Writing a pricer in a recombining tree for CDS Options using a HJM model(Cheyette)

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Abstract

Concerning credit derivatives, Credit Default Swaps (CDSs) and Credit Default Swap Options (CDS options or CDSwaptions) are amongst of the most popular credit products. The constant growth of financial markets rises the need of researching for new methods and models that would allow us to price these complex financial instruments. The main goal of this thesis is the computation of the price of a credit single-name option, in a recombining tree based on a Cheyette model.
Keywords: default intensity, forward survival probabilities, implied volatility, Credit Default Swaps, Credit Default Swap options.

1. Introduction

In the financial market world, credit derivatives, like Credit Default Swaps (CDSs) and Credit Default Swap Options (CDS options or CDSwaptions) have had an increasingly relevant role. So, arises the need of researching for new methods and models that would allow us to price these complex financial instruments in a more efficient, parsimonious and accurate way.

The Quantitative research team from BNP Paribas Corporate & Institutional Banking (CIB) proposed the topic of this thesis, which is to write a pricer in a recombining tree for a CDS option based on a Markov representation of a Heath, Jarrow and Morton (HJM) model (i.e, Cheyette model[1]). In other words, based on a default intensity model [2], our main goal is to create an algorithm that computes the price for a CDS option at the valuation date. In order to do that, we will adapt the numerical procedure of Li, Ritchken and Sankarasubramanian (LRS) [3] to our particular credit derivative. Then, we will perform a sensitivity study where we pretend to study how a change in the input parameters impacts important features of CDS options.

As an inspiration and guidance of our study, the HJM model, given the (default-free) zero coupon bond prices, developed a general framework for modelling the interest rate dynamics, in continuous time. However, their model does not ensure that the term structure evolution of the forward rates is Markovian with respect to a finite number of state variables [2]. Since the forward and short rate processes may not have a Markovian character, implies some problems in numerical implementations, for instance, non-recombining trees.

In order to overcome this issues, Ritchken and Sankarasubramanian (RS) and Cheyette proposed two similiar approaches. In the first case, RS [4]provided some conditions on the volatility structure of forward rates that allows the evolution of the term structure to be represented by a two-dimensional state variable Markov process. In the second case, given an arbitrary initial term structure, Cheyette [1] derived a class of non-arbitrage term structure models with a Markovian character for a finite number of state variables by restricting the forward rate volatility function. By doing that, his framework avoids some numerical problems and guarantees a Markovian-character.

In the paper from Li, Ritchken and Sankarasubramanian (LRS) [3], an efficient algorithm were built for pricing European and American claims, based on the HJM model, with the same class of volatility structures presented in RS [4]. Finally, in Krekel [2], they used the model developed by Cheyette in order to model the default intensity and applied to real data, using two different approaches involving the finite-difference method.

This thesis is structured as follows. In Chapter 2, we will provide some fundamental financial concepts and conventions about credit derivatives. Then, we will present a default intensity model (derived in [2]). In Chapter 3, we will describe how to adapt the numerical method from LRS for the case of CDS options and the implementation of the algorithm developed. In Chapter 4, we will perform a sensitivity study where we intend to analyse how a change in the input parameters impacts important features of CDS options. Finally, in Chapter 5, we will provide some final remarks and suggest some future works.

2. Concepts and Methodology

We start by providing some basic financial concepts, results, definitions and notations, that will be essential in the derivation of the models that we will use to accomplish our goal. Also, we describe the approach used in order to model the default intensity and we explain the mechanism of a CDS contract and the valuation of CDS options.

