Entanglement-induced deviation from the geodesic motion in quantum gravity:

Gravity-matter entanglement and the weak equivalence principle

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To my father

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Resumo

Estudamos a dedução de uma equação de movimento efetiva para uma partícula pontual, em modelos de gravidade quântica. De forma idêntica à dedução do movimento geodésico de uma partícula clássica que advém do acoplamento, entre a teoria clássica de campo e a relatividade geral, introduzimos uma equação de movimento efetiva, mas a partir de uma descrição abstrata de gravidade quântica. Desta forma, na presença de *entanglement* entre gravidade e matéria, os efeitos quânticos dão origem a modificações da trajetória geodésica, que se devem principalmente à sobreposição não-nula entre vários estados coerentes do sistema gravidade-matéria. Por fim, discutimos o estatuto do princípio de equivalência fraco nas teorias de gravidade quântica e a sua possível violação devido ao movimento não-geodésico.

Palavras-chave: Gravidade quântica, *entanglement* gravidade-matéria, equação da geodésica, princípio de equivalência fraco, relatividade geral, mecânica quântica.

Abstract

We study the derivation of the effective equation of motion for a pointlike particle in the framework of quantum gravity. Just like the geodesic motion of a classical particle is a consequence of classical field theory coupled to general relativity, we introduce the similar notion of an effective equation of motion, but starting from an abstract quantum gravity description. In the presence of entanglement between gravity and matter, quantum effects give rise to modifications of the geodesic trajectory, primarily as a consequence of the nonzero overlap between various coherent states of the gravity-matter system. Finally, we discuss the status of the weak equivalence principle in quantum gravity and its possible violation due to the nongeodesic motion.

Keywords: Quantum gravity, gravity-matter entanglement, geodesic equation, weak equivalence principle, general relativity, quantum mechanics.

Contents

Preface

The research of this thesis lead to the production of one article, which is on the arXiv, and that was submitted for publication:

- F. Pipa, N. Paunković and M. Vojinović. Entanglement-induced deviation from the geodesic motion in quantum gravity. arXiv:1801.03207, 2018.

It also lead to three poster presentations in international conferences:

- F. Pipa, N. Paunković and M. Vojinović, Deviation from the geodesic motion due to gravity-matter entanglement, Emergent Quantum Mechanics, poster presented at the conference at the conference: Towards Ontology of Quantum Mechanics and the Conscious Agent, October 26th-28th 2017, University of London, 2017.
- F. Pipa, N. Paunković, and M. Vojinović, Violation of the weak equivalence principle due to gravity-matter entanglement, poster presented at the conference: Quantum Networks 2017, August 1th-4th 2017, University of Oxford, 2017.
- F. Pipa, N. Paunković and M. Vojinović, The violation of the weak equivalence principle due to gravity-matter entanglement, poster presented at the conference: Gravitational Decoherence seminar, June 26th-28th 2017, PhysikzentrumBadHonnef, Germany, 2017.

Chapter 1

Introduction

1.1 Goals and motivation

The formulation of the theory of quantum gravity (QG) is one of the most fundamental open problems in modern theoretical physics. In models of QG, as in any quantum theory, superpositions of states are allowed. In a tentative "theory of everything", which includes both gravity and matter at a fundamental quantum level, superpositions of product gravity-matter states are particularly interesting. Entangled states are highly nonclassical, and as such are especially relevant because they give rise to a drastically different behavior of matter from what one would expect based on classical intuition, as confirmed by numerous examples from the standard quantum mechanics (QM). Therefore, it is interesting to study such states in the context of a QG coupled to matter, in particular the Schrödinger cat-like states, i.e., the superpositions of macroscopic states. Moreover, a recent study [30] suggests that physically allowed states of a gravity-matter system are generically entangled due to gauge invariance, providing additional motivation for our study.

In standard QM, entanglement is generically a consequence of the interaction. Nevertheless, there exist situations which give rise to entanglement even without interaction. For example, the Pauli exclusion principle in the case of identical particles generates entanglement without an interaction, giving rise to an effective force (also called the "exchange interaction"). We investigate in detail whether an entanglement between gravity and matter could also be described as a certain type of an effective interaction, and if so, what are its aspects and details. In order to study this problem, we analyze the motion of a free test particle in a gravitational field. In general relativity (GR), this motion is described by a geodesic trajectory. However, we show that in the presence of the gravity-matter entanglement, the resulting effective interaction causes a deviation from a classical geodesic trajectory. In particular, we generalize the standard derivation of a geodesic equation from the case of classical gravity to the case of a full QG model, and derive the equation of motion for a particle which contains a non-geodesic term, reflecting the presence of the entanglement-induced effective interaction. The effects we discuss are purely quantum with respect to both gravity and matter, unlike previous studies of quantum matter in classical curved spacetime [11, 22, 31, 44].

As a consequence of the modified equation of motion for a particle, we also discuss the status of the equivalence principle in the context of QG, and a possible violation of its weak flavor.

1.2 Thesis overview

The thesis is organized as follows. Chapter 2 is devoted to a review of the main ideas needed to understand our results. We give a brief review of classical field theory in section 2.1, spacetime symmetries and the

Noether's theorem in section 2.2, the multipole formalism in section 2.3, and in section 2.4 the derivation of geodesic equation in classical gravity, particularly in GR, where the multipole formalism is employed and the geodesic equation for a particle is derived from the covariant conservation of the stress-energy tensor. In chapter 3, after a brief review of the canonical quantization in section 3.1, we generalize this procedure and derive our main results. Section 3.2 contains the general setup, the abstract quantum gravity framework that will be used, and the main assumptions. In section 3.3 we discuss the effective covariant conservation equation, which receives a correction to the classical one, due to the quantum gravity effects. In section 3.4 we put everything together and derive our main result — the effective equation of motion for a point particle, with the leading quantum correction. In section 3.5 we discuss the consistency of the assumptions that enter the approximation scheme used to derive the effective equation of motion. Chapter 4 is devoted to the discussion of the consequences of our results in the context of the weak equivalence principle. For the purpose of clarity, in section 4.1 we first provide the definitions of various flavors of the equivalence principle in the classical scenario. Then, in section 4.2 we discuss the status of the equivalence principle in the context of quantum gravity and the results obtained in chapter 3. Section 4.3 provides further analysis of universality and equality between inertial and gravitational masses, in the context of the Newtonian approximation. Finally, chapter 5 contains our conclusions, discussion of the results and possible lines of further research.

1.3 Notation and conventions

Our notation and conventions are as follows. We will work in the natural system of units in which $c = \hbar = 1$ and $G = l_p^2$, where l_p is the Planck length and G is the Newton's gravitational constant. By convention, the metric of spacetime will have the spacelike Lorentz signature $(-,+,+,+)$. The spacetime indices are denoted with lowercase Greek letters $\mu, \nu, \dots$ and take the values $0, 1, 2, 3$. These can be split into the timelike index 0 and the spacelike indices denoted with lowercase Latin letters $i, j, k, \dots$ which take the values $1, 2, 3$. The Lorentz-invariant metric tensor is denoted as $\eta_{\mu\nu}$. Quantum operators always carry a hat, $\hat{\phi}(x)$, $\hat{g}(x)$, etc. The parentheses around indices indicate symmetrization with respect to those indices, while brackets indicate antisymmetrization:

$$A_{(\mu\nu)} \equiv \frac{1}{2} \left(A_{\mu\nu} + A_{\nu\mu} \right) \,, \qquad A_{[\mu\nu]} \equiv \frac{1}{2} \left(A_{\mu\nu} - A_{\nu\mu} \right) \,.$$

Finally, we will systematically denote the values of functions with parentheses, $f(x)$, while functionals will be denoted with brackets, $F[\phi]$.

Chapter 2

The geodesic equation in General Relativity

In this chapter, we will start by giving an introduction to classical field theory, which we will need to understand GR as a classical field theory. Then, we will also give a brief introduction to spacetime symmetries and the Noether's theorem, since we will need it to understand local Poincaré invariance and the conservation of stress-energy tensor, which is one of our assumptions in this thesis. Next, we will give an introduction to the multipole formalism, which we will need to derive the geodesic equation. Finally, in the last section, we will derive the geodesic equation in general relativity.

2.1 Classical field theory

In this section, we describe briefly the Lagrangian formalism behind classical field theory. In mechanics, the Lagrangian is a function of coordinates and velocities, and is given by the difference between the kinetic and potential energies of the particle,

$$L(q,\dot{q}) = T - V. \tag{2.1}$$

The action functional is given by:

$$S = \int_{t_0}^{t_1} dt L(q,\dot{q}). \tag{2.2}$$

Finding the extremum of the action functional (i.e., the so-called principle of *least action*), allows one to arrive to the equation of Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \tag{2.3}$$

where the index i denotes the degrees of freedom of a system for the general d-dimensional case. Given a Lagrangian, the above equation becomes the equation of motion. Another equivalent formal procedure to derive the equations of motion is the Hamiltonian formalism. Assuming $\frac{\partial^2 L}{\partial \dot{q}} \neq 0$, we define the conjugate momentum by

$$p \equiv \frac{\partial L}{\partial \dot{q}}, \tag{2.4}$$

and the Hamiltonian H by

$$H(q,p) \equiv p\dot{q}(q,p) - L(q,\dot{q}(q,p)). \tag{2.5}$$

Here $H(q,p)$ corresponds to the total energy of the system. Then, for the d-dimensional case, we have the Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}\,. \tag{2.6}$$

The physics of elementary particles and gravitation is described by the Lagrangian field theory. We pass from discrete number of degrees of freedom, labeled by i, to continuous number of degrees of freedom, labeled by the coordinate(s) x, such that the quantity $q_i(t)$ is replaced by the field $\phi(x,t)$, both parametrized by the time t. Then, we have the following substitutions:

$$\begin{aligned} q(t) &\longrightarrow \phi(x,t), \\ \dot{q}(t) &\longrightarrow \partial_\mu \phi(x,t), \\ L(q,\dot{q}) &\longrightarrow \mathcal{L}(\phi, \partial_\mu\phi), \\ \int dt &\longrightarrow \int d^4x. \end{aligned} \tag{2.7}$$

We want to make the formalism of field theory relativistic. For this purpose, coordinates are combined into four-vectors $x^\mu = (t, x)$ and we require that the action is Lorentz invariant (i.e., invariant under Lorentz transformations, see the next section for the definition of Lorentz transformation), expressing it in the following form

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi), \tag{2.8}$$

where the Lagrangian density $\mathcal{L}$ is also Lorentz invariant. The equations of motion are given as the Euler–Lagrange equations of the variational problem $\delta_\phi S = 0$ for the action functional S. We can then deduce the Euler-Lagrange equations for a classical field theory:

$$\begin{aligned} 0 = \delta_\phi S &= \int d^4x \Big\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\Big\} \\ &= \int d^4x \Big\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\delta\phi + \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big)\Big\}. \end{aligned} \tag{2.9}$$

By the Stokes theorem the third term can be turned into a surface integral over the boundary of the spacetime region of integration. Since the initial and final configurations are assumed to be given, considering deformations that vanish over the spatial boundary and assuming that the fields vanish fast enough at infinity, this term is zero. Then, since the integral must vanish for arbitrary $\delta\phi$, we arrive to the Euler-Lagrange equation of motion for a field:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = 0. \tag{2.10}$$

Continuing the analogy with classical mechanics, we define the conjugate momentum

$$\pi \equiv \frac{\partial \mathcal{L}(\phi, \partial_\mu\phi)}{\partial\dot{\phi}} \tag{2.11}$$

Then, we define the Hamiltonian density as

$$\mathcal{H}(\phi, \pi) = \pi\dot{\phi} - \mathcal{L}. \tag{2.12}$$

2.2 Spacetime symmetries and the Noether's theorem

Since it is important for the core of our argument, in this section, we will explain the most important spacetime symmetries and how they connect to conserved quantities via the Noether's theorem. We will start by explaining the main spacetime symmetries, i.e., the Lorentz and the Poincaré symmetries. First, we note that we can have two types of symmetries: global and local. Global symmetries deal with transformations which are the same at all points of spacetime, while local symmetries deal with transformations which can be different at different points of spacetime. We will focus on global spacetime symmetries, readers who want a rigorous introduction to global and local symmetries, can see for example [7].

The observer in spacetime uses some reference frame, normally associated with an imagined extension of a physical object, and equipped with coordinates x^μ that serve to identify physical events. The connection between space and time coordinates of two inertial frames, moving with respect to one other with constant velocity, is given by the Lorentz transformations, which express a connection between two inertial reference frames S and S'. These transformations include spatial rotations and boosts to another reference frame with constant velocity, and they are described by the linear transformation

$$x^{\mu'} = \Lambda^{\mu'}_{\ \nu} x^\nu, \tag{2.13}$$

where $\Lambda^{\mu'}_{\ \nu}$ is called a Lorentz tensor, leaving the length of the four-vectors x^μ invariant, i.e., $s^2 \equiv x^\nu x_\nu = x^{\mu'} x_{\mu'} = \eta_{\mu'\rho'} x^{\mu'} x^{\rho'} = \boldsymbol{x}^2 - t^2$, with $\eta_{\nu\sigma} = \eta_{\mu'\rho'} \Lambda^{\mu'}_{\ \nu} \Lambda^{\rho'}_{\ \sigma}$. Lorentz transformations are defined by six parameters, three rotations, and three boosts. The Lorentz group may be described as the orthogonal group $O(1,3)$, a Lie group that is constituted by the set of transformations that leave the orthogonal form s^2 invariant on $\mathbb{R}^4$. The fact that the Lorentz group leaves the norm s^2 of a vector invariant is not enough since for physical reasons we need the interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.14}$$

to be invariant. This ensures that the speed of light is the same in every inertial reference frame, and it allows us to add constant translations to the Lorentz transformation. Then, we need to include also spatial and time translations to the Lorentz transformations. The resultant set of transformations is known as Poincaré transformations,

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu, \tag{2.15}$$

and it can be shown that it forms a 10-parameter (six boosts and rotations plus four translations) Lie group.

The Minkowski space that we will denote by $\boldsymbol{M_4}$ is a four-dimensional space possessing the metric $\eta_{\mu\nu}$, and used to describe all physical phenomena except gravitation. In $\boldsymbol{M_4}$ there exists a preferred set of reference frames called inertial reference frames. An inertial observer can always choose coordinates, called global inertial coordinates, where we have that the infinitesimal interval between points $P(x)$ and $Q(x+dx)$ is given by (2.14). Coordinate transformations which do not change the form of the metric define the isometry group of a given space. From what we have seen, the isometry group of $\boldsymbol{M_4}$ is the group of global Poincaré transformations. Thus, when the gravitational field is absent, the underlying symmetry of fundamental interactions is given by the Poincaré group. For a general spacetime where gravity is present, such as the one described by GR, this is no longer the case. GR is instead invariant under local Poincaré transformations (see [7] for more details).

The Noether's theorem shows that the invariance of the action under a continuous symmetry transformation implies the existence of conserved quantities. Global symmetries are typically described by finite-dimensional Lie groups like the ones seen above and give rise to conserved currents and conserved charges (first Noether's theorem). Local symmetries are typically described by infinite-dimensional Lie groups and give rise to more complicated statements, such as covariantly conserved currents (second Noether's theorem). Theories for which have local symmetries are called gauge theories. Here we will review the first Noether's theorem, serving as a basis to understand by analogy the second Noether's theorem which the full rigorous treatment is technically too complex and out of the scope of this thesis. Readers can find a full treatment of this theorem in [7].

