Structured Population Dynamics in a Real World Context

Ana Martins
anaclmartins@tecnico.ulisboa.pt

Instituto Superior Técnico, Lisboa, Portugal

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Abstract

An introduction to the state-of-the-art relating population models is presented. The concept of Leslie matrix is explained and subsequently adapted in order to include migration flows. The growth rate sensitivity is analysed. An algorithm that compresses Leslie matrices is presented. The models and measures are applied and compared in a real world open system context.
Keywords: Population Models, Leslie Matrices, Evolutionary Entropy, Application to Real World Context, Compression of Leslie Matrices.

1. Introduction

The forecast of an ecosystem allows a better knowledge of its dynamics with time and, consequently, it is a tool for decision makers to plan in a long term. However, these predictions are subject to change according to perturbations, which is what sensitivity analysis measures.

Suppose that the age of inhabitants is known. In the past, two main methods to study the evolution of the population with time were used: a continuous-time integral equation, first introduced by F. R. Sharpe and A. J. Lotka [13]; or a matrix formulation with age classes, presented by H. Bernardelli [1], E.G. Lewis [12] and by P.H. Leslie [10]. Afterwards, some extensions of these basic models were published by M.H. Williamson [15], M.B. Usher [14], L.P. Lefkovitch [9] and C.A. Bosch [2].

An analysis of the population dynamics is essencial since it gives intrinsic details about the ecosystem. For example, L. Demetrius showed in [6] that, for populations characterized by a stable size or by small fluctuations of it, an unidirectional increase in population entropy for a large period of time leads to an evolutionary change caused by mutation and natural selection. Moreover, for populations who experienced an exponential growth, the same effect can be achieved by an unidirectional increase in growth rate and a decrease in entropy for periods of time.

Our work presented here will explain the last approach and use it as basis to the model here exposed. We will also analyse the growth rate sensitivity, based on the work of L. A. Demetrius and V. M. Gundlach [7], and L.A. Demetrius and H.M. Oliveira [5]. Afterwards, a brief explanation of an algorithm that, given a Leslie matrix of c age classes, returns another correspondent Leslie matrix with $n \times c$ age classes, that is the algorithm compresses a Leslie matrix with null-error. We will apply our studies in population dynamics to a parish in Portugal, called Arroios (Lisbon).

2. Models

To obtain projections of a population, there is the need to create and adapt several models that mimic its behaviour with time and dynamics.

This work uses as basis Leslie matrices, since only regular age classes are considered, according to the data retrieved from the Portuguese Census. For the sake of simplification, and since only women can produce new human beings and their number is highly correlated to the growth of a population, we shall use a single gender projection, namely the projection of the female individuals [3]. Also, we suppose that the migration flows are steady with time.

2.1. Model without migratory flows

The Leslie matrix is used to analyse the age distribution of survivors and of descendants, considering successive intervals of time and supposing that the rates of fertility and mortality are applied to all elements equally and constant over time.

Let:

- n_{x_t} be the number of females alive in the age group x to $x+1$ at time t;
- P_x be the survival probability of a female from an age class between x and $x+1$ at time t to the next age class $x+1$ to $x+2$ at time $t+1$.

It has values between $0 < P_x < 1$;

- $F_x \geq 0$ be the number of daughters born between t and $t+1$ that are still alive in the age class from 0 to 1 at time $t+1$ and whose mothers lived aged x to $x+1$.

Hence, the age distribution at the end of one unit interval can be rewritten as the following $m+1$ square matrix $L \times n_0 = n_1$, where n_0 and n_1 are column vectors providing the age distribution at $t = 0, 1$:

$$\mathbf{L} = \begin{bmatrix} F_0 & F_1 & F_2 & \dots & F_{m-2} & F_{m-1} & F_m \\ P_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & P_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & P_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P_{m-2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & P_{m-1} & 0 \end{bmatrix}. \tag{1}$$

The model without migratory flows, considered on H. Leslie's article [10], is written as:

$$\mathbf{n}(t+1) = \mathbf{L} \times \mathbf{n}(t), \tag{2}$$

with $\mathbf{n}(t=0) = n_0$ and where $\mathbf{n}(t+1)$ is the vector of the female population subdivided by age-classes at time $t+1$, $\mathbf{L}$ is the Leslie Matrix calculated at time t and $\mathbf{n}(t)$ is the vector of the female population subdivided by age-classes at time $t+1$.