2.1. Some Fundamental Financial Concepts

We let τ denote the default time and we consider the following indicator function:

$$\mathbf{1}_{\{\tau>t\}} = \begin{cases} 1, & \text{if } \tau > t \\ 0, & \text{if } \tau \leq t \end{cases} \quad (1)$$

Firstly, we introduce the concept of zero-coupon bond. A default-free zero-coupon bond has a face value of one unit of currency and a defaultable zero-coupon bond (i.e, subject to default risk) has a payoff at maturity T of $\mathbf{1}_{\{\tau>T\}}$[1]. For both cases, zero-coupon bonds have no intermediate payments. We let,

- $P(t,T)$ denotes the price of a default-free zero-coupon bond at current time t, when its maturity is T, where $t < T$.
- $\bar{P}(t,T)$ denotes the price of a defaultable zero-coupon bond at current time t, when its maturity is T, where $t < T$, given that default did not occur before time t.

There are many types of interest rates, for instance, forward rates are interest rates that can be established in current time t for an investment in a future time period. We present some useful definitions about forward rates [2].

Definition 2.1. *Let $f(t,T_1,T_2)$ denote the continuously compounded default-free forward rate over the period $[T_1,T_2]$ at time t, computed as follows (for $t \leq T_1 < T_2$):*

$$f(t,T_1,T_2) = \frac{lnP(t,T_1) - lnP(t,T_2)}{T_2 - T_1} \quad (2)$$

[1]We will use the *"overline"* notation for the defaultable (financial) instruments and all the definitions of this defaultable quantities, that we will present next, are only valid for times before default ($\tau > t$). Also, we assume zero recovery in case of default before time t.

[2]The definitions and notations are inspired in Schönbucher [5].

Definition 2.2. *Let $\bar{f}(t,T_1,T_2)$ denote the continuously compounded defaultable forward rate over the period $[T_1,T_2]$ at time t, computed as follows (for $t \leq T_1 < T_2$):*

$$\bar{f}(t,T_1,T_2) = \frac{ln\bar{P}(t,T_1) - ln\bar{P}(t,T_2)}{T_2 - T_1} \quad (3)$$

Definition 2.3. *The continuously compounded instantaneous default-free forward rate at time t with maturity T is defined as:*

$$f(t,T) = -\frac{\partial lnP(t,T)}{\partial T} \quad (4)$$

where $t < T$ and in case the derivative of $P(t,T)$ w.r.t T exists.

Definition 2.4. *The continuously compounded instantaneous defaultable forward rate at time t with maturity T, is defined as:*

$$\bar{f}(t,T) = -\frac{\partial ln\bar{P}(t,T)}{\partial T} \quad (5)$$

where $t < T$ and in case the derivative of $\bar{P}(t,T)$ w.r.t T exists.

Also we define, respectively, the instantaneous default-free short rate and instantaneous defaultable short rate at time t, as follows:

$$r(t) = f(t,t) \text{ and } \bar{r}(t) = \bar{f}(t,t) \quad (6)$$

Another crucial concept in this thesis is the default intensity ($\lambda(t)$). We let $s(t,T)$ denote the instantaneous forward default intensity at time t for maturity T, given as follows:

$$s(t,T) = -\frac{\partial}{\partial T} ln(S(t,T)) \quad (7)$$

where $S(t,T)$ denote the forward survival probability at time t for maturity T and can be also written as:

$$S(t,T) = \frac{\bar{P}(t,T)}{P(t,T)} \quad (8)$$

Then, we can rewrite the instantaneous forward default intensity at time t for maturity T as follows:

$$s(t,T) = \bar{f}(t,T) - f(t,T) \quad (9)$$

In particular, the default intensity at time t is given by:

$$\lambda(t) = s(t,t) = \bar{r}(t) - r(t) \quad (10)$$

2.2. CDS and CDS options

A Credit Default Swap (CDS) is an over-the-counter contract between two parties, the protection buyer and the protection seller, where the purpose is to protect one of them, the protection buyer, against default of a specific company or sovereign entity (issuer). We let,

- $\delta_i = T_{i+1} - T_i$ be the time (in years) interval between payments of a CDS contract, for $i = 0, ..., n-1$.
- ξ be the credit swap rate or also called the premium payment rate [3].