The continuous transformation on the fields ϕ and the Lagrangian density in an infinitesimal form can be written as

$$\begin{aligned} \phi(x) \to \phi'(x) &= \phi(x) + \alpha \Delta\phi(x), \\ \mathcal{L}(x) &\to \mathcal{L}(x) + \alpha \Delta\mathcal{L}(x), \end{aligned} \tag{2.16}$$

where α is an infinitesimal parameter and $\Delta\phi$ and $\Delta\mathcal{L}$ represents some deformation of the field configuration and the Lagrangian density, respectively. If the action is invariant under this transformation, we call it a symmetry since it will leave the equations of motion invariant. We can allow also the action to change by a surface term, since as we have seen in section 2.1, the surface term didn't affect the derivation of the Euler-Lagrange equations of motion. Then, we have that the Lagrangian density should be invariant up to a four-divergence:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x), \tag{2.17}$$

for some $\mathcal{J}^\mu$. Comparing $\Delta\mathcal{L}$ to the variation of the fields, we get

$$\begin{aligned} \alpha\Delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi}(\alpha\Delta\phi) + \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Big)\partial_\mu(\alpha\Delta\phi) \\ &= \alpha\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\Big) + \alpha\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\Big]\Delta\phi \end{aligned} \tag{2.18}$$

By equation (2.10), the third term vanishes. Setting the remaining term equal to $\alpha\partial_\mu\mathcal{J}^\mu(x)$, we get the following expressions for the current and its conservation:

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu, \qquad \partial_\mu j^\mu(x) = 0. \tag{2.19}$$

Note that if the symmetry involves more than one field, the first term in the equation for $j^\mu(x)$ should be substituted by a sum of analogous terms for each field. This expression states that for each continuous global symmetry of $\mathcal{L}$, we have a conservation law, i.e., the conservation of the current, and it is called the first Noether's theorem. Moreover, this conservation law can be expressed by stating that the charge is constant in time. By defining charge as

$$Q(t) \equiv \int_{all\ space} j^0(x,t)d^3x, \tag{2.20}$$

we can see that is constant by writing the time derivative of the charge as an integral and adding on the right-hand side a spatial integral over a total three divergence, which vanishes due to boundary conditions,

$$\frac{d}{dt}Q(t) = \int \partial_0 j^0(\boldsymbol{x},t)d^3x = \int [\partial_0 j^0(\boldsymbol{x},t) + \partial_i j^i(\boldsymbol{x},t)]d^3x = 0. \tag{2.21}$$

Thus, the charge is conserved:

$$\frac{d}{dt}Q(t) = 0. \tag{2.22}$$

For the purpose of this thesis, it is important to note that the Noether's theorem can be applied to spacetime transformations, such as translations in spacetime. We can make an infinitesimal translation, which corresponds to the field configuration:

$$x^\mu \to x^\mu - a^\mu \implies \phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x). \tag{2.23}$$

Since the Lagrangian density is also a scalar, it transforms in the same way,

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}). \tag{2.24}$$

Then, invoking the Noether's theorem, we get four currents,

$$T^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\delta^\mu_\nu, \tag{2.25}$$

which represent the stress-energy tensor. Moreover, the theorem implies the conservation of the stress-energy tensor,

$$\partial_\mu T^{\mu\nu} = 0, \tag{2.26}$$

and the existence of four conserved quantities:

$$E = \int T^{00} d^3x, \qquad P^i = \int T^{0i} d^3x. \tag{2.27}$$

Thus, the Noether's theorem allows us to arrive to the conservation of stress-energy tensor, which leads us to calculate the total energy of the field configuration E, and the total momentum of the field configuration P^i.

Let us just note that, in a qualitatively analogous way to the first Noether's theorem, the second Noether's theorem establishes the correspondence between the local translation symmetry and the covariant conservation of the stress-energy tensor:

$$\nabla_\nu T^{\mu\nu} = 0. \tag{2.28}$$

2.3 Multipole formalism

In the context of the classical theories of gravity, like GR, the question of deriving the geodesic equation for a particle has initially been studied by Einstein, Infeld and Hoffmann [17], Mathisson [24], Lubánski [23], Fock [18], and others. Slightly later, the question was revisited in the seminal paper by Papapetrou [29], with generalizations followed by a number of authors [12–16, 26, 27, 32, 33, 40, 42, 43], developing the so-called *multipole formalism*. Recently, the multipole formalism has been reformulated in a manifestly covariant language and extended from pointlike objects to strings, membranes and further to p-branes, with general equations of motion studied in Riemann and Riemann-Cartan spaces [2, 35–39]. Today, the multipole formalism and the resulting classes of effective equations of motion have found applications in a wide range of topics, from string theory [41] to cosmology [34] to blackbrane dynamics [4–6] to elasticity and the studies of the shape of red blood cells in biological systems [3].

In this section we give a short review of the multipole formalism, providing some basic motivation for its introduction and a few elementary properties. A more rigorous treatment and more details can be found in [36].

The multipole formalism revolves around the idea of expanding a function into a series of derivatives of the Dirac δ function, or δ series for short. Perhaps the easiest way to understand the δ series is to introduce it as a Fourier transform of a power series. For example, given a real-valued function $f(x)$, one can write it as a Fourier transform of $\tilde{f}(k)$ as

$$f(x) = \int_{\mathbb{R}} dk\, \tilde{f}(k)\, e^{ikx}\,. \tag{2.29}$$

In principle, we can expand $\tilde{f}(k)$ into power series as

$$\tilde{f}(k) = \sum_{n=0}^{\infty} c_n k^n\,, \tag{2.30}$$

where c_n are some coefficients, substitute the expansion back into (2.29), and integrate term by term. Using the identity

$$k^n e^{ikx} = (-i)^n \frac{\partial^n}{\partial x^n} e^{ikx} \tag{2.31}$$

and the integral representation of the Dirac δ function

$$\delta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{ikx}\,, \tag{2.32}$$

we obtain

$$f(x) = \sum_{n=0}^{\infty} c_n \int_{\mathbb{R}} dk\, k^n\, e^{ikx} = \sum_{n=0}^{\infty} (-i)^n c_n \frac{d^n}{dx^n}\int_{\mathbb{R}} dk\, e^{ikx} = \sum_{n=0}^{\infty} 2\pi(-i)^n c_n \frac{d^n}{dx^n}\delta(x) \equiv \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n}\delta(x)\,. \tag{2.33}$$

In the last step, we have merely renamed the coefficients in the expansion.

The above example is the most elementary construction of the δ series, providing some intuition. It is straightforward to see that one can generalize the procedure to perform the expansion around an arbitrary point z instead of zero, such that

$$f(x) = \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n}\delta(x-z)\,. \tag{2.34}$$

The coefficients b_n can be evaluated using the inverse formula,

$$b_n = \frac{(-1)^n}{n!}\int_{\mathbb{R}} dx\, (x-z)^n f(x)\,, \tag{2.35}$$

and are usually called *n-th order moments* of the function $f(x)$. From (2.35) one sees that the δ series is well defined for every function $f(x)$, which falls off to zero faster than any power of x at both infinities.

Let us study an instructive example. Let the function $f(x)$ be an ordinary Gaussian, peaked around the point x_0,

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-(x-x_0)^2}\,. \tag{2.36}$$

One can evaluate the coefficients in the corresponding δ series using (2.35) to obtain:

$$b_n = \begin{cases} \displaystyle\sum_{k=0}^{n/2} \frac{(z-x_0)^{n-2k}}{4^k\, k!\,(n-2k)!} & \text{for even } n\,, \\[2ex] \displaystyle -\sum_{k=0}^{(n-1)/2} \frac{(z-x_0)^{n-2k}}{4^k\, k!\,(n-2k)!} & \text{for odd } n\,. \end{cases} \tag{2.37}$$

It is important to note the following property — if the expansion point z does not coincide with the peak of the Gaussian, x_0, the magnitude of the coefficients b_n in general grows with n. For example, if

$z - x_0 = 2$, we have

$$f(x) = \delta(x-z) - 2\frac{d}{dx}\delta(x-z) + \frac{9}{4}\frac{d^2}{dx^2}\delta(x-z) - \frac{11}{6}\frac{d^3}{dx^3}\delta(x-z) + \frac{115}{96}\frac{d^4}{dx^4}\delta(x-z) + \dots \tag{2.38}$$

However, if the expansion point coincides with the peak, $z - x_0 = 0$, the magnitude of the coefficients falls off as n grows:

$$f(x) = \delta(x-z) + \frac{1}{4}\frac{d^2}{dx^2}\delta(x-z) + \frac{1}{32}\frac{d^4}{dx^4}\delta(x-z) + \frac{1}{384}\frac{d^6}{dx^6}\delta(x-z) + \dots \tag{2.39}$$

From this simple example one can infer an important property of δ series — the coefficients b_n decrease as n grows, if the expansion point is near the peak of the function $f(x)$. Turning the argument around, if we require that the coefficients decrease with n,

$$|b_n| > |b_{n+1}|\,, \qquad \forall n \in \mathbb{N}_0\,, \tag{2.40}$$

this places *a restriction on the possible values of the expansion point* z. This is the crucial property of the δ series, and is being used to define the "position of the particle" which corresponds to a distribution of matter fields described by a localized function $f(x)$.

Also, assuming that the expansion point z has been chosen to be near the peak of the function, the decreasing nature of the coefficients b_n allows one to approximate the function $f(x)$ by a truncated series. This formalizes the intuitive idea that if one looks at some localized distribution of matter fields from "far away", it will look roughly as a point particle. The truncation point then quantifies the amount of "internal structure" that is known about $f(x)$. One can therefore study the function $f(x)$ at various approximation levels: the *single pole* approximation,

$$f(x) \sim b_0\delta(x-z)\,, \tag{2.41}$$

the *pole-dipole* approximation,

$$f(x) \sim b_0\delta(x-z) + b_1\frac{d}{dx}\delta(x-z)\,, \tag{2.42}$$

the *pole-dipole-quadrupole* approximation,

$$f(x) \sim b_0\delta(x-z) + b_1\frac{d}{dx}\delta(x-z) + b_2\frac{d^2}{dx^2}\delta(x-z)\,, \tag{2.43}$$

and so on.

It is completely straightforward to generalize the δ series to three (or more) dimensions, with the δ series of a function $f(\vec{x})$ around the point $\vec{z}$ defined as

$$f(\vec{x}) = \sum_{n=0}^{\infty} b_n^{i_1\dots i_n}\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_n}}\delta^{(3)}(\vec{x}-\vec{z})\,. \tag{2.44}$$

Here the indices $i_1, \dots i_n$ take values 1, 2 and 3, and the inverse formula for the coefficients is

$$b_n^{i_1\dots i_n} = \frac{(-1)^n}{n!}\int_{\mathbb{R}^3} d^3x\,(x^{i_1}-z^{i_1})\dots(x^{i_n}-z^{i_n})f(\vec{x})\,. \tag{2.45}$$

For example, in electrostatics, one can expand the charge density $\rho(\vec{x})$ localized around the point $\vec{z}=0$ as

$$\rho(\vec{x}) = b_0\delta^{(3)}(\vec{x}) + b_1^i\frac{\partial}{\partial x^i}\delta^{(3)}(\vec{x}) + \dots \tag{2.46}$$

According to (2.45), the coefficients are

$$b_0 = \int_{\mathbb{R}^3} d^3x\, \rho(\vec{x}) \equiv Q\,, \qquad \vec{b}_1 = -\int_{\mathbb{R}^3} d^3x\, \vec{x}\rho(\vec{x}) \equiv -\vec{p}\,, \tag{2.47}$$

where we recognize the total charge Q and the electrostatic dipole moment $\vec{p}$ of the source. Thus we have

$$\rho(\vec{x}) = Q\delta^{(3)}(\vec{x}) - \vec{p}\cdot\nabla\delta^{(3)}(\vec{x}) + \dots \tag{2.48}$$

Substituting the δ series expansion of $\rho(\vec{x})$ into the formula for the electrostatic potential,

$$\varphi(\vec{r}) = \int_{\mathbb{R}^3} d^3x\, \frac{\rho(\vec{x})}{|\vec{r}-\vec{x}|}\,, \tag{2.49}$$

and evaluating the integral, one obtains the familiar expression for the multipole expansion in electrostatics [21]:

$$\varphi(\vec{r}) = \frac{Q}{|\vec{r}|} + \frac{\vec{p}\cdot\vec{r}}{|\vec{r}|^3} + \dots \tag{2.50}$$

This example also illustrates what type of approximation is achieved with the truncation of the δ series.

Next we generalize to time-dependent functions. If the function $f(\vec{x})$ evolves in time, while remaining localized in space, one can expand it into δ series by choosing the most convenient reference point $z(t)$ at each moment of time,

$$f(\vec{x},t) = \sum_{n=0}^{\infty} b^{i_1\dots i_n}(t)\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_n}}\delta^{(3)}(\vec{x}-\vec{z}(t))\,, \tag{2.51}$$

where $t\in\mathbb{R}$ is a time variable, and the coefficients b are now time-dependent. Then one can introduce the proper time τ, and use the identity

$$\int_{\mathbb{R}} d\tau\, \delta(t-\tau) = 1 \tag{2.52}$$

to rewrite (2.51) in a 4-dimensional manifestly Lorentz-invariant form

$$f(x) = \int_{\mathbb{R}} d\tau \sum_{n=0}^{\infty} b^{\mu_1\dots\mu_n}(\tau)\partial_{\mu_1}\dots\partial_{\mu_n}\delta^{(4)}(x-z(\tau))\,, \tag{2.53}$$

where we have relabeled $(\vec{x},t)\equiv x$, introduced $z^0(\tau)=\tau$, used shorthand notation $\partial_\mu \equiv \partial/\partial x^\mu$, and defined $b^0 = b^{00} = b^{000} = \dots = 0$, since the time derivatives do not actually appear in (2.51). The introduction of these auxiliary timelike components of the b-coefficients, demanded by Lorentz invariance, gives rise to an additional gauge symmetry of the expansion coefficients, since only the "spatial" components carry nontrivial information about the function $f(x)$. This additional gauge symmetry is called *extra symmetry 1*, and is studied in detail in [36].

Finally, one can make one more generalization, and introduce the notion of a δ series around a p-brane, a $(p+1)$-dimensional submanifold living in a D-dimensional spacetime manifold. Namely, we have seen that one can expand a function into a δ series around a point and around a one-dimensional line (equations (2.44) and (2.53), respectively). Generalizing in that direction, one can introduce the world-trajectory of a p-dimensional object through D-dimensional spacetime $\mathcal{M}$, with parametric equations $x^\mu = z^\mu(\xi^a)$ describing the trajectory as a $(p+1)$-dimensional submanifold $\Sigma\subset\mathcal{M}$. Here $\mu\in\{0,\dots,D-1\}$ and $a\in\{0,\dots p\}$, where x^μ are coordinates on $\mathcal{M}$ while ξ^a are intrinsic coordinates on Σ. Then, given a function $f(x)$ whose support is localized near the submanifold Σ, one can write its δ series expansion around Σ in a fully diffeomorphism- and reparametrization-invariant way as:

$$f(x) = \int_\Sigma d^{p+1}\xi\sqrt{-\gamma}\sum_{n=0}^{\infty}\nabla_{\mu_1}\dots\nabla_{\mu_n}\left[B^{\mu_1\dots\mu_n}(\xi)\frac{\delta^{(D)}(x-z(\xi))}{\sqrt{-g}}\right]. \tag{2.54}$$

Here γ is the determinant of the induced metric $\gamma_{ab} = g_{\mu\nu}u^\mu_a u^\nu_b$ on Σ, where $g_{\mu\nu}$ is the metric on $\mathcal{M}$ and $u^\mu_a \equiv \partial z^\mu/\partial\xi^a$ are the tangent vectors of Σ. Note that, in order to ensure the correct tensorial behavior, the B-coefficients have been moved inside the action of the covariant derivatives. Namely, despite the fact that the covariant derivatives act with respect to x and B's do not depend on x, covariant derivatives still act nontrivially on B's with the connection terms. For similar reasons, the term $\sqrt{-g}$ has been introduced to combine with the δ function into a quantity which transforms as a scalar under diffeomorphisms. Its introduction amounts merely to a suitable redefinition of B's and does not modify the δ series in any nontrivial way.