This model is the simplest approach one can obtain when modelling a population that must be a closed system, that is, a system without any kind of flows, e.g. immigration and emigration.

2.2. Model with Leslie matrix that includes immigration and emigration

The following population model was adapted in a way that it accounts for internal flow, from a parish to another inside the same country, and external flow, from a country to another. These migration flows were added to the survival probability of a female from one class to the next one. Also, if we consider the intrinsic and migratory inhabitants, the fertility rates must also account them.

On that account, the Leslie matrix is given by:

$$L' = \begin{bmatrix} F'_0 & F'_1 & F'_2 & \dots & F'_{m-2} & F'_{m-1} & F'_m \\ P'_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & P'_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & P'_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P'_{m-2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & P'_{m-1} & 0 \end{bmatrix}, \tag{3}$$

where $F'_x \geq 0$ is the number of daughters born between t and $t+1$ that are still alive in the age class from 0 to 1 at time $t+1$ and whose mothers lived aged x to $x+1$ and were an inhabitant or a immigrant at time $t+1$, and P'_x is the survival probability plus the migration flows of a female from an age class between x and $x+1$ at time t to the next age class $x+1$ to $x+2$ at time $t+1$.

Furthermore, we obtain the following model:

$$\mathbf{n}(t+1) = \mathbf{L'} \times \mathbf{n}(t). \tag{4}$$

This method can be computationally described as:

- start with a Leslie matrix for closed systems, where add unknowns parameters to the survival rates P_i;
- obtain the resulting vector of the population after two time steps (i.e. 10 years);
- compare the number of daughters resultant from this altered matrix and the corresponding first entry of the population;
- place an unknown in the first entry of the matrix F'_0;
- calculate a system with all the unknowns;
- distribute the unknown value placed in F'_0 between all non-zero entries referring to fertility according to their previous proportion between each other;
- place a 0 in the first entry of the matrix, F'_0 (no human daughters with less than 4 years old can reproduce themselves).

Therefore, this matrix explores an open system, even though it considers stable migration flows with time, which may not apply to an ecosystem with large fluctuations on its migration flows.

3. Partial Derivatives of Evolutionary Entropy

Evolutionary entropy is a function of the change caused by perturbations on a system that can be a biological, a metabolic or even an economic system. Therefore, allows the specification of the rate at which, after a perturbation, the macroscopic variables of the system return to their steady state, that is, allows the specification of its robustness. This section is based in the work of L.A. Demetrius and V. M. Gundlach [7], and L.A. Demetrius and H.M. Oliveira[5].

3.1. **Evolutionary Entropy**

Let us start by considering the diagonal matrix $\mathbf{U}$

$$\mathbf{U} = \begin{bmatrix} u_0 & 0 & \dots & 0 & 0 \\ 0 & u_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{d-1} & 0 \\ 0 & 0 & \dots & 0 & u_d \end{bmatrix}. \quad (5)$$

Consider also the stochastic matrix

$$\mathbf{P} = (p_{ij}) = \frac{a_{ij}u_j}{\lambda u_i}, \quad (6)$$

describing a Markov process with transition rate $(p_{ij})$and stationary distribution $\Pi = (\pi_i)$, where $\Pi\mathbf{P} = \Pi$ and $\pi_i = v_i u_i$, that can be represented by

$$\mathbf{P} = \frac{1}{\lambda}\mathbf{U}^{-1}\mathbf{A}\mathbf{U}. \quad (7)$$

The evolutionary entropy $\mathbf{H}$, weighted average of H_i, is the rate at which $\mathbf{P}$ generates information, which analytically translates into

$$\mathbf{H} = \sum_{i=1}^{d} \pi_i H_i = -\sum_{i,j=1}^{d} \pi_i p_{ij} \log p_{ij}, \quad (8)$$

where

$$H_i = -\sum_{j=1}^{d} p_{ij} \log p_{ij}. \quad (9)$$

Notice that, as a consequence, H_i is the Shannon-entropy associated with the state X_i of the Markov chain.