Regarding that n^* can be equal to 0,1, ..., n-1, a CDS contract is constitute by two payment legs which are defined as follows:

- **Fixed leg**: the protection seller receives regular payments of $\xi\delta_i$, at times $T_1, T_2, \ldots, T_{n^*}$ from the protection buyer, where $n^* = n$ if default didn't occur until maturity (T_n); otherwise ($0 < n^* < n$), the regular payments end as soon as default happens and, in this case, n^* is such that $T_{n^*} < \tau < T_{n^*+1}$ [4].
- **Floating leg**: if a default occurs before the contract maturity date (T_n) then the protection seller has to compensate the protection buyer in $(1-R)$ times the notional value at time T_{n^*+1}, where R is the recovery rate [5].

In this thesis we mainly focus on forward CDS contracts and CDS options. A forward CDS contract is a CDS contract starting on a future date, T_k, say (with $T_k < T_n$). Then the above definitions hold, except that the payments start only from time T_{k+1} onwards. A CDS option is an option on a underlying forward CDS contract. Concerning the CDS option, we have the following formal definitions,

Definition 2.5. *A payer (receiver) CDSwaption with expiry (or exercise) date T_k and default swap rate ξ is a call (put) option on a forward CDS starting at time T_k. So the holder of the option has the right, not the obligation, to buy (sell) a CDS at the predefined value ξ, if there has been no credit event until time T_k.*

2.3. Valuation of CDS Options

For an arbitrary time t, the value of the fixed leg and the floating leg of a forward CDS are, respectively, given by:

$$V_{fixed}(t) = \xi \sum_{i=k}^{n-1} \delta_i \bar{P}(t, T_{i+1}) \quad (11)$$

$$V_{floating}(t) = \xi_{k,n}(t) \sum_{i=k}^{n-1} \delta_i \bar{P}(t, T_{i+1}) \quad (12)$$

Where we denote by $\xi_{k,n}(t)$ the value for ξ such that the value of the floating leg is equal to the value of the fixed leg of a forward CDS, both at time t. $\xi_{k,n}(t)$ is usually called the forward credit swap rate. Since in the next section we consider that the default-free short rate process $\{r(t), t\}$ is deterministic, then $\xi_{k,n}(t)$ is given by:

$$\xi_{k,n}(t) = (1-R)\frac{\sum_{i=k}^{n-1}[S(t,T_i)P(t,T_{i+1}) - \bar{P}(t,T_{i+1})]}{\sum_{i=k}^{n-1}\delta_i \bar{P}(t,T_{i+1})} \quad (13)$$

Then, by (11) and (12), the payoff of the payer CDSwaption at the maturity T_k, hereby denoted by $V_{payer}(T_k)$, is given by:

$$V_{payer}(T_k) = (\xi_{k,n}(T_k) - \xi)^+ \sum_{i=k}^{n-1} \delta_i \bar{P}(T_k, T_{i+1}) \quad (14)$$

Therefore, under the survival measure $\bar{Q}$ derived in Krekel [2] [6], we have that the price of a payer CDSwaption at time 0 is given by,

$$V_{payer}(0) = \mathrm{E}_{\bar{Q}}\Big[exp\Big(-\int_0^{T_k} \bar{r}(s)ds\Big) \sum_{i=k}^{n-1} \delta_i \bar{P}(T_k, T_{i+1})(\xi_{k,n}(T_k) - \xi)^+\Big] \quad (15)$$

The price of the payer CDSwaption at time 0, under the survival measure $\bar{Q}$, is the discounted expected payoff of the CDSswaption at expiry T_k where the discount factor is $\bar{r}(.)$.

3. Numerical Procedure for CDS Options

We derive an adaptation of the numerical method derived in LRS [3], for the particular case of CDS options.

3.1. Pricer in a Recombining Tree

Based on the default intensity model from Krekel [2], we adapt the method described in LRS [3], using, in our case, the default intensity $\lambda(t)$, instead of the short rate $r(t)$ process (used in the original paper). Therefore, we transform the default intensity process in a process that has a constant volatility parameter, and we build a recombining tree for this new process. For non-arbitrage arguments, we need to consider values of the elasticity parameter (γ, with $\gamma > 0$) such that the default intensity is always positive.

