY

Based on the theory of elasticity and satisfying the equilibrium, compatibility and costituve relations the governative solution of the problem is described.

The representation of the equilibrium of an infinitesimal axisymmetric plate element is presented in figure 2.

From the equilibrium of forces in vertical direction, $\sum F_z = 0$, we obtain (Reddy 2006):

$$\frac{d}{dr}(Q_r r) + qr = 0 \qquad (2.1)$$

By the equilibrium of moments, $\sum M_o = 0$, the shear force is given by (Reddy 2006):

$$Q_r r = \frac{d}{dr}(M_{rr} r) - M_{\theta\theta} \qquad (2.2)$$

Replacing (2.2) in (2.1), the equilibrium can be written by:

$$\frac{d^2 M_{rr}}{dr^2} + 2\frac{dM_{rr}}{dr} - \frac{dM_{\theta\theta}}{dr} + qr = 0 \qquad (2.3)$$

Based on constitutive and compatibilty relations the expressions for the bending moments are the following:

$$M_{rr} = -D\left(\frac{\mathrm{d}^2 w_0}{\mathrm{d}r^2} + \frac{v}{r}\frac{\mathrm{d}w_0}{\mathrm{d}r}\right) \qquad (2.4)$$

$$M_{\theta\theta} = -D\left(\frac{1}{r}\frac{dw_0}{\mathrm{d}r} + v\frac{\mathrm{d}^2 w_0}{\mathrm{d}r^2}\right) \qquad (2.5)$$

Replacing (2.4) and (2.5) in (2.2) we have the following expression for the shear force for axisymmetrical bending written in terms of displacements:

$$Q_r = -D\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0}{\mathrm{d}r}\right)\right] \qquad (2.6)$$

where *D* is given by: (Leitão e Castro 2018)

$$D = \frac{E\mathrm{h}^3}{12(1-v^2)} \qquad (2.7)$$

After, replacing (2.7) in (2.3), we obtain the governing equation which can be written by:

$$\frac{D}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_0}{\mathrm{d}r}\right)\right]\right\} = 0 \qquad (2.8)$$

In the case of a slab with annular geometry, as shown in figure 3, the location of the supports can be made on the inner or outer edge. Therefore, it is interesting to carry out the study for the two possible cases.

Considering the geometry parameters (figure 3), the boundary conditions for these cases are described in table 1.

Table 1 - Boundary conditions.

	Support on outer edge	**Support on inner edge**
$\boldsymbol{r = R_{int}}$	$Q_r = 0$ $M_{rr} = 0$	$w_0 = 0$ $M_{rr} = 0$
$\boldsymbol{r = R_{ext}}$	$w_0 = 0$ $M_{rr} = 0$	$Q_r = 0$ $M_{rr} = 0$

The displacements field can be obtained by making a fourth-order integration of equation (2.8) and imposing the appropriate boundary conditions.

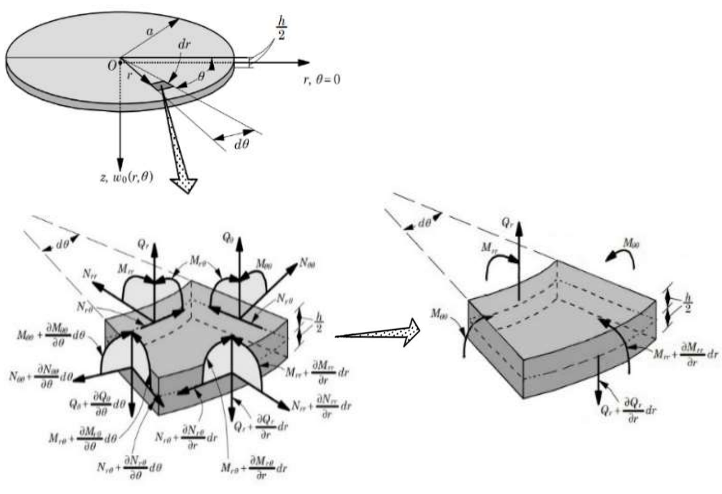

Figure 2 - Equilibrium of an infinitesimal elemento (Reddy 2006).