The fully general δ series (2.54) has been studied in detail in [36]. For the purpose of the discussion given in the main text of this thesis, we are interested in the case of a particle, i.e., a $(p = 0)$-brane, moving along a 1-dimensional timelike curve $\mathcal{C}$ which is a submanifold of the $(D = 4)$-dimensional spacetime $\mathcal{M}$. In this case, there is only one intrinsic coordinate on $\mathcal{C}$, denoted $\xi^0 \equiv \tau$, only one tangent vector

$$u^\mu_0 \equiv \frac{\partial z^\mu(\xi)}{\partial\xi^0} = \frac{dz^\mu(\tau)}{d\tau} = u^\mu, \tag{2.55}$$

while the induced metric tensor is a 1×1 matrix $\gamma_{00} = g_{\mu\nu}u^\mu_0 u^\nu_0$. The parametrization of the curve $\mathcal{C}$ with the coordinate τ can be chosen to fix the reparametrization gauge symmetry via the gauge-fixing condition $\gamma_{00} = -1$, which is actually the natural normalization of the tangent vector, $g_{\mu\nu}u^\mu u^\nu = -1$. Finally, one can then apply the δ series expansion (2.54) to the stress-energy tensor $T^{\mu\nu}(x)$ of the matter fields as

$$T^{\mu\nu}(x) = \int_\mathcal{C} d\tau\sum_{n=0}^{\infty}\nabla_{\rho_1}\dots\nabla_{\rho_n}\left[B^{\mu\nu\rho_1\dots\rho_n}(\tau)\frac{\delta^{(D)}(x-z(\tau))}{\sqrt{-g}}\right]. \tag{2.56}$$

Note that the coefficients B now carry two additional indices inherited from the stress-energy tensor. In the single pole approximation, one drops all terms in the sum except the $n = 0$ term, truncating the series to the form

$$T^{\mu\nu}(x) = \int_\mathcal{C} d\tau\, B^{\mu\nu}(\tau)\frac{\delta^{(D)}(x-z(\tau))}{\sqrt{-g}}. \tag{2.57}$$

2.4 Geodesic equation in general relativity

In this section we will demonstrate the application of the multipole formalism in its crudest *single pole* approximation, and employ it to derive the geodesic equation of motion for a point particle in classical Riemannian spacetime. The results presented in this section are well known in the literature, and illustrate the derivation procedure of the geodesic motion for a point particle. After reviewing the standard results in this section, in chapter 3 the same procedure will be utilized to study the quantum gravity case.

The derivation procedure is based on two main assumptions. The first assumption is that the matter fields have internal dynamics such that they form particle-like kink solutions which are stable (i.e., non-decaying) across the spacetime regions under consideration. If that is the case, one can employ the multipole formalism and expand the stress-energy tensor into a series of derivatives of the Dirac δ function as (see section 2.3 for details):

$$T^{\mu\nu}(x) = \int_\mathcal{C} d\tau\left[B^{\mu\nu}(\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}} + \nabla_\rho\left(B^{\mu\nu\rho}(\tau)\frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}}\right) + \dots\right]. \tag{2.58}$$

Here we assume that the stress-energy tensor of matter fields has nonzero value only near some timelike curve $\mathcal{C}$ represented by parametric equations $x^\mu = z^\mu(\tau)$, where τ is a parameter counting the points along the curve $\mathcal{C}$. In that case, the B-coefficients in the δ series will be smaller and smaller with each new term in the series. We introduce a series of smallness scales for the coefficients,

$$B^{\mu\nu} \sim \mathcal{O}_0 \,, \qquad B^{\mu\nu\rho} \sim \mathcal{O}_1 \,, \qquad B^{\mu\nu\rho\sigma} \sim \mathcal{O}_2 \,, \qquad \dots \tag{2.59}$$

such that one can consider the multipole scales to behave as

$$\mathcal{O}_0 \gg \mathcal{O}_1 \gg \mathcal{O}_2 \gg \dots \tag{2.60}$$

Next we choose to work in the so-called *single pole* approximation, in which all quantities of order $\mathcal{O}_1$ and higher can be neglected. It is also assumed that the typical radius of curvature of spacetime near the curve $\mathcal{C}$ will be large enough not to interfere in the internal dynamics of the matter fields along $\mathcal{C}$ and break the kink configuration apart. Physically speaking, the sequence of inequalities (2.60) states that one can systematically approximate the full solution of the matter field equations of motion by neglecting various degrees of freedom which describe the "size" and "shape" of the kink compared to its orbital motion (i.e., motion along the curve $\mathcal{C}$). Given this setup, in the single pole approximation the matter fields are in a configuration that looks like a point particle traveling along a worldline curve $\mathcal{C}$, and terms of order $\mathcal{O}_1$ and higher can be dropped from the stress-energy tensor, giving:

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \, B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \,. \tag{2.61}$$

The second assumption is the validity of the local Poincaré invariance for the matter field equations. Namely, the classical action which describes the gravity-matter system can be generally written as

$$S[g,\phi] = S_G[g] + S_M[g,\phi] \,, \tag{2.62}$$

where g and ϕ denote gravitational and matter degrees of freedom, respectively, and it is generally considered to feature local Poincaré invariance. Our assumption is that the matter action S_M and the gravitational action S_G are invariant even taken separately. If this is the case, the Noether theorem gives us the covariant conservation of the stress-energy tensor of matter fields,

$$\nabla_\nu T^{\mu\nu} = 0 \,. \tag{2.63}$$

Taken together, assumptions (2.61) and (2.63) are sufficient to establish two results:

(a) that the parametric functions $z(\tau)$ of the curve $\mathcal{C}$ satisfy the geodesic equation,

$$\frac{d^2 z^\lambda(\tau)}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dz^\mu(\tau)}{d\tau} \frac{dz^\nu(\tau)}{d\tau} = 0 \,, \tag{2.64}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the Christoffel connection for the background spacetime metric $g_{\mu\nu}$, and

(b) that the leading order coefficient $B^{\mu\nu}(\tau)$ in the stress-energy tensor for the particle has the form

$$B^{\mu\nu}(\tau) = m \, u^\mu(\tau) u^\nu(\tau) \,, \tag{2.65}$$

where $m \in \mathbb{R}\backslash\{0\}$ is an arbitrary constant parameter, while u^μ is the normalized tangent vector to the curve $\mathcal{C}$,

$$u^\mu \equiv \frac{dz^\mu(\tau)}{d\tau}\,, \qquad u^\mu u^\nu g_{\mu\nu} = -1\,. \tag{2.66}$$

In order to demonstrate these two statements, we start from (2.63), contract it with an arbitrary test function $f_\mu(x)$ of compact support, and integrate over the whole spacetime,

$$\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, f_\mu \nabla_\nu T^{\mu\nu} = 0\,. \tag{2.67}$$

Then we perform the partial integration to move the covariant derivative from the stress-energy tensor to the test function. The boundary term vanishes since the test function has compact support, giving

$$\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, T^{\mu\nu}\nabla_\nu f_\mu = 0\,. \tag{2.68}$$

Then we substitute (2.61), switch the order of integrations and perform the integral over spacetime $\mathcal{M}_4$, ending up with

$$\int_{\mathcal{C}} d\tau B^{\mu\nu}\nabla_\nu f_\mu = 0\,. \tag{2.69}$$

The spacetime covariant derivative of the test function can be split into a component tangent to the curve $\mathcal{C}$ and a component orthogonal to it, in the following way. Using the identity

$$\delta^\lambda_\mu = -u^\lambda u_\mu + P_\perp{}^\lambda{}_\mu\,, \tag{2.70}$$

where $-u^\lambda u_\mu$ and $P_\perp{}^\lambda{}_\mu$ are projectors along u^μ and orthogonal to u^μ, respectively, we rewrite the derivative of f_ν as

$$\nabla_\nu f_\mu = -u_\nu \nabla f_\mu + f^\perp_{\nu\mu}\,, \tag{2.71}$$

where $\nabla \equiv u^\lambda\nabla_\lambda$ is the covariant derivative in the direction of the curve $\mathcal{C}$, while $f^\perp_{\nu\mu} \equiv P_\perp{}^\lambda{}_\nu \nabla_\lambda f_\mu$ is a quantity orthogonal to the curve $\mathcal{C}$ with respect to its first index. Substituting (2.71) into (2.69), and performing another partial integration, we find

$$\int_{\mathcal{C}} d\tau \Big[f_\mu \nabla(B^{\mu\nu}u_\nu) + B^{\mu\nu} f^\perp_{\nu\mu} \Big] = 0\,, \tag{2.72}$$

where the boundary term again vanishes due to the compact support of the test function.

Given that the values of f_μ and $f^\perp_{\nu\mu}$ are both arbitrary and mutually independent along the curve $\mathcal{C}$, the coefficients multiplying them must each be zero. The first term gives us

$$\nabla(B^{\mu\nu}u_\nu) = 0\,, \tag{2.73}$$

while the second term, knowing that $f^\perp_{\nu\mu}$ is orthogonal to the curve $\mathcal{C}$ in its first index, gives

$$B^{\mu\nu} P_\perp{}^\lambda{}_\nu = 0\,. \tag{2.74}$$

Focus first on (2.74). Knowing that $B^{\mu\nu}$ is symmetric, we can use (2.70) to decompose it into orthogonal and parallel components with respect to its two indices,

$$B^{\mu\nu} = B_\perp^{\mu\nu} + B_\perp^\mu u^\nu + B_\perp^\nu u^\mu + B u^\mu u^\nu\,, \tag{2.75}$$

where $B_\perp^{\mu\nu}$, $B_\perp^\mu$ and B are unknown coefficients, the first two being orthogonal to the curve $\mathcal{C}$ in all their indices. Substituting this expansion into (2.74), one finds that

$$B_{\perp}^{\mu\nu} = 0 \,, \qquad B_{\perp}^{\mu} = 0 \,, \tag{2.76}$$

leaving the scalar B as the only nonzero component of $B^{\mu\nu}$. Changing the notation from B to m, one obtains

$$B^{\mu\nu}(\tau) = m(\tau) u^\mu u^\nu \,. \tag{2.77}$$

This equation looks very similar to (2.65) but is still not equivalent to it, since the coefficient $m(\tau)$ is still not known to be a constant.

Next, focus on (2.73). Substituting (2.77), it reduces to

$$\nabla (m u^\mu) = 0 \,. \tag{2.78}$$

Projecting onto the tangent direction u_μ and using the identity $u_\mu \nabla u^\mu = 0$, one obtains

$$\nabla m \equiv \frac{dm}{d\tau} = 0 \,, \tag{2.79}$$

establishing that the parameter m is actually a constant. Given this, equation (2.78) reduces to

$$\nabla u^\mu = 0 \,. \tag{2.80}$$

Remembering that $\nabla \equiv u^\lambda \nabla_\lambda$ and expanding the covariant derivative, we see that this is the geodesic equation (2.64). Finally, (2.79) and (2.77) together give (2.65), which completes the proof of statements (a) and (b).

There are three general remarks one should make regarding the above procedure. The first remark is about the physical interpretation and properties of the free parameter m. Namely, it can be given the interpretation of the total mass of the particle — substituting (2.65) into the stress-energy tensor (2.61) and integrating the T^{00} component over the volume of the spatial hypersurface orthogonal to u^μ, one can easily verify that the total rest-energy of the matter fields at a given time is equal to m. Note, however, that the sign of m is not fixed to be positive. This is not surprising, since the covariant conservation equation (2.63) and the stress-energy tensor (2.61) do not contain any information (or assumption) about the positivity of energy. Instead, the positive energy condition $m > 0$ has to be established from the full matter field equations, which take into account the internal dynamics of the matter fields that make up the particle. Finally, note that the case $m = 0$ cannot be covered by the single pole approximation, since the latter assumes that $m \sim B^{\mu\nu} \sim \mathcal{O}_0 \gg \mathcal{O}_1$. In other words, in the case $m = 0$ the leading order coefficient in the stress-energy tensor is the dipole term $B^{\mu\nu\rho} \sim \mathcal{O}_1$, so one needs to work in the pole-dipole approximation to consistently discuss massless particles.

The second remark is about the metric $g_{\mu\nu}$ of the background geometry. When discussing the motion of a particle, the background geometry is usually assumed to be fixed, and backreaction of the gravitational field of the particle itself is not taken into account, leading to the notion of a "test particle". However, ignoring the backreaction is not a necessary assumption. Namely, one can take the full stress-energy tensor of the matter fields which form the kink solution (as opposed to the approximate single pole stress-energy tensor (2.61)), put it as a source into the Einstein's field equations and solve for the metric $g_{\mu\nu}$. The resulting metric does include the backreaction, and can then be reinserted into the geodesic equation for the particle. Note that this procedure is self-consistent, since the geodesic motion of the particle is a consequence of the covariant conservation equation (2.63) which is in turn itself a consequence of Einstein's field equations. Also note that the metric $g_{\mu\nu}$ obtained in this way does not necessarily give rise to the black hole geometry in the neighborhood of the particle. This is because the Schwarzschild radius of the kink may be (and usually is) much smaller than the scale $\mathcal{O}_1$ which defines the precision of the single pole approximation (2.61). A simple example would be the motion of a planet around the Sun — in the single pole approximation, the radius of the planet (itself far larger than the planet's gravitational radius) is considered to be of the order $\mathcal{O}_1$ and the planet is treated as a pointlike object, but the spacetime metric used in the geodesic equation can still take into account the planet's gravitational field in addition to the field of the Sun.

The third remark is about going beyond the single pole approximation. This has been studied in detail in the literature [12–16, 26, 27, 29, 32, 33, 36–39, 42, 43], so here we merely point out the main physical interpretation. Namely, keeping the second term in the multipole expansion (2.58) physically amounts to giving the particle a nonzero "thickness", in the sense that its internal angular momentum can be considered nonzero. In the resulting equation of motion for the particle, this angular momentum couples to the spacetime curvature tensor, giving rise to a deviation from the geodesic motion. This can intuitively be understood as an effect of tidal forces acting across the scale of the kink's width, pushing it off the geodesic trajectory. Similarly, including quadrupole and higher order terms in (2.58) takes into account additional internal degrees of freedom of the kink, which also couple to spacetime geometry and produce further deviation from geodesic motion.

The above review of the multipole formalism, and its application to the derivation of the geodesic equation in GR, will be used in the next chapter to discuss the corrections to the motion of a particle stemming from quantum gravity. As we shall see, these quantum corrections will give rise to additional terms in the effective equation of motion for a particle, pushing it slightly off the geodesic trajectory, even in the single pole approximation.

Chapter 3

Geodesic equation in quantum gravity

In this chapter, we discuss the motion of a particle within the framework of quantum gravity. The exposition is structured into five parts — first, we give a short introduction to the canonical quantization procedure; second, we introduce the abstract quantum gravity formalism and give some technical details about the description of the states. In the third part, we discuss the quantum version of the covariant conservation equation of the stress-energy tensor. In the fourth part, we adopt the derivation presented in section 2.4 to the quantum formalism and obtain the effective equation of motion for the particle. Finally, in the fifth part, we discuss the self-consistency assumptions that go into the calculation.

3.1 Quantization procedure

We now give a short description of the canonical quantization of a theory, which allows us to go to a quantum field theory from a classical theory. An alternative method to quantize a theory would be using path integrals.

In quantum theory, the time evolution of a system in the Schrödinger picture is given by the Schrödinger equation,

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \tag{3.1}$$

where the states are time dependent. We are interested instead in the description in which the operators carry the time dependence, since this is the concept that generalizes more readily to a field theory. This is done using through the Heisenberg picture.