From the default intensity process $\{\lambda(t), t\}$ we consider the following transformation,

$$Y(t, \lambda(t)) = \int_{\lambda_0}^{\lambda(t)} \frac{1}{\sigma l^\gamma} dl \quad (16)$$

where σ and γ(usually called the elasticity parameter) are constant.

[3] Usually is represented as base points in the market quotes.

[4] Usually, these payments are made quarterly, that is, in intervals of 3 month.

[5] Usually for CDS contracts the recovery rate is 40%.

[6] For future details about the change of measure, please see Krekel [2].

Then, we apply the It's lemma to (16) and we obtain the following equation [7],

$$dY(t,\lambda(t)) = m(Y,\phi_\lambda,t)dt + dW(t) \tag{18}$$

Therefore, we consider the following two-state variable process,

$$\begin{aligned} dY(t,\lambda(t)) &= m(Y,\phi_\lambda,t)dt + dW(t) \\ d\phi_\lambda(t) &= (\sigma_\lambda^2(t) - 2k_\lambda\phi_\lambda(t))dt \end{aligned} \tag{19}$$

We can only guarantee non-arbitrage when $\gamma = 1$ [2]; in this case the resulting model is usually called proportional model [3]. Then it follows, by straightforward calculations, that the process Y is given by,

$$Y(t,\lambda(t)) \approx \frac{ln(\lambda(t))}{\sigma} \tag{20}$$

From now on, all the calculations and implementations will be for the proportional model.

3.2. Lattice Approximation

Similarly to LRS approach [3], we start by partitioning the interval between the valuation date and the expiry date into subintervals of equal length, Δt, and we let N denote the number of such subintervals, and also the number of (time) steps considered in the recombining tree. The tree is a recombining one, and we need two indexes to describe the state of the tree, namely: the first index (i, say) is related with the number of time steps since the valuation date until the expiry date of the CDS option, so $i \in \{0, 1, \ldots, N\}$. The second index ($j$, say) is the number of nodes of the tree at each time step, where $j \in \{0, 1, \ldots, i\}$. As the tree is a recombining one, at time step i we have $(i+1)$ nodes, and therefore at the expiry date ($i = N$) we have $N+1$ nodes. So, for instance, (i, j) represents the jth node of the ith time step of the tree.

Following the terminology of LRS, we call the values of the join process (Y, ϕ_λ) of the tree as the *approximation variables*, and we use the following notation: if at a certain node ((i, j), say) we assume that the values of the join process are (y^a, ϕ_λ^a), then in the next time step we will have the values $(y^{a+}, \phi_\lambda^{a+})$ and $(y^{a-}, \phi_\lambda^{a-})$ (respectively at nodes $(i+1, j)$ and $(i+1, j+1)$) where, y^{a+} and y^{a-} are, respectively, given by [8],

$$y^{a+} = y^a + (J+1)\sqrt{\Delta t} \tag{23}$$

$$y^{a-} = y^a + (J-1)\sqrt{\Delta t} \tag{24}$$

such that,

$$y^{a-} \leq y^a + m(y^a, \phi_\lambda^a, t)\Delta t \leq y^{a+} \tag{25}$$

Regarding ϕ_λ^a, one can see that $\{\phi_\lambda(t), t\}$ is a deterministic process, because only depends on t, then the values ϕ_λ^{a+} and ϕ_λ^{a-} are equal. So, we denote ϕ_λ^{a*} as their common value, which can be obtained by integrating the process (19) until time Δt,

$$\phi_\lambda^{a*} = \phi_\lambda^a + [(\sigma\lambda^a)^2 - 2k_\lambda\phi_\lambda^a]\Delta t \tag{26}$$

where $\lambda^a = e^{\sigma y^a}$.