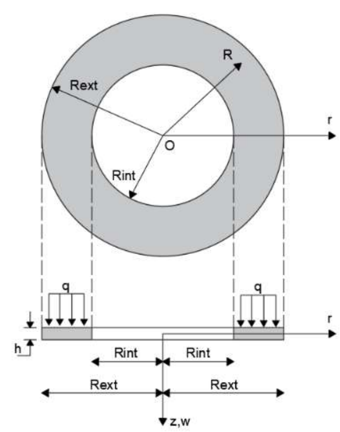

Figure 3 - Parametrization of the problem.

II. PRATICAL STUDY CASE

The practical case that motivated the theme of this dissertation was the staircase of the entrance hall of the Hotel Savoy Pallace located on the Madeira island (see figure 4 and 5). The structure, shown in figure 4 and 5, refers to a circular staircase connecting the floor -2 to the ground floor. It is not connected to the near collumns, so its support is made by 5 ties anchored to the slab of the floor 4. The supports are placed along the inner edge of the intermediate span, so this situation will be the base for the study.

Figure 4 - Structure of the staircase.

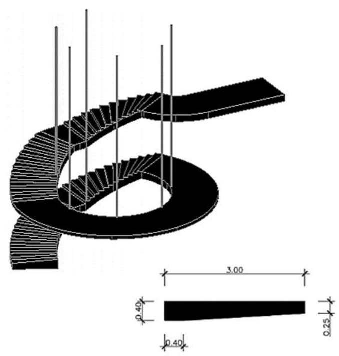

Figure 5 -Three-dimensional schematic model of stairs and cross section of the slab.

B. MODELING DESCRIPTION

In order to study the static and dynamic behavior of the above structure, a numerical analysis was developed using the Finite Element Method (MEF) program, SAP2000 (Computers & Structures, 2017).

As previously mentioned, the plan zone is the only one to study in the context of this work. Thus, it was decided to model only this portion of the structure assuming, as a simplification, that the remaining part does not influence significantly the structural behavior. In figure 6 it is showed the geometry and the boundary conditions assumed.

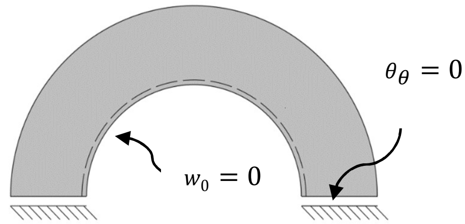

Figure 6 - Model's boundary conditions.

C. DEFLECTION LIMIT STATE

Through the numerical analysis carried out, it was possible to obtain the structure deformation for the quasi-permanent combination of actions. This is shown in figure 7.

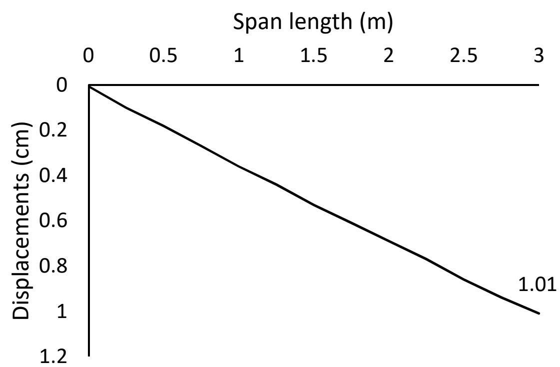

Figure 7 – Displacements for inner suported structure.

The global coefficient method was used to evaluate the long term displacements. The value obtained was approximately 5 times greater than the elastic displacements. The value obtained for the maximum slope was about 2 times greater than the EC2 limtit. In this way it was necessary to adopt measures to minimize long-term deformation.

D. VIBRATION LIMITE STATE

In many flexible structures, such as stairs or passereles, excessive vibrations may cause disconfort to users. The natural frequencies (see table 2) of the structure are evaluated, as well as their vibration modes (see figure 8) in order to study the dynamic characteristics of the structure.

Table 2 - 1º,2º and 3º vibration mode.

Mode	Period (seg)	Frequency (Hz)
1º Mode	0.19	5.34
2º Mode	0.14	6.91
3º Mode	0.09	11.03

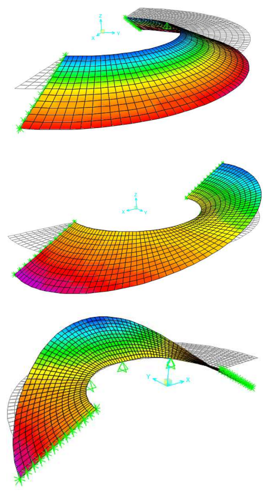

Figure 8 - 1º,2º and 3º vibration mode.