The Heisenberg picture is defined by acting with the inverse evolution operator $U^\dagger(t, t_0)$ onto the Schrödinger picture entities $|\Psi(t)\rangle$ and $\hat{O}$. Using the integral form of the equation of motion in the Schrödinger picture, $|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle$, we get:

$$|\Psi_H\rangle \equiv U^\dagger(t, t_0)|\Psi(t)\rangle = |\Psi(t_0)\rangle, \tag{3.2}$$

where the operator $\hat{O}_H(t)$ in the Heisenberg picture is given by

$$\hat{O}_H(t) \equiv U^\dagger(t, t_0)\hat{O}U(t, t_0). \tag{3.3}$$

These expressions lead to the same form of the expectation value for both the Heisenberg and the Schrödinger picture:

$$\langle\Psi_H|\hat{O}_H(t)|\Psi_H\rangle = \langle\Psi_H|U^\dagger(t, t_0)\hat{O}U(t, t_0)|\Psi_H\rangle = \langle\Psi(t)|\hat{O}|\Psi(t)\rangle. \tag{3.4}$$

In case of a time-independent Hamiltonian, as in (3.1), the solution of the Schroedinger equation is $U(t, t_0) = e^{-i\hat{H}(t-t_0)}$. From this last equation, we can derive the equation of motion in the Heisenberg picture for a time-independent Hamiltonian:

$$i\frac{d\hat{O}_H(t)}{dt} = [\hat{O}_H(t), \hat{H}]. \tag{3.5}$$

Now, we will give a short recapitulation of the procedure developed by Dirac and others (see also [7] and [30] for a review), which allows one to quantize in a standard way an arbitrary physical system with constraints, and it is normally used in gauge theories. This procedure is important, since gravity has local symmetries and is a gauge theory. While this topic is important, the discussion of the details is out of scope of the thesis.

In this procedure we start by classifying the constraints of the theory into first and second-class (see [7] for a technical definition of these constraints). Then, the second class constraints are eliminated when we pass from the Poisson brackets to the Dirac brackets. The constraints that remain are the first class constraints that represent the generators of the gauge symmetry. Generally, we can write the Hamiltonian of the theory as

$$H = H_0 + \lambda^A \mathcal{C}_A, \tag{3.6}$$

with the Lagrange multipliers λ_A , the first class constraints $\mathcal{C}_A$, and with the term in the Hamiltonian describing the evolution of the physical degrees of freedom H_0. Then, we perform the quantization in the Heisenberg picture, where fundamental fields $\phi(x)$ and respective conjugate momenta are promoted to quantum mechanical operators,

$$\phi(x) \to \hat{\phi}(x) \qquad \pi(x) \to \hat{\pi}(x). \tag{3.7}$$

Moreover, we introduce the state vectors $|\Psi\rangle \in \mathcal{H}_{kin}$, where $\mathcal{H}_{kin}$ is the kinematical Hilbert space of the theory. The Dirac brackets between the fields and their momenta are then promoted to the commutators of the corresponding operators. The Hamiltonian, being a functional that depends on the fields and momenta, also becomes an operator, and it is used in the Heisenberg equations of motion for the field operators 3.5. The kinematical Hilbert space $\mathcal{H}_{kin}$ is projected onto its gauge invariant subspace $\mathcal{H}_{phys}$, by requiring that every state vector $|\Psi\rangle \in \mathcal{H}_{phys}$ is annihilated by the generators of the gauge symmetry group,

$$\langle\Psi|\hat{\mathcal{C}}_A|\Psi\rangle = 0. \tag{3.8}$$

In quantum electrodynamics these conditions are known as the Gupta-Bleuler quantization conditions [8, 20], and they ensure that the gauge symmetry of the classical theory remains to be a symmetry of the quantum theory as well.

3.2 Preliminaries and the setup

We work in the so-called generic abstract quantum gravity setup, as follows. Starting from the Heisenberg picture for the description of quantum systems, we assume that gravitational degrees of freedom are described by some gravitational field operators $\hat{g}(x)$, while matter degrees of freedom are described by matter field operators $\hat{\phi}(x)$, where x represents the coordinates of some point on a 4-dimensional spacetime manifold $\mathcal{M}_4$. Both sets of operators have their corresponding canonically conjugate momentum operators, $\hat{\pi}_g(x)$ and $\hat{\pi}_\phi(x)$, such that the usual canonical commutation relations hold. The total (kinematical) Hilbert space of the theory is $\mathcal{H}_{\text{kin}} = \mathcal{H}_G \otimes \mathcal{H}_M$, where the gravitational and matter Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_M$ are spanned by the bases of eigenvectors for the operators $\hat{g}$ and $\hat{\phi}$, respectively.

The total state of the system, $|\Psi\rangle \in \mathcal{H}_{\text{kin}}$, does not depend on x, in line with the Heisenberg picture framework.

There are several important points that need to be emphasized regarding the above setup. First, we do not explicitly state what are the fundamental degrees of freedom $\hat{g}$ for the gravitational field. They can be chosen in many different ways, giving rise to different models of quantum gravity. Since we aim to present the analysis of geodesic motion which is model-independent, we refrain from specifying what are the fundamental degrees of freedom $\hat{g}$. Instead, we merely assume that the expectation values of the operators describing the effective spacetime geometry, i.e., the metric, connection, curvature, etc., depend somehow on $g = \langle\Psi|\hat{g}|\Psi\rangle$ and $\pi = \langle\Psi|\hat{\pi}_g|\Psi\rangle$, and are expressible as operator functions in terms of them:

$$\langle\Psi|\hat{g}_{\mu\nu}|\Psi\rangle = g_{\mu\nu}(g,\pi_g)\,, \qquad \langle\Psi|\hat{\Gamma}^{\lambda}{}_{\mu\nu}|\Psi\rangle = \Gamma^{\lambda}{}_{\mu\nu}(g,\pi_g)\,, \qquad \langle\Psi|\hat{R}^{\lambda}_{\mu\nu\rho}|\Psi\rangle = R^{\lambda}_{\mu\nu\rho}(g,\pi_g)\,, \qquad \dots \tag{3.9}$$

When discussing these geometric operators, for simplicity we will usually not explicitly write their (g,π_g)-dependence.

Second, in order for any operator function to be well defined, some operator ordering has to be assumed. However, since we aim to work in an abstract model-independent QG formalism, we do not choose any particular ordering, but merely assume that one such ordering has been fixed. In a similar fashion, we also simply assume that all operators and spaces are well defined, convergent, and otherwise specified in enough mathematical detail to have a well defined and unique QG model. In a nutshell, our calculations are formal, in the sense that one should be able to repeat them in a detailed fashion if one is given a specific model of QG. This also means that our analysis and results should not depend on any of these details, but are rather common to a large class of QG models, and are based only on very few assumptions given above.

Third, we employ a natural distinction between gravitational and matter degrees of freedom, assuming that the separation between gravity and matter present in the classical theory, described by action of the form

$$S_{\text{total}}[g,\phi] = S_{\text{gravity}}[g] + S_{\text{matter}}[g,\phi]\,, \tag{3.10}$$

remains present also in the full quantum regime. That is to say, we assume that one can construct a theory of quantum gravity without matter fields, using only gravitational degrees of freedom g, so that this theory gives sourceless Einstein's equations of GR in the classical limit. Once such a pure-QG model has been constructed, we assume one can couple matter ϕ to it without changing the structure of the gravitational sector, obtaining the full QG model which features Einstein's equations with appropriate matter sources in the classical limit. While we do not consider this to be a strong assumption, we feel that it is nevertheless important to spell it out explicitly, since there may exist some QG models which fail to satisfy it, and our analysis may be inapplicable to such models.

After the introduction of the above conceptual setup, we turn to some more practical details. For the purpose of discussing geodesic motion, we are mostly interested in the effective classical theory of the abstract QG introduced above. To that end, the main objects of attention are *coherent states* of gravity and matter. Denoting them as $|g\rangle \in \mathcal{H}_G$ and $|\phi\rangle \in \mathcal{H}_M$, respectively, by definition they are assumed to saturate Heisenberg inequalities for the gravitational and matter field operators,

$$\Delta\hat{g}\Delta\hat{\pi}_g \approx \frac{\hbar}{2}\,, \qquad \Delta\hat{\phi}\Delta\hat{\pi}_\phi \approx \frac{\hbar}{2}\,. \tag{3.11}$$

Moreover, we want these coherent states to be *classical* in the sense that all four uncertainties $\Delta\hat{g}$, $\Delta\hat{\pi}_g$, $\Delta\hat{\phi}$ and $\Delta\hat{\pi}_\phi$, are simultaneously minimal, compared to some scale. In other words, we do not want $|g\rangle$ and $|\phi\rangle$ to be just arbitrary squeezed states, but rather only the squeezed states which are also coherent. Working with coherent states is convenient for the analysis of the effective classical theory because then the notion of a "point in a phase space" of the theory has minimal uncertainty, allowing one to reconstruct the classical theory from the full quantum description.

Given a full coherent state vector as

$$|\Psi\rangle = |g\rangle \otimes |\phi\rangle \,, \tag{3.12}$$

we wish to evaluate and study the "effective classical" values for the metric tensor and the matter stress-energy tensor as expectation values of the corresponding operators:

$$g_{\mu\nu} = \langle\Psi|\hat{g}_{\mu\nu}|\Psi\rangle \,, \qquad T_{\mu\nu} = \langle\Psi|\hat{T}_{\mu\nu}|\Psi\rangle \,. \tag{3.13}$$

However, in quantum gravity, one should not discuss only states of the form (3.12), but also linear combinations of such states. In particular, an arbitrary state $|\boldsymbol{\Psi}\rangle \in \mathcal{H}_{\text{kin}}$ can be written as a linear combination of coherent states of the form

$$|\boldsymbol{\Psi}\rangle = \sum_{i,j} c_{ij} |g_i\rangle \otimes |\phi_j\rangle \,, \tag{3.14}$$

where indices i and j count all possible coherent states in the theory. The coefficients c_{ij} are in general not unique, since coherent states typically form an overcomplete basis of the Hilbert space. However, while the state (3.14) is the most general state, it is also quite cumbersome to work with. Therefore, for the purpose of this thesis, we will restrict to a *toy example state*, defined by having only two terms in the sum (3.14), as

$$|\boldsymbol{\Psi}\rangle = \alpha|\Psi\rangle + \beta|\tilde{\Psi}\rangle \,, \tag{3.15}$$

where $|\tilde{\Psi}\rangle \equiv |\tilde{g}\rangle \otimes |\tilde{\phi}\rangle$ is some other coherent state analogous to (3.12) but giving different expectation values for the classical metric and stress-energy tensors:

$$\tilde{g}_{\mu\nu} = \langle\tilde{\Psi}|\hat{g}_{\mu\nu}|\tilde{\Psi}\rangle \,, \qquad \tilde{T}_{\mu\nu} = \langle\tilde{\Psi}|\hat{T}_{\mu\nu}|\tilde{\Psi}\rangle \,. \tag{3.16}$$

One can see that our toy-example state (3.15) is a Schrödinger-cat type of state, describing a coherent superposition of two classical configurations of gravitational and matter fields.

In addition to discussing superpositions in general, one also needs to address the issue of gauge invariance. Namely, assuming that a theory of quantum gravity ought to obey local Poincaré invariance, which is a gauge symmetry, the proper Hilbert space of the theory cannot be the full space $\mathcal{H}_{\text{kin}}$, since it contains state vectors which may fail to be gauge invariant. The *physical* Hilbert space $\mathcal{H}_{\text{phys}}$ of gauge invariant states is therefore a proper subset of $\mathcal{H}_{\text{kin}}$, specified by the Gupta-Bleuler-like conditions which enforce gauge invariance at the quantum level. Therefore, one cannot simply assume that the coherent state (3.12) is gauge invariant. In fact, it was argued in [30] that $\mathcal{H}_{\text{phys}}$ actually does not contain any separable states of the form (3.12), rendering the particular choice (3.12) non-invariant and thus unphysical. This represents an additional argument to study states of type (3.14) and, as a particularly convenient toy example, states given by (3.15).

Before continuing, it is important to emphasize two points. First, we are assuming that the state (3.15) is gauge invariant, i.e., an element of the physical Hilbert space $\mathcal{H}_{\text{phys}}$. This assumption is benign, in the sense that all main results of the thesis will continue to hold qualitatively even for the more general state of type (3.14), as long as it is chosen to belong to $\mathcal{H}_{\text{phys}}$. It will become evident later on that qualitative conclusions of the thesis do not depend on the fact that (3.15) has precisely two terms in the sum. Choosing the state with three, four or more terms will lead to analogous conclusions, although quantitative details of the computation may become technically more involved.

Second, given that (3.15) is a Schrödinger-cat type of state, there are some phenomenological restrictions on the values of the coefficients α and β. Namely, in the ordinary experimental situations we basically never observe these kind of states, which means that the overall entangled state $|\boldsymbol{\Psi}\rangle$ looks pretty much like a classical coherent state, say the state $|\Psi\rangle$. In other words, we want the fidelity between these two states to be large, $F(|\boldsymbol{\Psi}\rangle, |\Psi\rangle) = |\langle\boldsymbol{\Psi}|\Psi\rangle| \approx 1$. We will therefore choose to call the state $|\Psi\rangle$ the *dominant state*, while the $|\tilde{\Psi}\rangle$ will be called the *sub-dominant state*. Since we want to study the case of a general classical sub-dominant states $|\tilde{\Psi}\rangle$, without restricting the value of $|\langle\Psi|\tilde{\Psi}\rangle|$, the assumption of large fidelity boils down to that of small β, and consequently large α. Therefore, we will be systematically working in the limit $\beta \to 0$. It should be clear that, from the point of view of the classical limit, the most natural choice would be to take $\alpha = 1$ and $\beta = 0$, i.e., the state (3.12). But, as argued in [30], there is a danger that such a state may fail to be gauge invariant, so we need to introduce at least a small sub-dominant state, in order to ensure the gauge invariance of the total state. So the simplest possible candidate state which describes the classical physics sufficiently well, and simultaneously stands a chance of being gauge invariant, is the state (3.15), with β very small but nonzero.

Having adopted the state in the form (3.15), let us now introduce some technical apparatus to study it efficiently. To begin with, coherent states in general have nonzero overlap. Therefore, we introduce the overlaps as follows:

$$S_G \equiv \langle g|\tilde{g}\rangle\,, \qquad S_M \equiv \langle \phi|\tilde{\phi}\rangle\,, \qquad S \equiv \langle\Psi|\tilde{\Psi}\rangle = S_G S_M\,. \tag{3.17}$$

Note that, since in (3.15) only the relative phase between $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ is important, we can reabsorb the phases of the coefficients α and β into $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$, respectively. In this way, we have $\alpha, \beta \in \mathbb{R}$, while only the overlap S between the two coherent states carries the information about the relative phase, and is therefore complex. Moreover, since S is a product between S_G and S_M, the phase of S can be distributed between S_G and S_M in an arbitrary way. A convenient choice is to have the phase in the matter sector, so that $S_G \in \mathbb{R}$ and $S_M \in \mathbb{C}$.

Next, we can decompose $|\tilde{g}\rangle$ and $|\tilde{\phi}\rangle$ into parts proportional to and orthogonal to $|g\rangle$ and $|\phi\rangle$, respectively,

$$|\tilde{g}\rangle = S_G|g\rangle + \epsilon_G|g^\perp\rangle\,, \qquad |\tilde{\phi}\rangle = S_M|\phi\rangle + \epsilon_M|\phi^\perp\rangle\,, \tag{3.18}$$

where $\langle g|g^\perp\rangle \equiv 0$, $\langle\phi|\phi^\perp\rangle \equiv 0$, and

$$\epsilon_G \equiv \sqrt{1-(S_G)^2}\,, \qquad \epsilon_M \equiv \sqrt{1-|S_M|^2}\,. \tag{3.19}$$

Note that $\epsilon_G, \epsilon_M \in \mathbb{R}$.