We can easily see that the total number of distinct values of ϕ_λ^a at each node will be equal to the total number of unique paths that reach that node. Following the simplification proposed by LRS, we chose to keep at each node the interval $[\underline{\phi}_\lambda^a, \bar{\phi}_\lambda^a]$ where $\underline{\phi}_\lambda^a$ (resp. $\bar{\phi}_\lambda^a$) is the minimum (resp. maximum) value of ϕ_λ^a from the valuation date until the respective node. We assume that at a certain node ((i, j), say) we have the interval $[\underline{\phi}_\lambda^a, \bar{\phi}_\lambda^a]$. So, we let $\mathtt{m}$ be the number of elements in this interval and then we partition $[\underline{\phi}_\lambda^a, \bar{\phi}_\lambda^a]$ into subintervals of equal length $\Delta\phi$, i.e,

$$\underline{\phi}_\lambda^a = \phi_\lambda^a(1) < \phi_\lambda^a(2) < \ldots < \phi_\lambda^a(\mathtt{m}) = \bar{\phi}_\lambda^a$$

where, $\phi_\lambda^a(u), u = 1, \ldots, \mathtt{m}$ denotes the uth point of the partition and the partition step is given by $\Delta\phi = \frac{\phi_\lambda^a(\mathtt{m}) - \phi^a(1)}{\mathtt{m}-1}$.

Then for each value of this partition ($\phi_\lambda^a(u), u = 1, \ldots, \mathtt{m}$) we calculate the respective successor value ($\phi_\lambda^{a*}(u), u = 1, \ldots, \mathtt{m}$) and we proceed in the same way for all the nodes at ith step [9].

We let $p = p(y^a, \phi_\lambda^a)$ be the probability of moving from y^a to y^{a+} in the next time increment, where p depends on the value of the approximate variables as follows:

$$p(y^{a+} - y^a) + (1-p)(y^{a-} - y^a) = m(y^a, \phi_\lambda, t)\Delta t \tag{27}$$

Moreover, from (24) the probability p can be written in function of the integer J,

$$p = \frac{m(y^a, \phi_\lambda^a, t)\Delta t + (1-J)\sqrt{\Delta t}}{2\sqrt{\Delta t}} \tag{28}$$

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[7] Where, in particular, the drift term can be written as follows,

$$m(Y,\phi_\lambda,t) = \frac{1}{\sigma}\Big\{\frac{k_\lambda[s(0,t) - \lambda(t)] + \phi_\lambda(t) + \frac{d}{dt}s(0,t)}{\lambda(t)}\Big\} - \frac{\sigma}{2} \tag{17}$$

[8] Where J is an integer defined as follows,

$$J = \begin{cases} |Z|, & \text{if } Z \text{ is even} \\ Z+1, & \text{otherwise} \end{cases} \tag{21}$$

with

$$Z = \Big\lfloor m(y^a, \phi_\lambda^a, t)\sqrt{\Delta t} \Big\rfloor \tag{22}$$

[9] Clearly, at a certain time step i, when $j = 0$ or $j = i$ we denote these nodes as edge nodes.

We note that on the implementation of the method we will use this formula for calculating this probability.

3.3. Backward Recursion and Interpolation

After building the lattice approximation or the recombining tree we present a general procedure to price an option where we use a general backward recursion approach and interpolation to calculate the option price at the valuation date.

We let, $g_i(y^a, \phi^a_\lambda)$ be the option value at the i^{th} step of the recombining tree conditional on the state variables (y^a, ϕ^a_λ). Also, we assume that all the option values at $(i+1)^{th}$ step are available, then the option prices at i^{th} step are as follows,

$$\begin{aligned} g_i(y^a, \phi^a_\lambda) = [pg_{i+1}(y^{a+}, \phi^{a*}_\lambda) + (1-p) \\ g_{i+1}(y^{a-}, \phi^{a*}_\lambda)]e^{-\bar{r}^a_i \Delta t} \end{aligned} \tag{29}$$

However, the value of the accumulated variance at the $(i+1)^{th}$ (ϕ^{a*}_λ) is not always available, because we have chosen to only keep the minimum and the maximum values of the successor values of ϕ^a_λ, that is, it is completely determined by the current values of (y^a, ϕ^a_λ). Therefore, in the case that we do not have the exact values of $g_{i+1}(y^{a+}, \phi^a_\lambda)$ or $g_{i+1}(y^{a-}, \phi^a_\lambda)$, we apply linear interpolation in order to determine these option prices.