The lowest frequency, called the fundamental frequency (5.34 Hz), refers to the first mode of vibration. Its configuration corresponds to a deformed one without any inflection point. The second mode of vibration has a frequency of 6.91 Hz and the deformed shape has a single inflection point. The third mode of vibration analyzed is associated to a deformed shape with two points of inflection with a corresponding frequency of 11.03 Hz.

The study shows that the structure is not vibration sensitive.

E. ULTIMATE LIMIT STATE

The guarantee of non-collapse of structural elements is a fundamental point in the design of structures.

Through the developed numerical model it was possible to determine the moment distributions in the several directions, as shown in figures 9, 10 and 11.

Regarding the radial moments, it is verified that these have reduced values. The maximum value is 10 kNm/m, and therefore they are not significant for the flexion safety check of the slab. Once the slab isn't symmetrical, torsional moments occur. For radial moments, their value are reduced and so they are not determinant in the flexion safety verification.

Under these conditions we can claim that the circumferential direction is the one that matters the most concerning the ultimate limite state verification.

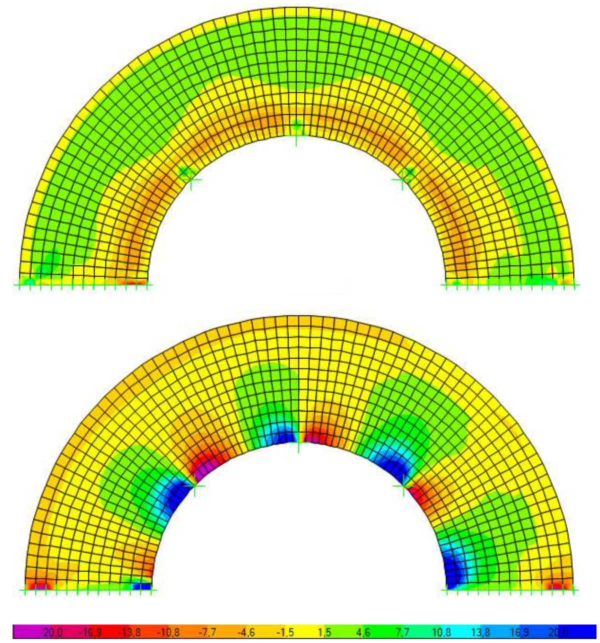

Figure 9 – Radial bending moment, M_{rr}, (up) and torsional, M_{rr}, (down) bending moments distribution [kNm/m].

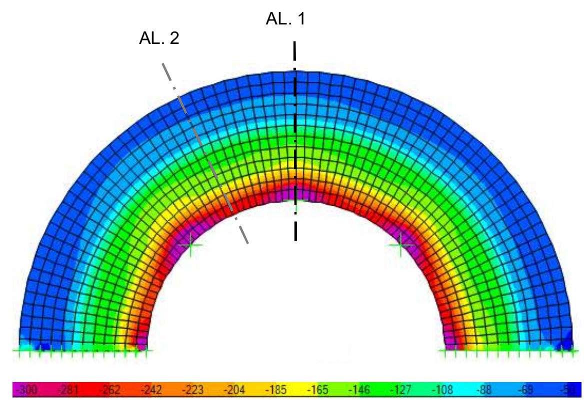

Figure 10 - Circumferencial bending moments distribution [kNm/m].

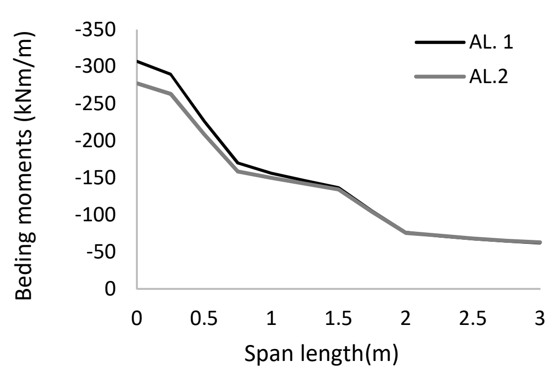

Figure 11 – Circumferencial moments distribution along the alignment 1 and 2.

It is possible to notice that important circunferential negative moments are generated, having a maximum value next to the support and decreasing along the width.This is a major distinction in comparision to a cantilever.