Since the state (3.15) must be normalized, $\langle\boldsymbol{\Psi}|\boldsymbol{\Psi}\rangle = 1$, we have

$$\alpha^2 + \beta^2 + 2\alpha\beta\,\mathrm{Re}(S) = 1\,, \tag{3.20}$$

where $\mathrm{Re}(S)$ denotes the real part of S. This equation can be treated as a quadratic equation for α, and solved to give

$$\alpha = -\beta\,\mathrm{Re}(S) \pm \sqrt{1+\beta^2\left[\mathrm{Re}(S)^2-1\right]}\,. \tag{3.21}$$

Choosing the positive solution without loss of generality, we can expand α into power series in the limit $\beta \to 0$ as:

$$\alpha = 1 - \beta\,\mathrm{Re}(S) + \mathcal{O}(\beta^2)\,. \tag{3.22}$$

Additionally, one can use (3.18) to rewrite $|\tilde{\Psi}\rangle$ into the form

$$|\tilde{\Psi}\rangle = S|\Psi\rangle + \epsilon|\Psi^\perp\rangle\,, \tag{3.23}$$

where

$$\epsilon = \sqrt{\epsilon_M^2 + \epsilon_G^2 - \epsilon_M^2 \epsilon_G^2}\,, \tag{3.24}$$

and

$$|\Psi^\perp\rangle = \frac{\epsilon_M S_G}{\epsilon} |g\rangle \otimes |\phi^\perp\rangle + \frac{\epsilon_G S_M}{\epsilon} |g^\perp\rangle \otimes |\phi\rangle + \frac{\epsilon_G \epsilon_M}{\epsilon} |g^\perp\rangle \otimes |\phi^\perp\rangle\,. \tag{3.25}$$

Finally, given all of the above, we can rewrite the total state (3.15) as:

$$|\boldsymbol{\Psi}\rangle = \Big[1 + i\beta S_G \,\mathrm{Im}(S_M)\Big] |\Psi\rangle + \beta\epsilon |\Psi^\perp\rangle + \mathcal{O}(\beta^2) \tag{3.26}$$

Note that in the cases when S_G and S_M are large (when the sub-dominant gravity and matter states are both similar to the dominant ones), and consequently ϵ_G and ϵ_M are small, we can neglect the final term from (3.25), obtaining the Schmidt form of the "orthogonal correction" of the sub-dominant state, with respect to the dominant one. It is interesting to observe that such a state is always necessarily entangled, as its entanglement entropy[1] is always bigger than zero. In other words, to obtain a nearby classical product state of gravity and matter $|\tilde{\Psi}\rangle$, one has to perturb the original (classical product) state $|\Psi\rangle$ with an entangled state $|\Psi^\perp\rangle \approx \epsilon^{-1}\epsilon_M S_G |g\rangle \otimes |\phi^\perp\rangle + \epsilon^{-1}\epsilon_G S_M |g^\perp\rangle \otimes |\phi\rangle$.

At this point we can evaluate the expectation values for the metric and stress-energy operators in the state (3.26). Taking into account that the operator $\hat{g}_{\mu\nu}$ depends only on the gravitational degrees of freedom, we use (3.25) and (3.26) to obtain

$$\boldsymbol{g}_{\mu\nu} = \langle\boldsymbol{\Psi}|\hat{g}_{\mu\nu}|\boldsymbol{\Psi}\rangle = g_{\mu\nu} + \beta h_{\mu\nu} + \mathcal{O}(\beta^2)\,, \tag{3.27}$$

where $g_{\mu\nu}$ is the dominant classical metric (3.13), while the correction term $h_{\mu\nu}$ is given as

$$h_{\mu\nu} = 2\epsilon_G \,\mathrm{Re}\left(S_M \langle g|\hat{g}_{\mu\nu}|g^\perp\rangle\right)\,. \tag{3.28}$$

We see that the correction term is of purely quantum origin, without a classical analog — it is a function of the off-diagonal matrix elements of the metric operator and of the overlap between the two coherent states in (3.15).

One can perform a similar calculation of the expectation value for the stress-energy operator, noting that $\hat{T}_{\mu\nu}$ depends on both gravitational and matter degrees of freedom, to obtain

$$\boldsymbol{T}_{\mu\nu} = \langle\boldsymbol{\Psi}|\hat{T}_{\mu\nu}|\boldsymbol{\Psi}\rangle = T_{\mu\nu} + \beta\bar{T}_{\mu\nu} + \mathcal{O}(\beta^2)\,, \tag{3.29}$$

where $T_{\mu\nu}$ is the dominant classical stress-energy (3.13), while the correction term $\bar{T}_{\mu\nu}$ is given as

$$\bar{T}_{\mu\nu} = 2\epsilon \,\mathrm{Re}\Big(\langle\Psi|\hat{T}_{\mu\nu}|\Psi^\perp\rangle\Big)\,. \tag{3.30}$$

Again we see that the correction term is of purely quantum origin, being a function of the off-diagonal matrix elements of the stress-energy operator and of the overlap.

Regarding the effective entangled metric and stress-energy tensors (3.27) and (3.29), it is important to stress that they do not satisfy classical Einstein's equations of GR. Namely, we assume that Einstein's equations are separately satisfied by the dominant metric and stress-energy (3.13) coming from the coherent state $|\Psi\rangle$, and by the sub-dominant metric and stress-energy (3.16) coming from the other

[1]Entanglement entropy is a measure of the entanglement of a system. For the case of a pure bipartite state, it is found through the calculation of the Von Neumann entropy of one of its reduced states. When it is bigger than zero, the two subsystems are entangled.

coherent state $|\tilde{\Psi}\rangle$, as two different classical solutions:

$$R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = 8\pi G\, T_{\mu\nu}\,, \qquad R_{\mu\nu}(\tilde{g}) - \frac{1}{2} \tilde{g}_{\mu\nu} R(\tilde{g}) = 8\pi G\, \tilde{T}_{\mu\nu}\,. \tag{3.31}$$

However, due to the nonlinearity of Einstein's equations, and due to the presence of the overlap terms $h_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ in (3.27) and (3.29), quantities $\boldsymbol{g}_{\mu\nu}$ and $\boldsymbol{T}_{\mu\nu}$ do not satisfy Einstein's equations, as long as $\beta \neq 0$.

This leads us to the following physical interpretation. Given that the sub-dominant classical solution $(\tilde{g}_{\mu\nu}, \tilde{T}_{\mu\nu})$ is quadratic in $|\tilde{\Psi}\rangle$ which enters (3.15) multiplied by β, in the limit $\beta \to 0$ the sub-dominant solution is of the order $\mathcal{O}(\beta^2)$ in the equations (3.27) and (3.29). Therefore, in our approximation the only nontrivial effect of its presence enters through the overlap terms (3.28) and (3.30). Consequently, the only classical spacetime-matter configuration which satisfies Einstein's equations and which is present in our description — is the dominant one $(g_{\mu\nu}, T_{\mu\nu})$. From a phenomenological point of view, therefore, it is natural to expand all quantities as corrections to the dominant classical configuration $(g_{\mu\nu}, T_{\mu\nu})$, including the equation of motion for a point particle. As we shall see in the remainder of the text, given that $(\boldsymbol{g}_{\mu\nu}, \boldsymbol{T}_{\mu\nu})$ contains quantum gravity corrections through the overlap terms, the presence of these quantum corrections in (3.27) and (3.29) will introduce an "effective force" term into the effective equation of motion for the particle, pushing it off the geodesic trajectory defined by the classical dominant metric $g_{\mu\nu}$.

3.3 Effective covariant conservation equation

After the discussion of the general QG setup and the state (3.15), we move on to the discussion of the quantum analog of the covariant conservation equation (2.63). As in the classical theory, our basic assumption is that the matter sector of our QG model features local Poincaré invariance, i.e., that this symmetry is preserved at the quantum level. This assumption gives rise to a Gupta-Bleuler-like condition on the physical states, in the form

$$\langle \boldsymbol{\Psi} | \hat{\nabla}_\nu \hat{T}^{\mu\nu} | \boldsymbol{\Psi} \rangle = 0\,, \tag{3.32}$$

where $\hat{\nabla}_\mu$ is the covariant derivative operator, defined by promoting the metric appearing in the Christoffel symbols into a corresponding operator. In general, the action of the stress-energy operator on the state $|\boldsymbol{\Psi}\rangle$ can be written[2] as

$$\hat{T}^{\mu\nu} |\boldsymbol{\Psi}\rangle = \boldsymbol{T}^{\mu\nu} |\boldsymbol{\Psi}\rangle + \boldsymbol{\Delta T}^{\mu\nu} |\boldsymbol{\Psi}^\perp\rangle\,, \tag{3.34}$$

where $\boldsymbol{T}^{\mu\nu}$ and $\boldsymbol{\Delta T}^{\mu\nu}$ are the expectation value and the uncertainty of the operator $\hat{T}^{\mu\nu}$ in the state $|\boldsymbol{\Psi}\rangle$, respectively,

$$\boldsymbol{T}^{\mu\nu} \equiv \langle \boldsymbol{\Psi} | \hat{T}^{\mu\nu} | \boldsymbol{\Psi} \rangle\,, \qquad \boldsymbol{\Delta T}^{\mu\nu} \equiv \sqrt{\langle \boldsymbol{\Psi} | \left(\hat{T}^{\mu\nu}\right)^2 | \boldsymbol{\Psi} \rangle - \left(\langle \boldsymbol{\Psi} | \hat{T}^{\mu\nu} | \boldsymbol{\Psi} \rangle\right)^2}\,, \tag{3.35}$$

while $|\boldsymbol{\Psi}^\perp\rangle$ is some state orthogonal to $|\boldsymbol{\Psi}\rangle$. Note that the equation of the form (3.34) is completely general, holding for any stress-energy operator acting on an arbitrary state. Substituting (3.34) into (3.32), we obtain

$$\boldsymbol{\nabla}_\nu \boldsymbol{T}^{\mu\nu} + \langle \boldsymbol{\Psi} | \hat{\nabla}_\nu | \boldsymbol{\Psi}^\perp \rangle \boldsymbol{\Delta T}^{\mu\nu} = 0\,, \tag{3.36}$$

[2] Given any self-adjoint operator $\hat{A}$ and any state $|\Psi\rangle$, one can write

$$\hat{A}|\Psi\rangle = a|\Psi\rangle + b|\Psi^\perp\rangle\,, \tag{3.33}$$

where $\langle\Psi|\Psi^\perp\rangle \equiv 0$ and $a, b \in \mathbb{C}$. Multiplying this equation by $\langle\Psi|$ and by $\langle\Psi|\hat{A}$ from the left, one easily obtains that a and b are the expectation value and the uncertainty of the operator $\hat{A}$ in the state $|\Psi\rangle$, respectively.

where $\boldsymbol{\nabla}_\nu$ is the expectation value of the operator $\hat{\nabla}_\nu$,

$$\boldsymbol{\nabla}_\nu \equiv \langle \boldsymbol{\Psi} | \hat{\nabla}_\nu | \boldsymbol{\Psi} \rangle \,. \tag{3.37}$$

At this point we need to make one more assumption. Namely, we assume that the error scale of the single pole approximation is bigger than the uncertainty of the stress-energy operator, $\boldsymbol{\Delta T}^{\mu\nu}$. Symbolically,

$$\mathcal{O}_1 \gtrsim \boldsymbol{\Delta T}^{\mu\nu} \,. \tag{3.38}$$

This means that in the single pole approximation we do not see the effects of the quantum fluctuations of matter fields. Intuitively, this is a reasonable assumption in most cases. For example, in the case of the kink solution describing the hydrogen atom, the scale on which one can detect quantum fluctuations (i.e., the Lamb shift effects) is much smaller than the size of the atom itself (i.e., the radius of the first Bohr orbit). Therefore, we expect that if our single pole approximation ignores the size of the atom itself, it also ignores the corresponding quantum fluctuations. An analogous assumption is made in relation to the uncertainty of the metric operator $\hat{g}_{\mu\nu}$,

$$\mathcal{O}_1 \gtrsim \boldsymbol{\Delta g}_{\mu\nu} \,, \tag{3.39}$$

given that the quantum gravity fluctuations can arguably also be ignored in the single pole approximation.

Applying (3.38) to (3.36), in the single pole approximation the second term can be dropped, leading to the effective classical covariant conservation equation,

$$\boldsymbol{\nabla}_\nu \boldsymbol{T}^{\mu\nu} = 0 \,. \tag{3.40}$$

In a similar fashion, one can employ (3.39) to drop the off-diagonal components in the Christoffel symbol operators, leading to an effective classical expression

$$\boldsymbol{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} \boldsymbol{g}^{\lambda\sigma} \left(\partial_\mu \boldsymbol{g}_{\sigma\nu} + \partial_\nu \boldsymbol{g}_{\sigma\mu} - \partial_\sigma \boldsymbol{g}_{\mu\nu} \right) \,, \tag{3.41}$$

where $\boldsymbol{g}_{\mu\nu} \equiv \langle \boldsymbol{\Psi} | \hat{g}_{\mu\nu} | \boldsymbol{\Psi} \rangle$ is the effective classical metric and $\boldsymbol{g}^{\mu\nu}$ is its inverse matrix.

With effective classical expressions (3.40) and (3.41) in hand, we can now employ (3.27) and (3.29) to expand them into the dominant and correction parts. First we use (3.27) and $\boldsymbol{g}_{\mu\lambda} \boldsymbol{g}^{\lambda\nu} = \delta^\nu_\mu$ to find the inverse entangled metric $\boldsymbol{g}^{\mu\nu} = g^{\mu\nu} - \beta g^{\mu\rho} g^{\nu\sigma} h_{\rho\sigma} + \mathcal{O}(\beta^2)$, and then substitute into (3.41) to obtain

$$\boldsymbol{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + \frac{\beta}{2} g^{\lambda\sigma} \left(\nabla_\mu h_{\sigma\nu} + \nabla_\nu h_{\sigma\mu} - \nabla_\sigma h_{\mu\nu} \right) + \mathcal{O}(\beta^2) \,, \tag{3.42}$$

where the Christoffel symbols in ordinary ∇_μ are defined with respect to the dominant classical metric $g_{\mu\nu}$. Then, expanding (3.40) into the form

$$\partial_\nu \boldsymbol{T}^{\mu\nu} + \boldsymbol{\Gamma}^\mu{}_{\sigma\nu} \boldsymbol{T}^{\sigma\nu} + \boldsymbol{\Gamma}^\nu{}_{\sigma\nu} \boldsymbol{T}^{\mu\sigma} = 0 \,, \tag{3.43}$$

we substitute (3.29) and (3.42), and after a bit of algebra we rewrite it as:

$$\nabla_\nu T^{\mu\nu} + \beta \left[\nabla_\nu \bar{T}^{\mu\nu} + T^{\sigma\nu} \left(\nabla_\sigma h^\mu{}_\nu - \frac{1}{2} \nabla^\mu h_{\nu\sigma} \right) + \frac{1}{2} T^{\mu\sigma} \nabla_\sigma h^\nu{}_\nu \right] + \mathcal{O}(\beta^2) = 0 \,. \tag{3.44}$$

This equation is the one we sought out — it represents the analog of the classical covariant conservation equation (2.63), while taking into account the overlap between the two coherent states in (3.15), approximated to the linear order in β.

As a final step, (3.44) can be rewritten in a more compact form. For convenience, introduce the following shorthand notation (see our conventions from the last paragraph of the Introduction),

$$F^{\mu}{}_{\nu\sigma} \equiv \nabla_{(\sigma} h^{\mu}{}_{\nu)} - \frac{1}{2}\nabla^{\mu} h_{\nu\sigma} \,, \tag{3.45}$$

and also note that

$$F^{\nu}{}_{\nu\sigma} = \frac{1}{2}\nabla_{\sigma} h^{\nu}{}_{\nu} + \frac{1}{2}\nabla_{\nu} h^{\nu}{}_{\sigma} - \frac{1}{2}\nabla^{\nu} h_{\nu\sigma} = \frac{1}{2}\nabla_{\sigma} h^{\nu}{}_{\nu} \,, \tag{3.46}$$

so that, dropping the term $\mathcal{O}(\beta^2)$, equation (3.44) is rewritten as:

$$\nabla_{\nu}\left(T^{\mu\nu} + \beta \bar{T}^{\mu\nu}\right) + 2\beta F^{(\mu}{}_{\nu\sigma} T^{\nu)\sigma} = 0 \,. \tag{3.47}$$

The equation (3.47) represents the effective classical covariant conservation law for the stress-energy tensor, with the included quantum correction, represented to first order in β. It is the starting point for the remainder of our analysis, and replaces equation (2.63) in the derivation of the equation of motion for a point particle.