Then, we apply the formula (29) in order to obtain the option price at i^{th} step [10].

3.4. Cheyette Algorithm

The procedure explained at the last section is called the Cheyette algorithm, that we have implemented, using the programming language C++. In this section we explain in more detail the implementation issues. The method will be derived for the particular case of a CDS option on a underlying forward CDS contract. Along the rest of the thesis, we assume the case of the proportional model. Furthermore we assume that the initial term structure of the forward default intensity $s(0,t)$ is flat initially, in the sense that $s(0,t) = \lambda_0$, and we assume that the default-free interest rates are deterministic, i.e, $r(t) = r_0 \; \forall t$.

Clearly, since we are constructing a recombining tree and at each node we only have up or down movements with respect to a probability, at the i^{th} step we have $i+1$ nodes. So, we divide the construction of the Cheyette algorithm in three parts, namely,

- Lattice approximation (where we construct the tree, for the whole steps and nodes, i.e, for $i = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, i$);
- Terminal nodes (where, as the name suggest, we compute for the last step of the tree, $i = N$, for the whole nodes, $j = 0, 1, \ldots, N$);
- Backward recursion and interpolation (once computed the value for the terminal nodes, we proceed backwards, or the steps $i = N-1, N-2, \ldots, 0$ and for the respective nodes $j = 0, 1, \ldots, i$).

We let T_k be the expiry date (as in Chapter 2) and t_0 be the valuation date of the CDS option. Then, we start by computing the time to expiry (in years) of this option at the valuation date as follows,

$$\tau^* = \frac{T_k - t_0}{365} \tag{30}$$

So, having our interval of interest, we now divide the time to expiry τ^*, in intervals of equal length, that is,

$$\Delta t = \frac{\tau^*}{N} \tag{31}$$

If we consider the (i,j) node and given that the value `m` is the number of subintervals of the partition of $[\underline{\phi}^a_\lambda, \bar{\phi}^a_\lambda]$, we implement the lattice approximation in the following way:

1. Regarding the initial conditions that we have taken, in particular, that $\gamma = 1$ and $s(0,t) = \lambda_0$ (i.e, $\frac{d}{dt}s(0,t) = 0$), we can rewrite the equation (17) as follows:
$$m(y^a, \phi^a_\lambda(u), t) = \frac{k_\lambda[\lambda_0 - \lambda^a] + \phi^a_\lambda(u)}{\sigma \lambda^a} - \frac{\sigma}{2} \tag{32}$$
where $u = 1, 2, \ldots,$`m` and λ^a denotes the default intensity at node (i,j).
2. For each value of $\phi^a_\lambda(u)$, we compute the value of the drift term of the transformed process, according to (32), and then the value of the integer J according to (21). So, we are now able to calculate the probability p of an up movement according to (28) [11].

   Taking into account that most of the time the value of J is equal to zero, we decide to send an error message and stop the procedure, when J is such that $J \neq 0$, to avoid larger jumps in the value of the default intensity, .
3. Next, given that we know the value of the default intensity (λ^a) at node (i,j), then the values at the nodes $(i+1,j)$ and $(i+1,j+1)$ are computed, respectively, as follows,
$$\lambda^{a+} = exp\Big\{\lambda^a + (J+1)\sqrt{\Delta t}\Big\} \tag{33}$$
$$\lambda^{a-} = exp\Big\{\lambda^a + (J-1)\sqrt{\Delta t}\Big\} \tag{34}$$

[10] Note that the way that the lattice approximation was built the values at nodes the successor nodes (y^{a+}, ϕ^{a*}_+) and (y^{a+}, ϕ^{a*}_-) will be available, where, $\phi^{a*}_- \leq \phi^{a*} \leq \phi^{a*}_+$ [3].