The safety check is made on the table 3.

Table 3 - Ulimate limite state verification.

	Value [KN/m²]	Area of required reinforcement (cm²)	Area of adopted reinforcement (cm²)
$\boldsymbol{M^*_{sd,rr}}$	30	6.24	7.54 **(OK)**
$\boldsymbol{M^*_{sd,\theta\theta}}$	290	20.0	28.25 **(OK)**

Where $\boldsymbol{M^*_{sd,rr}}$ and $\boldsymbol{M^*_{sd,\theta\theta}}$ are given by the following expressions:

$$M^*_{sd,rr} = M_{sd,rr} + |M_{sd,\theta r}| \tag{2.9}$$

$$\boldsymbol{M^*_{sd,\theta\theta}} = M_{sd,rr} + |M_{sd,\theta r}| \tag{2.10}$$

III. PARAMETRIC ANALYSIS

In order to obtain a range of results that allow to understand the field of applicability of this type of solutions a parametric analysis was made.

The parameters evaluated and here analysed are the following:

- Supports located on the outer edge
- Angle between supports
- Average radius of the structure
- Span length

F. SUPPORTS LOCATED ON THE OUTER EDGE

Up to this subchapter, the structural behavior of solutions supported along the inner edge was studied. Consequently the interest in study the opposite situation. Therefore, behavior of outer edge supported structures is summarized here.

The displacements distribution along the radius are compared in figure 12 from the supported edge (inner or outer situation) to the opposite.

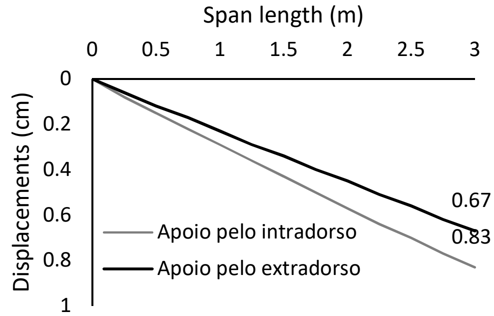

Figure 12 - Displacements for inner (“Apoio pelo intradorso”) & outer (“Apoio pelo extradorso”) suported structures.

The deformed shape does not present differences from the previously case studied. However the graph, shows a decrease of the maximum deflection by 1.6 mm (on the order of 20%) compared to the one obtained for the structure supported along the inner edge. Consequently, it is concluded that supporting the structure on the outer edge corresponds to a more rigid solution.

In the figure 13 the bending moments diagrams (in modulus) along the circumferential direction are presented, of the interior and exteriorly supported structures.

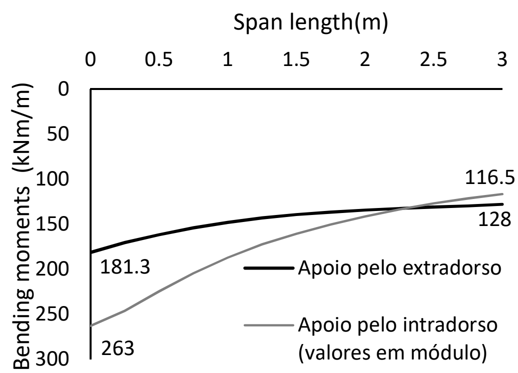

Figure 13 - Circumferencial moments distribution for inner (“Apoio pelo intradorso”) & outer (“Apoio pelo extradorso”) suported structures.

The circumferential bending moments are positive contrary to what has been seen in the previous case. In this way the tractions occur in the lower fibers of the slab, and so placement of the main flexural reinforcement must be done on the opposite face relatively to the previous case.

A reduction by 50% of the circumferential bending moments was verified, which leads naturally to less amounts of reinforcement.

G. VARIATION OF SLAB THE THICKNESS

In order to analyze the influence of this parameter on the slab behavior its height was varied by keeping the other parameters unchanged.

Figure 14 presents the variation of the displacements for different values of the slab thickness.