Finally, note that the quantum correction term in (3.47) has two distinct parts — one part comes from the quantum correction to the dominant classical stress-energy tensor, i.e., the overlap term $\bar{T}^{\mu\nu}$, while the other part comes from the quantum correction to the dominant classical metric, i.e., the overlap term $h_{\mu\nu}$. This latter quantum correction enters through the Christoffel connection terms present in the covariant derivative. As we shall see in the next subsection, its presence will be crucial for the "force term" in the equation of motion for the particle, responsible for the deviation from the classical geodesic trajectory.

3.4 Effective equation of motion

We are now ready to derive the equation of motion for a particle in the single pole approximation, using the technique presented in section 2.4. However, instead of (2.63), we start from the effective covariant conservation law (3.47), which contains the quantum correction terms. Throughout, we assume the following relation of scales,

$$\mathcal{O}(\beta) > \mathcal{O}_1 \geq \mathcal{O}(\beta^2) \,. \tag{3.48}$$

In other words, we assume that the quantum correction terms linear in β are not smaller than the width of our particle, since otherwise one could simply ignore them and recover the classical geodesic motion for the particle.

Repeating the method of section 2.4, we begin by contracting (3.47) with an arbitrary test function $f_{\mu}(x)$ of compact support, and integrating over the whole spacetime,

$$\int_{\mathcal{M}_4} d^4x \sqrt{-g}\, f_{\mu} \left[\nabla_{\nu}\left(T^{\mu\nu} + \beta\bar{T}^{\mu\nu}\right) + 2\beta F^{(\mu}{}_{\nu\sigma} T^{\nu)\sigma}\right] = 0 \,. \tag{3.49}$$

We then perform the partial integration to move the covariant derivative from the stress-energy tensors to the test function. As before, the boundary term vanishes since the test function has compact support, giving

$$\int_{\mathcal{M}_4} d^4x \sqrt{-g}\left[-\left(T^{\mu\nu} + \beta\bar{T}^{\mu\nu}\right)\nabla_{\nu} f_{\mu} + 2\beta F^{(\mu}{}_{\nu\sigma} T^{\nu)\sigma} f_{\mu}\right] = 0 \,. \tag{3.50}$$

Now we need to model the dominant and correction parts of the stress-energy tensor. For the dominant part, it is straightforward to assume the single pole approximation, as was done in the classical case:

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} . \tag{3.51}$$

Regarding the correction term, we also use the single pole approximation,

$$\bar{T}^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \bar{B}^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} , \tag{3.52}$$

but one should note that in the case of $\bar{T}^{\mu\nu}$ it is less obvious why this approximation is adequate, and requires some justification. However, in order to focus on the derivation of the particle equation of motion, for the moment we simply adopt (3.52), and postpone the analysis and the meaning of this approximation for subsection 3.5.

Then we substitute (3.51) and (3.52) into (3.50), switch the order of integrations and perform the integral over spacetime $\mathcal{M}_4$, ending up with

$$\int_{\mathcal{C}} d\tau \Big[- \left(B^{\mu\nu} + \beta \bar{B}^{\mu\nu} \right) \nabla_\nu f_\mu + 2\beta F^{(\mu}{}_{\nu\sigma} B^{\nu)\sigma} f_\mu \Big] = 0 . \tag{3.53}$$

The next step is to employ the identity (2.71) to separate the tangential and orthogonal components of the derivative of the test function. Substituting it into (3.53), and performing another partial integration, we find

$$\int_{\mathcal{C}} d\tau \Big[\left(B^{\mu\nu} + \beta \bar{B}^{\mu\nu} \right) f^{\perp}_{\nu\mu} + \Big[\nabla \left(B^{\mu\nu} u_\nu + \beta \bar{B}^{\mu\nu} u_\nu \right) - 2\beta F^{(\mu}{}_{\nu\sigma} B^{\nu)\sigma} \Big] f_\mu \Big] = 0 , \tag{3.54}$$

where the boundary term again vanishes due to the compact support of the test function.

After these transformations, we make use of the same argument that both f_μ and $f^{\perp}_{\nu\mu}$ are arbitrary and mutually independent along the curve $\mathcal{C}$, concluding that the coefficients multiplying them must each be zero. The first term gives us

$$\nabla \left(B^{\mu\nu} u_\nu + \beta \bar{B}^{\mu\nu} u_\nu \right) - 2\beta F^{(\mu}{}_{\nu\sigma} B^{\nu)\sigma} = 0 , \tag{3.55}$$

while the second term, knowing that $f^{\perp}_{\nu\mu}$ is orthogonal to the curve $\mathcal{C}$ in its first index, gives

$$\left(B^{\mu\nu} + \beta \bar{B}^{\mu\nu} \right) P^{\lambda}_{\perp\nu} = 0 . \tag{3.56}$$

As in the previous case, given that $B^{\mu\nu}$ and $\bar{B}^{\mu\nu}$ are symmetric, one can decompose them into tangential and orthogonal components using (2.70), and then from (2.74) read off that all orthogonal components must be zero, concluding that

$$B^{\mu\nu} + \beta \bar{B}^{\mu\nu} = (B + \beta \bar{B}) u^\mu u^\nu \equiv m(\tau) u^\mu u^\nu , \tag{3.57}$$

where again we emphasized that the parameter m may depend on the particle's proper time τ.

Next, substituting this into (3.55) and neglecting the term $\mathcal{O}(\beta^2)$, we obtain

$$\nabla \left(m u^\mu \right) + \beta m u^\sigma \left(F^{\mu}{}_{\nu\sigma} u^\nu + F^{\nu}{}_{\nu\sigma} u^\mu \right) = 0 . \tag{3.58}$$

Projecting onto the tangent direction u_μ and using the identity $u_\mu \nabla u^\mu = 0$, one obtains

$$\nabla m = \beta m u^\sigma \left(u^\nu u_\lambda F^{\lambda}{}_{\nu\sigma} - F^{\nu}{}_{\nu\sigma} \right) , \tag{3.59}$$

establishing that, in contrast to the classical case, here the parameter m fails to be constant. Substituting this back into the equation (3.58), after some simple algebra we obtain

$$\nabla u^{\mu} + \beta u^{\nu} u^{\sigma} P^{\mu}_{\perp\lambda} F^{\lambda}{}_{\nu\sigma} = 0\,, \tag{3.60}$$

where the parameter m cancels out of the equation. As a final step, introducing the shorthand notation $F^{\mu}_{\perp\nu\sigma} \equiv P^{\mu}_{\perp\lambda} F^{\lambda}{}_{\nu\sigma}$, we can rewrite the equation of motion in its final form

$$\nabla u^{\mu} + \beta u^{\nu} u^{\sigma} F^{\mu}_{\perp\nu\sigma} = 0\,. \tag{3.61}$$

The presence of the orthogonal projector in the second term should not be surprising. Namely, since acceleration must always be orthogonal to the velocity, the second term in the equation must also be orthogonal to velocity, and this is guaranteed by the presence of the orthogonal projector.

Equations (3.57), (3.59) and (3.61) are the main result of this thesis, and we discuss them in turn. Equation (3.57) determines the structure of the stress-energy tensor describing the point particle, as a function of tangent vectors of its world line and a scalar parameter $m(\tau)$. Formally, it has the same form as its classical counterpart (2.77), and provisionally the parameter m may be even called *effective mass*. Namely, in the rest frame of the particle, integration of the $\boldsymbol{T}^{00}$ component of the entangled stress-energy tensor over the 3-dimensional spatial hypersurface can be interpreted as the total rest-energy of the kink configuration of fields that represents the particle. This terminology is of course provisional, since all these notions are merely a part of the semiclassical approximation of the full quantum gravity description.

Equation (3.59) determines the proper time evolution of the parameter $m(\tau)$. In contrast to the classical case, where $m(\tau)$ was determined by (2.79) to be a constant, here we see that its time derivative is proportional to (covariant derivatives of) the overlap $h_{\mu\nu}$ between the dominant and sub-dominant classical geometry, via (3.45). If one puts $\beta = 0$, (3.59) reduces to the classical case, as expected. The overlap between the two geometries gives rise to an effective force that is responsible for the change in time of the particle's effective mass. Since the particle is (effectively) not isolated, its total energy is therefore not conserved, in the sense of equation (3.59).

Finally, and most importantly, equation (3.61) represents the effective equation of motion of the particle, determining its world line. It has the form of the classical geodesic equation (2.80) with an additional correction term proportional to β and to the overlap term $h_{\mu\nu}$. This additional term represents an *effective force*, pushing the particle off the classical geodesic trajectory. It is analogous to the notion of the "exchange interaction" force in molecular physics, in the region where the wavefunctions of the two electrons in a molecule overlap.

In our case, however, the force term is determined by the overlap between the two classical spacetime and matter configurations superposed in the state (3.15), and in particular by the off-diagonal components of the metric operator $\hat{g}_{\mu\nu}$, see (3.28). It is thus a *pure quantum gravity effect*, a consequence of the nontrivial structure of the metric operator. Of course, the detailed properties and the magnitude of the force term depend on the choice of the two classical gravity-matter configurations and on the details of the quantization of the gravitational field.

3.5 Consistency of the approximation scheme

Regarding the analysis and the derivation of the effective equation of motion for a particle discussed in the previous subsection, there is one issue that we should reflect on. It is related to the additional consistency conditions that stem from our assumption that the quantum correction to the stress-energy tensor is approximated with a single pole term (3.52).

Namely, the two stress-energy tensors that enter the derivation of the effective equation of motion —

the classical dominant stress-energy $T^{\mu\nu}$, and the overlap stress-energy $\bar{T}^{\mu\nu}$ — can in general be written in the single pole approximation as:

$$T^{\mu\nu} = \int_{\mathcal{C}} d\tau \, B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \mathcal{O}_1(T) \,, \tag{3.62}$$

$$\bar{T}^{\mu\nu} = \int_{\mathcal{C}} d\tau \, \bar{B}^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \mathcal{O}_1(\bar{T}) \,. \tag{3.63}$$

Note that we have introduced two different $\mathcal{O}_1$ scales, one for each tensor. This is because, although we assume that both can be expanded into the δ series around the same curve $\mathcal{C}$, each tensor may have different "width", or in other words, the two configurations of matter fields may be such that they can be well approximated with a single pole term up to a priori two different $\mathcal{O}_1$ scales. In particular, if one chooses the $\mathcal{O}_1$ scale to write $T^{\mu\nu}$ in a single pole approximation, $\mathcal{O}_1 = \mathcal{O}_1(T)$, it is not obvious that $\bar{T}^{\mu\nu}$ can also be approximated by a single pole term, compared to the same scale, and vice versa. Therefore, there is an assumption about the relationship between scales that we have made when we used expressions (3.51) and (3.52) in the derivation of the effective equation of motion.

Looking at the structure of the equation (3.50) into which (3.51) and (3.52) have been substituted, the consistency condition for the approximation scheme, where we define $\mathcal{O}_1 \equiv \mathcal{O}_1(T)$, can be written as

$$\mathcal{O}_1 \geq \beta \mathcal{O}_1(\bar{T}) \,. \tag{3.64}$$

In particular, if this inequality were not valid, the dipole term in (3.63) would contribute to (3.50) with magnitude comparable to the single pole term of (3.62), and it would be inconsistent to ignore it in the derivation of the effective equation of motion.

The consistency condition (3.64) can be rewritten into a more explicit form. Substituting (3.63) and (3.30) into (3.64), we get

$$\mathcal{O}_1(T) \geq \beta \left[2\epsilon \, \mathrm{Re} \left(\langle \Psi | \hat{T}_{\mu\nu} | \Psi^\perp \rangle \right) - \int_{\mathcal{C}} d\tau \, \bar{B}^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \right] . \tag{3.65}$$

In addition, one can use (3.57) and (3.62) to eliminate the coefficient $\bar{B}^{\mu\nu}$ in favor of $T^{\mu\nu}$ and $m(\tau)$, which are arguably more observable, obtaining

$$\mathcal{O}_1(T) \geq 2\beta\epsilon \, \mathrm{Re} \left(\langle \Psi | \hat{T}_{\mu\nu} | \Psi^\perp \rangle \right) + T^{\mu\nu} - \int_{\mathcal{C}} d\tau \, m(\tau) u^\mu u^\nu \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \,. \tag{3.66}$$

This inequality should be interpreted as follows. Given an explicit model of quantum gravity, and within it an explicit configuration of matter fields that make up a particle, one can estimate all three quantities on the right-hand side of (3.66), namely the off-diagonal components of the stress-energy operator, its expectation value in the dominant coherent state, and the total mass of the particle, respectively. Then, the consistency condition (3.66) gives a lower bound on the scale $\mathcal{O}_1$, which represents an estimate of the error when discussing the effective equation of motion for the particle. In other words, the equation of motion can be considered to be approximately valid only across scales much larger than the $\mathcal{O}_1$ scale, bounded from below by inequality (3.66).

Finally, if one needs better precision than the scale determined by (3.66), one should take into account the dipole term in (3.63) and rederive a more precise form of the equation of motion. Still better precision would be obtained by including the dipole term in (3.62), which would amount to the equation of motion in the full pole-dipole approximation, and so on. This inequality should be interpreted as follows. Given an explicit model of quantum gravity, and within it an explicit configuration of matter fields that make up a particle, one can estimate all three quantities on the right-hand side of (3.66), namely the off-diagonal components of the stress-energy operator, its expectation value in the dominant coherent state, and the total mass of the particle, respectively. Then, the consistency condition (3.66) gives a lower bound on the scale $\mathcal{O}_1$, which represents an estimate of the error when discussing the effective equation of motion for the particle. In other words, the equation of motion can be considered to be approximately valid only across scales much larger than the $\mathcal{O}_1$ scale, bounded below by inequality (3.66).

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Chapter 4

Status of the weak equivalence principle

In light of the results of chapter 3, it is important to discuss the status of the equivalence principle (EP). Throughout the literature, one can find various different formulations of EP, in various flavors such as weak, medium-strong, strong, and so on (see [10, 28] for a review, and [1, 9, 11, 22, 31, 44] for various examples). Often these formulations and flavors are interpretation-dependent, and it is not always clear whether they are mutually equivalent or not, and what are the underlying assumptions and definitions used to express them.

Needless to say, such situation is less than satisfactory [10, 28], and in order to circumvent it, in this chapter we opt to specify one particular definition of the weak and strong equivalence principles (WEP and SEP, respectively) and to use this definition in the rest of the text. We do not aspire to claim that our definition is either equivalent to, or in any sense better than, other definitions present in the literature, and may not even correspond to the usual usage of the terminology. But for the purpose of clarity, it is prudent to fix one definition and stick to it. Therefore, in light of the results obtained in chapter 3, in this chapter we discuss the status of WEP defined as below.

4.1 Definition and flavors of the equivalence principle

The purpose of the equivalence principle is to prescribe the coupling of matter to gravity [25]. Its precise formulation therefore depends on the particular choice of the gravitational and matter degrees of freedom which one uses to describe matter and gravity. For the purpose of this thesis, we assume that the classical limit of quantum gravity corresponds to general relativity, which means that in this limit the fundamental gravitational degrees of freedom give rise to a nonflat spacetime metric. Given any choice of the gravitational degrees of freedom that belong to this class, in the classical framework one can formulate the equivalence principle as a two-step recipe to couple matter to gravity (we will discuss the quantum framework in subsection 4.2).

Start from the classical equation of motion for matter degrees of freedom in flat spacetime, written symbolically as

$$\mathcal{D}_{\text{flat}}[\phi, \eta_{\mu\nu}] = 0 \,, \tag{4.1}$$

where ϕ denotes the matter degrees of freedom, $\eta_{\mu\nu}$ is the Minkowski metric, while $\mathcal{D}_{\text{flat}}$ is an appropriate functional describing the equation of motion for ϕ in flat spacetime and is assumed to be local. Given this equation of motion, couple it to gravity as follows:

1. Rewrite the equation of motion in a manifestly diffeomorphism-invariant form, typically by a change

of variables to a generic curvilinear coordinate system,

$$\mathcal{D}_{\text{curvilinear}}[\phi, g^{(0)}_{\mu\nu}] = 0\,, \tag{4.2}$$

where $g^{(0)}_{\mu\nu}$ is still the flat spacetime metric, appropriately transformed from $\eta_{\mu\nu}$, and similarly for $\mathcal{D}_{\text{curvilinear}}$.