[11] We note that, on the edge nodes of the tree we only have one (distinct) value for ϕ^a_λ, because $\underline{\phi}^a_\lambda = \bar{\phi}^a_\lambda$.

4. Finally, given that we have at node (i, j) the interval $[\underline{\phi}_{\lambda}^{a}, \bar{\phi}_{\lambda}^{a}]$, we calculate the corresponding intervals at nodes $(i+1, j)$ and $(i+1, j+1)$, according to the procedure described previously.

After building the lattice approximation as described in the previous item, we are now able to calculate the CDS option price at the expiry date (that is, at the terminal nodes of the tree). At the terminal nodes we have all the information that we need to calculate the CDS option price. In particular, we proceed as follows:

1. Firstly, we compute the intervals of time between payments of the forward CDS:
$$\delta_{i^*} = \frac{T_{i^*+1} - T_{i^*}}{365}, \ \forall i^* = k, k+1, ..., n-1 \tag{35}$$
where the times $T_{k+1}, T_{k+2}, ..., T_n$ are the payment dates of the forward CDS contract;

2. Since we consider that the default-free interest rates are deterministic (r_0), then the zero-coupon bond price at expiry time T_k, for each date of the forward CDS, is calculated as follows,
$$P(T_k, T_{i^*+1}) = e^{-r_0 \frac{(T_{i^*+1} - T_k)}{365}}, \quad \forall i^* = k, k+1, ..., n-1 \tag{36}$$

3. For each value of $\phi_{\lambda}^{a}(u)$ with $u = 1, \ldots, \mathtt{m}$, we compute, respectively, the forward survival probability at time T_k for each date of the forward CDS (i.e, $S(T_k, T_{i^*+1})$, $\forall i^* = k, k+1, \ldots, n-1$) according to formula (**??**) and considering that at the valuation date the forward survival probability is calculated as follows,
$$S(t_0, T_{i^*}) = e^{-\lambda_0 \frac{(T_{i^*} - t_0)}{365}}, \quad \forall i^* = k, k+1, ..., n-1, n \tag{37}$$
Also, we calculate the defaultable zero-coupon bond price at time T_k for each date of the forward CDS, according to equation (8).

4. Next, we calculate the value of the forward credit swap rate at time T_k ($\xi_{k,n}(T_k)$) according to equation (13), because here the default-free short rates are deterministic.

Therefore, after calculating these quantities, we can now calculate the payoff at the terminal nodes of the payer CDS option according to formula 14.

3.5. Backward Recursion and Interpolation

After computing the CDS option prices at the expiry date, we use backward recursion and linear interpolation from $i = N-1$ to $i = 0$, to calculate the option prices in each one of the nodes of the tree. In order to do that, we proceed as described in the previous section 3.1.

4. Results

Regarding the Cheyette algorithm explained previously, now our goal is to study/analyse, in detail, how a change in the input parameters (or also called Cheyette parameters) impacts the computation of the price and calibration of the implied volatility of a payer CDS option. We chose only to perform this analysis for payer CDS options because the case of receiver CDS options will produce the obvious changes. Therefore it will not add any significant information to this study.

In order to perform this sensitivity study, we present three examples that show different numerical features/properties of this algorithm. In these examples we analyse,

- The evolution of the payer CDS option price with the number of steps of the tree (N), for different number of partitions ($\mathtt{m}$) of the accumulated variance (ϕ) interval;
- The convergence and the calibration of the implied volatility using two different root-finding methods, namely, the bisection and Newton's method;
- The effect of a change of the mean reversion parameter (k_{λ}) and also of the volatility parameter (σ) in the computation of the price and implied volatility of a payer CDS option, as a function of the strike price (ξ);
- The implied volatility for several values of the initial default intensity (λ_0), as function of the strike price (ξ);
- For an *at-the-money* (ATM) payer CDS option, how a change in the volatility parameter (σ) impacts the implied volatility, for several values of the mean reversion parameter (k_{λ}).