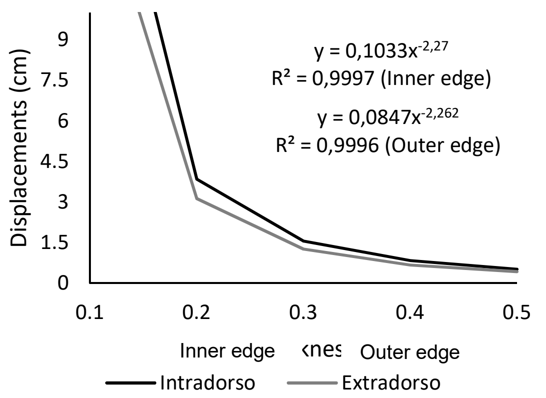

Figure 14 - Variation of the displacement with the increase of the slab thickness.

According to the analysis of the displacement graph, it is verified that these decrease with the increase of the height of the cross section, similar to what happens in the majority of the slabs, although with greater thickness this efficiency decreases.

Figure 15 shows the variation of the reduced circumferential bending moment diagram for an increase in slab thickness.

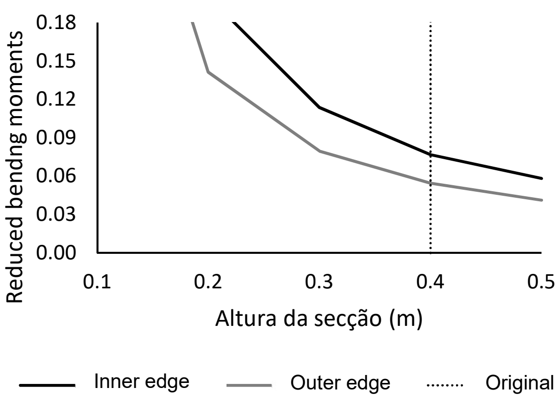

Figure 15 - Variation of the reduced circumferential bending moment diagram for an increase in slab thickness.

Thickening the cross-section leads to a decrease in the reduced bending moments witch can be explain by the increase of the effective depth. This solution is very effective if a redution of total amount of reinforcement is needed.

H. VARIATION OF THE MIDDLE RADIUS

Concerning the variation of the structure radius this parameter was studied in order to understand the consequences on the structure behavior.

In figure 16 the variation of the displacement for different values of radius is presented.

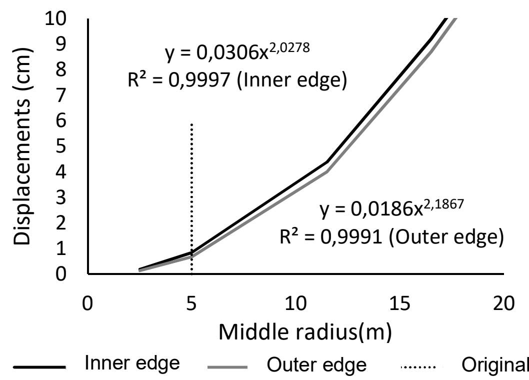

Figure 16 - Variation of the displacement to an increase of the middle radius.

By observing the previous graph it can be seen that the variation of this parameter influences in the same way a structure supported by the inner or outer edges and that the displacements tend to infinity with the radius, which is easily understood since it would correspond to a straight alignment.

Figure 17 shows the variation of the reduced circumferential bending moment diagram for an increase in middle radius.

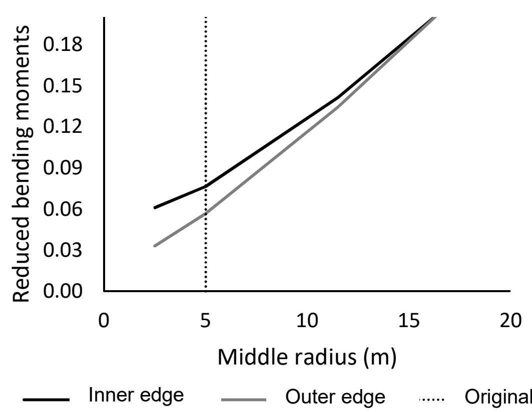

Figure 17 - Variation of the reduced circumferential bending moment diagram for an increase in middle radius.

It is also observed that with respect to the maximum moment there would be no significant limitations until 15 m radius, so also in the analysis of this parameter it is verified the greater importance of the service limit state in the design.

I. VARIATION OF THE SPAN LENGHT

In order to verify serviceability limits on the slab deformations, its span is important in order to enhance the conditions of its use. The results for the maximum displacements obtained with the increase of the span lenght is shown in figure 18.