2. Promote the curvilinear equation of motion to the equation of motion in curved spacetime by replacing the flat spacetime metric $g^{(0)}_{\mu\nu}$ with an arbitrary metric $g_{\mu\nu}$,

$$\mathcal{D}_{\text{curvilinear}}[\phi, g_{\mu\nu}] = 0\,, \tag{4.3}$$

thereby specifying the equation of motion for matter coupled to gravity.

The first step describes the matter equation of motion from a perspective of a generic curvilinear (or "arbitrarily accelerated") coordinate system, reflecting the principle of *general relativity*. The second step simply promotes that same equation to curved spacetime as it stands, with no additional coupling of any kind. This can be loosely formulated as a statement of *local equivalence between gravity and acceleration*, which is how the EP historically got its name. Also, note that these two steps operationally correspond to the standard *minimal coupling* prescription [25].

It is important to stress the *local* nature of EP, which manifests itself in the assumption that the initial equation of motion (4.1) is local, and that the EP essentially does not change it at all, at any given point in spacetime. This has one important implication — the gravitational degrees of freedom manifest themselves only through *nonlocal measurements*, as tidal effects induced by spacetime curvature. We will return to this point and comment more on it later in the text.

Depending on the further specification of the matter degrees of freedom, one can distinguish between various flavors of the EP. For example, if one talks about mechanics of point particles, one can start from the Newton's first law of motion, which states that in the absence of any forces, a particle has a straight-line trajectory in Minkowski spacetime. According to the step 1 above, the differential equation for a straight line in a generic curvilinear coordinate system is the geodesic equation,

$$\frac{d^2 z^\lambda}{d\tau^2} + \mathit{\Gamma}^\lambda_{(0)\mu\nu} \frac{dz^\mu}{d\tau}\frac{dz^\mu}{d\tau} = 0\,, \tag{4.4}$$

where the index (0) on the Christoffel symbol indicates that it is calculated using the metric $g^{(0)}_{\mu\nu}$, which is obtained by a curvilinear coordinate transformation from the Minkowski metric $\eta_{\mu\nu}$. Then, according to step 2, one again writes the same equation, only dropping the requirement of flat spacetime metric,

$$\frac{d^2 z^\lambda}{d\tau^2} + \mathit{\Gamma}^\lambda{}_{\mu\nu} \frac{dz^\mu}{d\tau}\frac{dz^\mu}{d\tau} = 0\,, \tag{4.5}$$

so that this time the Christoffel symbol is calculated using an arbitrary metric $g_{\mu\nu}$, and now encodes the interaction with the gravitational degrees of freedom. So one starts from the Newton's first law of motion for a particle in the absence of the gravitational field, and ends up with a geodesic equation in the presence of the gravitational field. We define this flavor of the EP as the *weak equivalence principle* (WEP).

Instead of mechanical particles, one can study matter degrees of freedom described by a field theory. For example, if one starts from the equation of motion for a single real scalar field,

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\right)\phi = 0\,, \tag{4.6}$$

according to the step 1 of the EP, one can rewrite it in a general curvilinear coordinate system as

$$\left(g^{\mu\nu}_{(0)} \nabla_\mu \nabla_\nu - m^2 \right) \phi = 0 \,, \tag{4.7}$$

where the Christoffel symbol inside the covariant derivative is again calculated using the flat-space metric $g^{(0)}_{\mu\nu}$. Then, according to step 2 of the EP, this equation is promoted to curved spacetime as it stands, leading to

$$\left(g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2 \right) \phi = 0 \,, \tag{4.8}$$

where now the covariant derivative is given with respect to an arbitrary metric $g_{\mu\nu}$ describing curved spacetime. Thus one arrives to the equation of motion for a scalar field coupled to gravity. We define this flavor of the EP as the *strong equivalence principle* (SEP).

So in short, WEP is a statement about mechanical systems such as particles and small bodies, while SEP is a statement about fields. We emphasize again that the above definitions may or may not correspond to what is known in other literature as WEP and SEP, depending on the particular source one compares our definitions to. For example, one can often find a definition of WEP as a statement about equality of inertial and gravitational masses. As another example, one can also find a definition of WEP as Galileo's statement that the acceleration of a particle due to the gravitational field is independent of the particle's internal details such as mass or chemical composition, a property also called *universality*, emphasizing the fact that gravitation interacts with all types of particles in the same way. For an excellent review of the various formulations and flavors of EP present in the literature, see [28].

In relation to these alternative formulations of WEP, one should note two comments. First, while the notion of "gravitational mass" may be useful in the context of Newtonian theory, in frameworks such as GR it is not useful, since the source in Einstein's equations is the whole stress-energy tensor, rather than any particular mass-like parameter. This renders any definition of WEP which relies on the notion of the gravitational mass unsuitable for analysis in a fundamental QG framework. Second, one can argue (see for example [28]) that the property of universality is implicitly present even without gravitational interaction, in the Newton's first law of motion. Namely, the first Newton's law can be formulated more precisely as follows: in the absence of any forces, a particle has a straight-line trajectory in Minkowski spacetime, *regardless of its internal details such as mass or chemical composition.* The Newton's first law is never spelled out in this way in textbooks, making room for a point of view that universality has something to do with gravity or the EP. However, if one accepts our definition of WEP given above, it is more natural to say that universality is a property of Newtonian mechanics, and is merely *being preserved* by the WEP when one lifts the straight-line equation of motion to curved spacetime. So from this point of view, one should arguably say that WEP is merely *compatible* with universality, rather than equivalent to it.

Given all these reasons, and despite the fact that these alternative definitions of WEP may be suitable in various other contexts, they are not quite adequate for the analysis given in this thesis. We therefore choose to retain our own definition of WEP, while the principles of universality and equality between inertial and gravitational mass will be called as such. They are discussed in more detail in subsection 4.3.

4.2 Equivalence principle and quantum theory

Adopting the above definitions of WEP and SEP, it is important to discuss their relationship. From the perspective of the classical field theory (CFT), the notion of a particle can be introduced as a localized kink-like configuration of matter fields, described as a solution of the (usually quite complicated) matter field equations. One can then employ the apparatus of multipole formalism and describe the evolution of this kink configuration in the single pole approximation, as was discussed in section 2.4. Using this method, one can recover the equation of motion for a particle in classical mechanics (CM) as an approximation of the field theory. Moreover, all this can be done before or after the application of the EP, leading to the following diagram:

$$
\begin{array}{ccc}
\mathrm{CFT}_\eta & \xrightarrow{\text{single pole approx.}} & \mathrm{CM}_\eta \\
\Big\downarrow{\scriptstyle \text{SEP}} & & \Big\downarrow{\scriptstyle \text{WEP}} \\
\mathrm{CFT}_g & \xrightarrow{\text{single pole approx.}} & \mathrm{CM}_g
\end{array}
$$

Here the indices η and g indicate that equations of motion in a given theory are written in flat and in curved spacetime, respectively.

The question whether this diagram commutes is nontrivial. Namely, on one hand, one can start from a flat-space classical field theory, approximate it to derive the equations of motion for a particle in flat-space classical mechanics, and then invoke WEP to reach classical mechanics coupled to gravity. On the other hand, one can first invoke SEP to couple matter to gravity at the field theory level, and then approximate it to derive the equation of motion for a particle in curved spacetime. A priori, there is no guarantee that one will reach the same equation of motion for a particle in curved spacetime using both methods.

It is in fact the existence of the local Poincaré symmetry that leads to the commutativity of the diagram. Namely, as was discussed in section 2.4, in the curved spacetime local Poincaré symmetry gives rise to the covariant conservation equation for the stress-energy tensor of matter fields, and this is all one needs to reach the geodesic equation as an equation of motion for the particle, in the sense that one does not need to know the full matter field equations in curved spacetime. This establishes the $\langle$SEP $\rightarrow$ single pole$\rangle$ path of the diagram. On the other hand, in flat spacetime one can also perform the calculation of section 2.4, this time using the ordinary (noncovariant) conservation equation for the stress-energy tensor, which is a consequence not of the local, but rather of the global Poincaré invariance of Minkowski spacetime. Repeating the calculation of section 2.4 with the symbolic substitutions $g \rightarrow \eta$ and $\nabla \rightarrow \partial$, it is not hard to conclude that one will obtain the equation of motion for a straight line in flat spacetime, again without knowing all details of the full matter field equations in flat spacetime. Then, applying WEP as discussed in section 4.1, one reaches the geodesic equation in curved spacetime. This establishes the $\langle$single pole $\rightarrow$ WEP$\rangle$ path of the diagram, concluding that the resulting equation of motion for the particle is the same in both cases, i.e., that the diagram commutes.

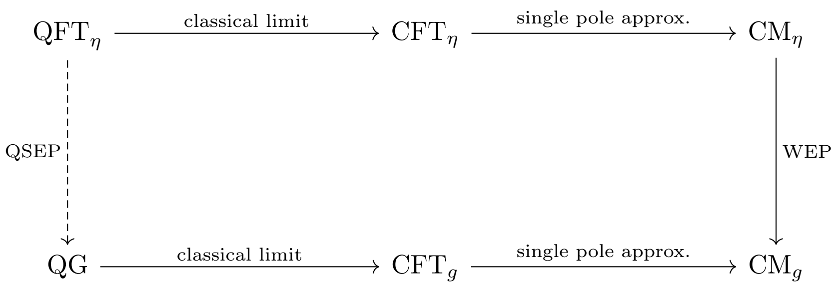
Let us also note that, going beyond the single pole approximation, WEP is known to be violated, with SEP remaining valid. For example, in the pole-dipole approximation, it is well known that the analogous diagram

$$
\begin{array}{ccc}
\mathrm{CFT}_\eta & \xrightarrow{\text{pole-dipole approx.}} & \mathrm{CM}_\eta \\
\Big\downarrow{\scriptstyle \text{SEP}} & & \Big\downarrow{\scriptstyle \text{WEP}} \\
\mathrm{CFT}_g & \xrightarrow{\text{pole-dipole approx.}} & \mathrm{CM}_g
\end{array}
$$

fails to commute. Namely, the $\langle$SEP $\rightarrow$ pole-dipole$\rangle$ path leads to an effective equation of motion for the particle in which there is explicit coupling of the particle's total angular momentum to the spacetime curvature [29]. On the other hand, the ⟨pole-dipole → WEP⟩ path produces the equation of motion without the curvature term. Thus, in the pole-dipole approximation, WEP fails to reproduce the correct equation of motion, since the particle is coupled to gravity in a nonminimal way, in spite of the fact that the fields which make up the particle are still minimally coupled to gravity, in line with SEP. Of course, this situation is to be expected, given that in the pole-dipole approximation the particle is no longer completely pointlike, and the coupling of the angular momentum to the curvature can be understood as a tidal effect of gravity across the "width" of the particle. On the other hand, one can instead argue that it would be wrong to apply WEP to the pole-dipole equation of motion for a particle. Namely, despite the fact that the latter is formally still local, it describes an object that is "less-than-perfectly pointlike", in the sense that its stress-energy tensor is proportional not only to a δ function but also to its derivative. From that point of view, one should not be allowed to apply the two-step prescription of EP defined above. Either way, the bottom line is that one can either declare WEP as violated or as inapplicable beyond the single pole approximation, but it cannot be declared as valid. This results in the noncommutativity of the above diagram.

Let us now turn to the quantum theory. Starting first from some quantum field theory (QFT_η) which describes the fundamental matter fields in Minkowski spacetime, one can take its classical limit, giving rise to some classical field theory (CFT_η). Then, assuming that the latter features kink solutions, one can describe those using the single pole approximation, leading to classical mechanics (CM_η) of the corresponding particles. Finally, applying WEP one couples those particles to gravity. The resulting equation of motion will always be a geodesic equation, assuming that the initial QFT and all subsequent approximations respect the global Poincaré invariance of Minkowski spacetime. This symmetry guarantees the conservation of the stress-energy tensor of the matter fields throughout the sequence of approximations, leading invariably to the geodesic equation of motion for the particle.

On the other hand, it is arguably more appropriate to take an alternative, more fundamental route — start from some fundamental quantum gravity (QG) model, and take the classical limit leading to some classical field theory (CFT_g) for both matter and gravity. Then, again assuming that this theory features kink solutions, employ the single pole approximation to obtain the classical mechanics for the particle in the gravitational field (CM_g). Note that this is in fact precisely the program that was performed in chapter 3, leading to the non-geodesic equation of motion (3.61) for the particle. In effect, one can conclude that the following diagram fails to commute:



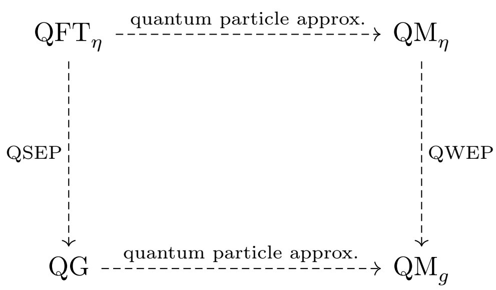
As a side comment, note that the dashed QSEP arrow represents some hypothetical map leading from a QFT in Minkowski spacetime to a full-blown model of QG, according to a notion that might be called a "quantum strong equivalence principle". It is unclear whether such a principle exists or not, let alone what its formulation is supposed to be, even if one is given precisely defined models of QFT and QG in question. We introduce it here simply for completeness, speculating that such a notion should exist, as a generalization of SEP from classical to quantum physics. It is also convenient to introduce it, in order to close the diagram and discuss its commutativity.

It is important to stress the reason why this diagram does not commute. Recalling the details of chapter 3, the local Poincaré symmetry is assumed to be respected at the fundamental level of QG and onwards, just like in the classical case. Moreover, the single pole approximation is used, avoiding any nonminimal coupling of the tidal forces that may be present. And yet, in spite of all that, the resulting equation of motion is not a geodesic. Looking at the equation of motion (3.61), the reason for this is the nontrivial overlap between coherent states describing two classical configurations of matter, and more importantly, of gravity. In other words, the deviation from the geodesic motion is a *pure quantum gravity effect* — it is not present in the classical case, nor in the case of quantum matter in classical Minkowski spacetime. A testimony of this fact is the quantum correction term for the metric (3.28), which features off-diagonal matrix elements of the metric operator $\hat{g}_{\mu\nu}$:

$$h_{\mu\nu} = 2\epsilon_G \, \mathrm{Re} \left(S_M \langle g|\hat{g}_{\mu\nu}|g^{\perp}\rangle \right) . \tag{4.9}$$

In this sense, due to the noncommutativity of the above diagram, one can argue that (within the discussed framework) *quantum gravity violates the weak equivalence principle.* Nevertheless, we would like to stress that our discussion regarding both strong and weak equivalence principles, based on the above prescription from subsection 4.1, is inherently *classical.* Indeed, in steps 1 and 2 which define the implementation of EP, one considers classical equations of motion. In our case, such definition suffices, as our entangled state (3.15) consists of a dominant and a sub-dominant term. Thus, we could expand our entangled equations (3.27) and (3.29) around the dominant classical terms, and discuss WEP in such a scenario. In fact, according to the definition of WEP, in general one can discuss its violation only *with respect to* some (perhaps unspecified, but assumed) classical spacetime metric. In our case, this role is played by the dominant classical metric $g_{\mu\nu}$.

In the more general case of superpositions of states which are more equally weighted, $\alpha \approx \beta$, and which consist of almost orthogonal states, $\langle\Psi|\tilde{\Psi}\rangle \approx 0$, the classical definitions of SEP and WEP are inapplicable. Therefore, both equivalence principles ought to be generalized to their respective *quantum* versions, denoted QSEP and QWEP respectively, in the sense of the following diagram:



Note that here all arrows are dashed, indicating the speculative nature of all these maps. Also, QM_g represents a hypothetical theory of quantum particles coupled to a quantum gravitational field.