Furthermore, we consider payer CDS options from 17-10-2008, whose expiry date can be either 20-12-2008 or 20-03-2009, on an underlying five year CDS contract with maturity date 20-12-2013 and quarterly payment dates [12]. In the following examples, we start by choosing k_{λ} and σ, such that, they have the same values as the ones obtained in [2] [2] for the optimal values of these parameters, for an iTRAXX S10 IG five year CDS or for an iTRAXX S10 Xover five year CDS. Moreover, depending on the choice of the values of k_{λ} and σ, we consider different values for the strike prices (ξ). In particular, when k_{λ} and σ are equal to the optimal parameters for an iTRAXX S10 IG five year CDS, the values, in basis points (bp), of the strike price (ξ) are: 80, 90, 100, 110, 120, 130, 140, 150, 160 and 170. On the other hand, when k_{λ} and σ are equal to the

[12] These dates are the same used in [2].

optimal parameters for an iTRAXX S10 Xover five year CDS, the values, in basis points (bp), of the strike price (ξ) are: 300, 325, 350, 375, 400, 425, 450, 475, 500, 525, 550, 575, 600 and 625.

Also, we note that,

1. Here, we always try to compute the payer CDS option price, at least, one time per day, from its valuation date until its expiry date, taking into account the computational time and the numerical stability of the option price;

2. We only show the effect of k_λ and σ on the pricing for the example 1, because for the other examples the evolution of the price have the same behaviour.

From the previous analysis, performed in detail in the full report, we can extract the following remarks:

1. As expected from LRS [3], we observed that the price of a payer CDS option tends to converge with the increasing of the number of steps of the recombining tree (N) and with the number of partitions ($\mathtt{m}$) of the interval of the accumulated variance;

2. As expected from Krekel [2], we observed that the mean reversion (k_λ) and the volatility (σ) parameters have a reversed effect on the pricing of a payer CDS option. In the sense that, for a fixed strike price, the price and the implied volatility increases with increasing σ but the price and the implied volatility increases with decreasing k_λ;

3. For the case of ATM payer CDS options, we observed the same patterns among the illustrative examples. This examples suggested that the implied volatility assumes an almost linear behaviour with σ;

4. However, we observed peculiar numerical behaviours when analysing the evolution of the implied volatility for different values of λ_0 in examples 1 and 3.

5. Conclusions

In this thesis, the main goal was to compute the price of a CDS option in a recombining tree based on a default intensity model derived in Krekel [2]. Since this is an Markovian model one and has a closed formula for the forward survival probability, we are in conditions to build a recombining tree for the computation of CDS option prices. In this sense, we developed the Cheyette algorithm inspired in the LRS [3] framework and made a sensitivity study in order to analyse the numerical features of this procedure.

Since we approached some concepts that are not usually used in the field of Mathematics, before drawing the algorithm, in Chapter 2, we explained some (basic) financial concepts for a better understanding of the problem. Also, we provide the default intensity model from Krekel [2]. This model was used to help us write an adaptation of the numerical method from LRS [3].

Therefore we believe that the major contributions of this dissertation were the formulation of a numerical procedure and the implementation of a parsimonious and efficient algorithm, that enable us to price CDS options.

Nevertheless, we consider that there is some aspects that need to be improved. So, as future work we do the following suggestions:

1. Concerning the last item of the previous remarks, a careful investigation should be done in order to find a justification for this numerical instability;

2. We believe that is extremely important that in future works, the Cheyette algorithm would be applied to recent real data in order to explore the performance and accuracy of this procedure in the financial markets. Due to the fact that the Cheyette algorithm seems to be a parsimonious and efficient method to compute the price of these financial instruments;

3. We consider that would be also very interesting to develop similar numerical procedures for other kinds of credit derivatives with the respective changes in the formulas and then perform a similar analysis. In this analysis, one should look for patterns and also numerical peculiarities as we did in this thesis.

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