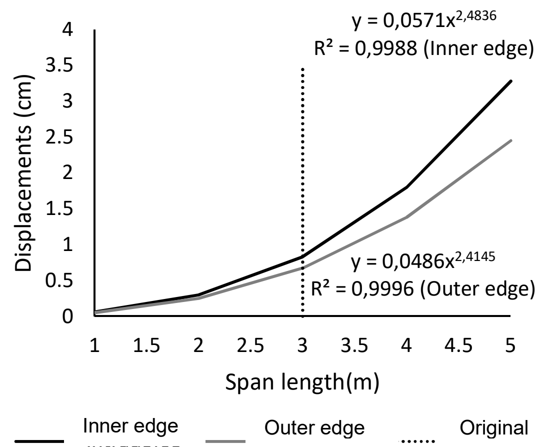

Figure 18 - Variation of the displacements along the radius.

Based on the analysis of the previous graph it can be seen that the displacements increase, more than linearly, with the increase of the span length.

From the graph of the displacement variation, it can be seen that from a platform width of more than 3.5 m, it is clearly advantageous to support the structure by its outer edge.

The reduced circumferential bending moment variation along the radius is showed on figure 19.

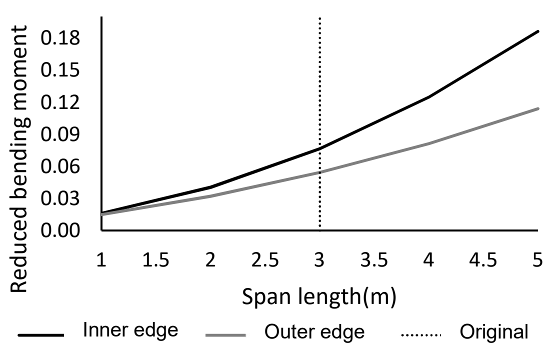

Figure 19 - Variation of the reduced circumferential bending moment diagram along the radius.

Observing the previous graph it's verified that the solutions are both "loose" in terms flexion safety verification, confirming that ultimate limit state does not condition the structural designing.

IV. STUDY CASE 2

Now the results obtained for the case of a structure with "S" geometry, supported along the inner edge, are resumed. For this purpose, a structure with geometric characteristics similar to the base problem is assumed. A middle radius of 5 m, a section with a constant thickness of 0.4 m and a support spacing of 50 ° was defined. The curvature in plan was also varied to quarters of circle, that is, from 90º to 90º. The simple supports placed along the edges block the vertical translations. Slide supports were used at the ends of the slab blocking the circumferential rotation.

A. DISPLACEMENTS ANALYSIS

Thus, in order to study one of the parameters that affect the serviceability behavior of the structures, the elastic displacements obtained for the case in question are showed in Figures 20 and 21.

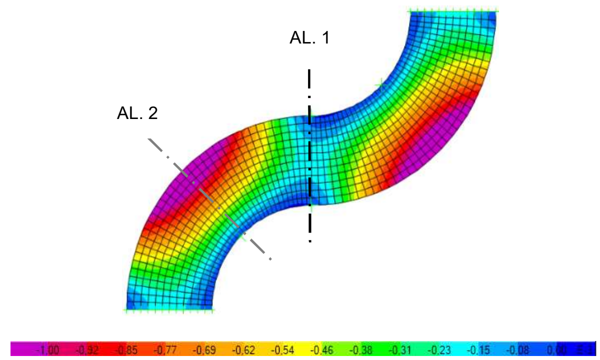

Figure 20 - Displacements along the radius [m].

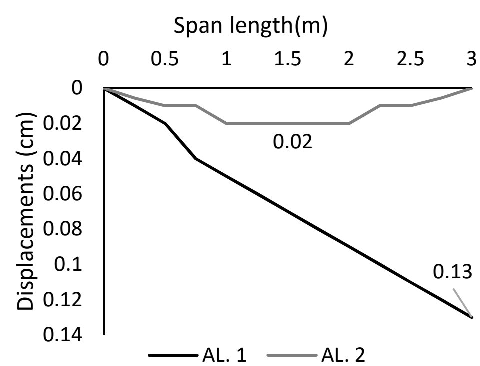

Figure 21 - Displacements along the alignment AL1&2.

By the analysis of the previous graph it can be seen, as would be expected given the arrangement of supports that displacements vary radially and circumferentially, contrary to what happened in the previously studied problems, where this variation only occurred along the radius.