Regarding the quantum weak equivalence principle (QWEP), one could try to define it in terms of the classical WEP, applied separately to each "branch" in the superposition. As long as the two "branches" $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ are themselves classical (coherent) states, corresponding to the respective solutions of Einstein's equations, such a definition might seem suitable. Note that this approach is compatible with the notion of a superposed observer, i.e., an observer that is, with respect to observed object(s), in a superposition of different states of motion (see recent work [19] and the references therein). However, the formulation of the quantum strong and weak equivalence principles for the case of generic non-coherent quantum states is an open question, outside of the scope of the current work.

Finally, the quantum version of the single pole approximation, called "quantum particle approximation" in the diagram above, is also not well defined — neither conceptually nor technically. Essentially, the whole diagram represents merely a speculation about the prescriptions which ought to map between the respective theories. In addition, like in the previous cases, the commutativity of the diagram (i.e., the violation of QWEP, given the validity of QSEP) would also be an open question. In some sense, the QSEP would represent a "true" equivalence principle, while QWEP would be a particle-like approximate image of QSEP. Being approximate, QWEP could possibly be violated in some cases, giving rise to noncommutativity of the diagram.

4.3 Universality, gravitational and inertial mass

In light of the results of chapter 3, in addition to the discussion of WEP violation, it is also important to discuss the status of the principle of universality, and the principle of equality between inertial and gravitational masses. In order to discuss them, it is instructive to study the Newtonian limit of the effective equation of motion (3.61), as follows.

We define the Newtonian limit in the standard way [25] — by assuming small spacetime curvature, a static field, nonrelativistic motion, and ignoring the backreaction of the particle on the background spacetime geometry. These approximations are implemented in the following way. First, ignoring the backreaction of the particle allows us to choose the dominant classical metric $g_{\mu\nu}$ as specified by the Newtonian line element

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + dx^2 + dy^2 + dz^2\,, \tag{4.10}$$

where $x^\mu \equiv (t, x, y, z)$ are spacetime coordinates, M is the mass of the gravitational source, $r \equiv \sqrt{x^2 + y^2 + z^2}$, and $G \equiv l_p^2$ is Newton's gravitational constant. We will discuss the motion of a test-particle in this background, given by the effective equation of motion (3.61). Second, the assumption of nonrelativistic motion of the particle allows us to neglect its spacelike velocity,

$$u^k \equiv \frac{dz^k}{d\tau} \approx 0\,, \tag{4.11}$$

leaving only the timelike component $u^0 \equiv dz^0/d\tau$ nonzero (the position of the particle $z^\mu(\tau)$ should not be confused with the label for the third spatial coordinate $z \equiv x^3$). Finally, the assumption of small spacetime curvature allows us to neglect all terms of order $\mathcal{O}(M^2)$ and higher.

Given this setup, one can easily calculate all nonzero Christoffel symbols corresponding to the dominant metric, obtaining:

$$\varGamma^0{}_{0k} = \varGamma^0{}_{k0} = \varGamma^k{}_{00} = \frac{GM}{r^3} x^k\,, \qquad k \in \{1, 2, 3\}\,. \tag{4.12}$$

One can then employ them to write the time and space components for the particle's effective equation of motion (3.61). Using (2.70) and (3.45), after some straightforward algebra, the time component of the equation of motion reduces to

$$\frac{d^2 z^0(\tau)}{d\tau^2} = 0\,, \tag{4.13}$$

owing to the normalization condition $u^\mu u^\nu g_{\mu\nu} = -1$ and the presence of the orthogonal projector in (3.61). Using convenient initial conditions, this equation can be integrated to make an identification between the proper time τ and the time component of the particle's parametric equation of trajectory $x^\mu = z^\mu(\tau)$ as

$$t = z^0(\tau) = \tau\,, \tag{4.14}$$

reflecting the notion of global universal time of Newtonian theory. Using this result, one can show that the space components of the particle's equation of motion obtain the following form (note that the spacelike indices can be raised and lowered at will, since the spatial part of the metric (4.10) is a unit matrix):

$$\frac{d^2 z^k}{d\tau^2} + \frac{GM}{r^3} z^k + \beta \left[\partial_0 h_{0k} - \frac{1}{2}\partial_k h_{00} - \frac{GM}{r^3} z^j h_{jk} \right] = 0 \,. \tag{4.15}$$

Note that here r has been evaluated at the position of the particle, $r = \sqrt{z^k z_k}$, and similarly for the gradients of h_{0k} and h_{00}. The first two terms in the equation come from the classical geodesic part ∇u^k in (3.61), while the third term is the quantum correction, coming from the effective force term $\beta u^\nu u^\sigma F^k_{\perp\nu\sigma}$.

The most important aspect of equation (4.15) is the similarity between the second term of the classical part and the final term of the quantum correction. The spacelike components h_{jk} can be separated into the trace and traceless part,

$$h_{jk} \equiv \frac{1}{3} h^i{}_i \, \delta_{jk} + \tilde{h}_{ij} \,, \qquad \tilde{h}^k{}_k \equiv 0 \,, \tag{4.16}$$

and the trace can be grouped together with the classical term, giving

$$\frac{d^2 z^k}{d\tau^2} + \frac{GM}{r^3} z^k \left(1 - \frac{1}{3}\beta h^i{}_i \right) + \beta \left[\partial_0 h_{0k} - \frac{1}{2}\partial_k h_{00} - \frac{GM}{r^3} z^j \tilde{h}_{jk} \right] = 0 \,. \tag{4.17}$$

Finally, multiplying the whole equation (4.17) with an arbitrary positive number, called the particle's *inertial mass* and denoted m_I, it takes the form of the Newton's second law of motion,

$$m_I \frac{d^2 z^k}{d\tau^2} = -m_I \left(1 - \frac{1}{3}\beta h^i{}_i \right) \frac{GM}{r^3} z^k - \beta m_I \left[\partial_0 h_{0k} - \frac{1}{2}\partial_k h_{00} - \frac{GM}{r^3} z^j \tilde{h}_{jk} \right] . \tag{4.18}$$

One can recognize two force terms on the right-hand side. The second term is of purely quantum origin, and represents the effective force acting on the particle ultimately due to the presence of the sub-dominant quantum state $|\tilde{\Psi}\rangle$ in (3.15). It has a non-Newtonian form, in the sense that none of its parts can be grouped together with the first force term, as was done with the trace part. The first force term, however, can be recognized as the classical Newton's gravitational force law, provided that one defines the ratio between the *gravitational mass* m_G and the *inertial mass* m_I of the particle as

$$\frac{m_G}{m_I} \equiv \left(1 - \frac{1}{3}\beta h^i{}_i \right) . \tag{4.19}$$

At this point we are ready to discuss the principles of universality and of the equality between gravitational and inertial masses. To begin with, it is obvious from (4.19) that the gravitational mass is equal to the inertial mass only up to a quantum correction term. This term contains the trace of spatial components of the metric overlap tensor $h_{\mu\nu}$, defined by equation (3.28), from which we obtain

$$h^i{}_i = 2\delta^{ij} \epsilon_G \, \mathrm{Re} \left(S_M \langle g | \hat{g}_{ij} | g^\perp \rangle \right) . \tag{4.20}$$

It is crucial to notice that, in addition to the dependence of the off-diagonal matrix element of the metric operator, this expression also depends on the overlap $S_M \equiv \langle \phi | \tilde{\phi} \rangle$ of matter fields, which include the particle itself. Therefore, the term in the parentheses in (4.19) cannot be reabsorbed into the constants G and M, since these describe the external source of gravity which should remain independent of the properties of the test particle. Thus, the only possibility to cast the first force term in (4.18) into the form of the Newton's law of gravitation, is to define the ratio between the gravitational and the inertial mass as in (4.19). As a consequence, the principle of equality between gravitational and inertial masses is violated by the presence of the correction term coming from quantum gravity.

A similar argument can be made regarding the principle of universality. One may cancel away the inertial mass from the Newton's law (4.18), returning to (4.17) which describes the acceleration of the particle in the presence of an external gravitational field. Again, the presence of (4.20) in the classical gravitational acceleration term guarantees that this term depends not only on the external gravitational source, but also on the structure of the test particle itself. Moreover, the remaining quantum correction terms also depend on $h_{\mu\nu}$, and therefore they too carry information about the internal structure of the particle. In this sense, test particles described by different matter configurations may have different overlap terms S_M, and therefore display different accelerations, given the same background gravitational field. This means that the principle of universality is violated by the presence of the quantum gravity correction terms.

As a final comment, we should also note that m_I (and consequently m_G as well) is a completely free parameter in the Newtonian setup, and should be determined by the interactions of nongravitational type. In particular, the Newtonian framework does not allow us to establish any connection between m_I, m_G and the effective mass m of the particle, discussed in the context of (3.57) and (3.59). This is because the total rest-energy of a particle is an inherently relativistic concept, not defined in Newtonian mechanics. On the other hand, if one goes to the relativistic framework, the notions of inertial and gravitational masses become ill-defined, since gravitational interaction cannot be described anymore by a mere force law in the Newtonian sense. Therefore, the relationship between m_I, m_G and m remains undefined.

Chapter 5

Conclusions

5.1 Summary of the results

In this thesis, we have discussed the effective motion of a point particle within the framework of quantum gravity, in particular the case where both matter and gravity are in a quantum superposition of the Schrödinger cat type. In chapter 2 we gave a recapitulation of the results of the classical theory, introducing classical field theory, spacetime symmetries and the Noether's theorem, the multipole formalism framework and illustrating the derivation of the geodesic equation for the motion of a particle in GR. Chapter 3 was devoted to the generalization of these results to the realm of the full quantum gravity. In section 3.1 we introduced the canonical quantization, in section 3.2 we introduced the abstract quantum gravity framework, discussed the model of the superposition of two coherent classical states, and established the main assumptions for the derivation of the effective equation of motion. In section 3.3 we have analyzed in detail the quantum version of equation for the covariant conservation of stress-energy tensor, which is a crucial ingredient in the derivation of the effective equation of motion. The explicit derivation of the equation of motion itself was then given in section 3.4, giving rise to the main results of the thesis — the equation for the stress-energy kernel (3.57), the equation for the time-evolution of the particle's mass (3.59), and the effective equation of motion for the particle (3.61). Most importantly, the effective equation of motion turns out to contain a non-geodesic term, giving rise to an effective force acting on the particle, as a consequence of the overlap terms between the two coherent states of the gravity-matter system. The last section 3.5 discusses the self-consistency of the assumptions used in the above analysis, giving rise to the equation (3.66) for the error estimate of the single pole approximation scale.

In light of the nongeodesic motion established in chapter 3, it is important to discuss it in the context of the equivalence principle. This topic was taken up in chapter 4. After we have defined various flavors of the equivalence principle in section 4.1, the main analysis was presented in section 4.2, establishing the violation of (various forms of) the weak equivalence principle, as a consequence of the nongeodesic correction to the equation of motion (3.61). Also, given the inherently classical nature of the equivalence principle, we have also speculated on possible generalizations to the quantum realm, introducing the notions of the quantum strong and weak equivalence principles, albeit without giving explicit statements about their definitions. Finally, in section 4.3 we have discussed the notions of universality and equality between inertial and gravitational masses in the context of quantum gravity, by studying the Newtonian limit of the equation of motion (3.61). This analysis gave a clear interpretation that both universality and the equality between gravitational and inertial masses are violated in our context, corroborating the conclusions of the abstract analysis of the EP given in section 4.2.

5.2 Discussion of the results

By far the most interesting topic to discuss in the context of the equation of motion (3.61) is how to estimate the magnitude of the nongeodesic term. As far as the analysis of this thesis goes, we can only say that this term is very small, given that it is proportional to β, which is in turn bounded from above by phenomenological argument that we do not observe superpositions of the gravitational field in nature. However, aside from this qualitative argument, in order to estimate the actual magnitude of the nongeodesic term one would need to go beyond the abstract quantum gravity formalism, and construct an explicit quantum gravity model coupled to matter fields, find some explicit kink solutions of the matter sector, and then calculate the overlap terms and the off-diagonal terms of the metric operator. Of course, any estimate obtained in such a way would be model-dependent. We consider this to be a feature of the abstract quantum gravity approach, since the magnitude of the nongeodesic term represents one way to operationally distinguish between different QG models. In other words, one could use equation (3.61) to experimentally test and compare these models, at least in principle. Probably the most obvious such test would employ equation (4.19) which relates the gravitational and inertial mass of the particle.

One result that was not discussed in detail is the nonconservation law for the effective mass of the particle, (3.59). However, it is not really surprising that the particle's total rest energy fails to be constant in the presence of gravity-matter entanglement. As (3.59) tells us, the nonconservation is actually a consequence of the additional effective force, which is itself a consequence of the quantum overlap between two classical geometries and matter states. Nevertheless, it would indeed be interesting to study the mass nonconservation in more detail.

It is also important to discuss the generalization of our results from the case of the superposition of two coherent states to many coherent states. In particular, one could discuss the case where the state $|\tilde{\Psi}\rangle$ in (3.15) is not a single coherent classical state, but a superposition of many coherent states,

$$|\tilde{\Psi}\rangle = \sum_i \gamma_i |\Psi_i\rangle \,. \tag{5.1}$$

As long as we maintain the assumption that $\alpha \gg \beta$ in (3.15), it is straightforward to see that all our results and conclusions still hold in the generic case. Therefore, there is no substantial difference in the analysis of a state which is a superposition of two coherent states, compared to the analysis of a superposition of many coherent states, as long as one of them is dominant while all others are subdominant. Detailed quantitative description is technically more complicated, but qualitatively all results will hold for both types of states.

5.3 Future lines of research

One of the main lines of future work would be to perform a similar analysis as was done in this thesis, but keeping the β^2 terms. This would naturally include the sub-dominant effective metric and stress-energy (3.16), giving qualitatively new insight into the notion of quantum superpositions of two classical geometries. That analysis might provide clues about the properties of quantum gravity which could arguably hold even in the equal-weight superpositions, defined by the choice $\alpha \approx \beta \approx 1/\sqrt{2}$ in (3.15).

Alternatively, one could repeat the analysis of this thesis, but in a pole-dipole approximation. This would also lead to novel effects, one of which might be a coupling of various quantum interference terms to the spacetime curvature and the angular momentum of the particle, generalizing the classical pole-dipole equation of motion [29].

Also, given that the multipole formalism is also applicable to Riemann-Cartan spacetimes [26, 27, 40, 43], the analysis of this thesis could be generalized to include coupling of quantum interference terms to spacetime torsion and the spin of the particle.

Finally, one could further discuss a more general setup in which the off-diagonal terms in the covariant conservation equation (3.36) are not ignored, in the sense of going beyond the approximations (3.38) and (3.39).

In addition to all of the above, one important line of research would be to study possible connections to experiments. First, one shoud study the counterpart of the so-called *geodesic deviation equation*. Namely, in GR, the geodesic motion as such is not observable, as a consequence of the equivalence principle. As we have emphasized in subsection 4.1, the EP dictates that the only way to observe gravitational degrees of freedom is via *nonlocal measurements*, which are not encoded in the geodesic equation. Therefore, what one can actually observe is the change in the relative separation of two nearby geodesic trajectories, due to the tidal effects. This is in turn described by the geodesic deviation equation, which explicitly features the Riemann curvature tensor. In our case, the equation of motion (3.61) is not a geodesic, but is still local in character, in the sense that it does contain gravitational degrees of freedom at the given point, but still it does not combine gravitational degrees of freedom of two or more points. Thus, one ought to compare the trajectories of two nearby particles, both following a trajectory determined by (3.61). The equation governing the separation between two particles in such a setup would be a counterpart to the geodesic deviation equation of GR with a corresponding quantum correction term. It should be derived and studied in detail, in order to better understand what effects could be in principle directly experimentally observable.

Second, one could also test our results by measuring the violation of the universality and of the equality of the gravitational and the inertial mass in the semiclassical Newtonian limit.

The above list of possible topics for further research is of course not exhaustive — one can probably study various additional aspects and topics related to this work, in particular giving more precise meaning to the notions of the quantum strong and weak equivalence principles.

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