Comparing the displacements along the section AL.1 and AL.2, they show very different variations. It is thus possible to conclude that there are two zones of different behaviors on the slab, the zone between curvature changes (in plan) and the zone of constant curvature.

Another interesting comparison is to note the clear increase in rigidity relative to the previous problem studied.

B. VIBRATION ANALYSIS

The fundamental frequency of the structure is 15.18 Hz which proves the above statement and makes this structural model free of any vibration problem.

C. MOMENTS ANALYSIS

The circumferencial moments distribution in the slab are now analyzed in figures 22 and 23.

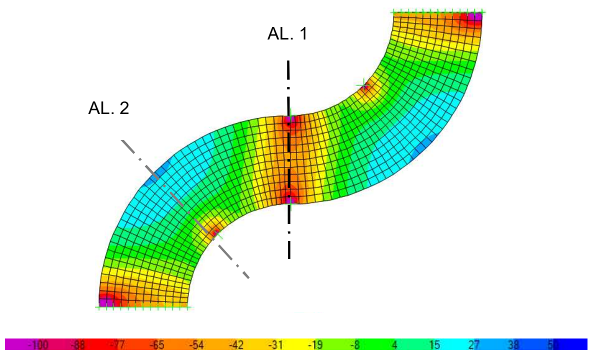

Figure 22 – Circumferencial moments distribution along the radius [kNm/m].

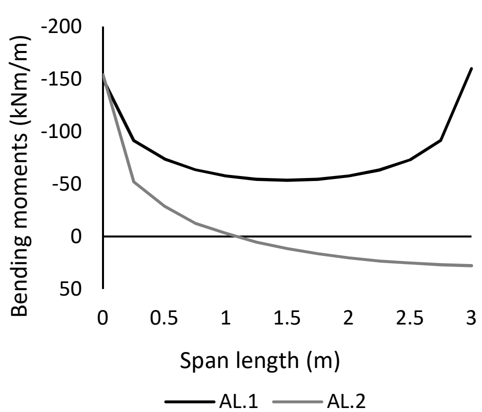

Figure 23 – Circumferencial bending moments distribution along the alignment AL1&2.

It is observed that the positive maximum values occur in the area of the free edge and the maximum negative values occur in the zone of change of curvature.

It is interesting to note that, contrary to what happened to a circular structure supported along the inner edge, the moments along the free edge show opposite sign to what was seen proving that this type of structure has a mixed behavior between continuous band and the previous slab studied.

The maximum values of moments obtained were smaller than the circular slab, which means a consequent gain in reinforcement quantities.

Structures with identical geometry were also studied with the following supporting conditions:

[1] Support on the outer edge;

[2] Continuous support along one edge.

Although the structure was considerably more rigid it was found that for case [1] the behavior was similar to the present case shown.

For the case [2] it was verified that the structural behavior was similar to the practical case studied.

V. CONCLUSIONS

What was seen and confirmed throughout this work was that for curved structures in plan it is possible to establish a structural sound solution either with inner or outer border support. This makes possible designing aesthetically interesting structures without an associated high cost.

Thus, based on a pratical example, supported along the inner edge, the validity of the solution was analyzed, showing that a adequate displacements control is needed to check serviceability requirements.

The analysis made for the structure supported on the outer edge show that this solution led to a relative increase in stiffness leading to lower value of displacements and higher frequencies, with respect to an internally supported solution. As for the bending moments, it was verified that the most important ones are circunferential, but in this case, they are positive and for that reason the bending reinforcement must be placed on the lower face of the slab. There were also gains in reinforcement amounts comparatively to the inner edge supported solution.

As for the study of the variation of the height of the cross section, it was verified that both solutions (inner and outer edge) behave in the same way with the variation of this parameter. There was a noticeable increase in structure stiffness with the increasing of the cross section height, leading to lower displacements. Another variable analyzed was the middle radius of the structure, that showed to have a big influence on the structural behavior.

There is a decrease of the rigidity of the structure with the increase of its radius and a consequent increase of the bending moments.

The variation of the span length was the parameter that had the most influence on the behavior of the structure. It was verified that its rigidity decreases drastically with the increase of this parameter, resulting in a consequent increase of the displacements

Regarding the case study 1 and 2, studied on the last chapter, it was verified that there was a very significant increase of rigidity of the structure when compared with the cases of circular geometry, making possible to design solutions with higher radius and spans.

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