3.2. **Characteristics of the Evolutionary Entropy**

[7][5]

The evolutionary entropy has relevant characteristics, such as its relation with generation time, its variational principles, and its robustness.

1. *Evolutionary entropy and generation time If S is the measure of the number of replicative cycles in the network and T is the cycle time (the mean return time of the Markov process associated with the matrix $\mathbf{P}$), the evolutionary entropy $\mathbf{H}$ can be displayed as:*

$$\mathbf{H} = \frac{S}{T}. \quad (10)$$

2. *Variational principles and evolutionary entropy The matrix $P = (p_{ij})$ when is derived from the interaction matrix $A = (a_{ij}) \geq 0$ represents a fundamental feature of its network. If $r = \log\lambda$ and $\mathbf{\Phi}$ is the reproductive potential, we obtain:*

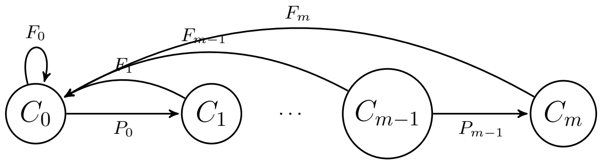

Figure 1: Graph G_c that has as an adjacency matrix the Leslie matrix L_t.

$$r = \mathbf{\Phi} + \frac{S}{T}. \quad (11)$$

3. *Evolutionary entropy and robustness An increase in entropy entails an increase in robustness and, therefore, a greater insensitivity of an observable to perturbations in the microlevel parameters that describe the network.*

3.3. **Evolutionary Entropy-Sensitivity Analysis**

[7][5]

It is relevant to derive and analyse the sensitivity of growth rate, which is related to evolutionary entropy. For example, in a demographic network, fluctuations on the fertility rates in lower age classes have a higher impact on the growth rate than in older age classes.

1. *Growth rate sensitivity The sensitivities S_{ij} are given by*

$$S_{ij} = \frac{\partial\lambda}{\partial a_{ij}} = v_i u_j, \quad (12)$$

where $v_i > 0$ and $u_j > 0$ are the components of the left and right dominant eigenvectors, respectively.

4. Compression of the Leslie matrices

For the sake of simplification, we shall use Leslie graphs, mentioned on 2, which have L_t as incidence matrix.

Imagine that we want to compress this matrix that can be seen as an adjacency matrix of a oriented and weighted graph. Therefore, the graph G_c, pictured in Figure 1, may provide a representation of L_t.

Our goal is to discover $F'_0, F'_1, \dots, F'_{\frac{m}{2}}$ and $P'_0, P'_1, \dots, F'_{\frac{m}{2}}$, illustrated in the Figure 2, such that they correspond to the values directly obtained using the fertility and survival rates for a Leslie matrix with $2c$-age classes.

We will have two cases: when m, the number of age classes, is an even number; and when m, the number of age classes, is an odd number.

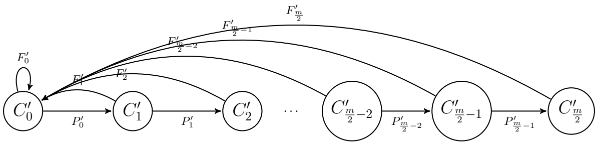

Figure 2: Graph G_{2c} that has as an adjacency matrix the Leslie matrix with $2c$-age classes equivalent to L_t, when m, the number of age classes, is an even number.

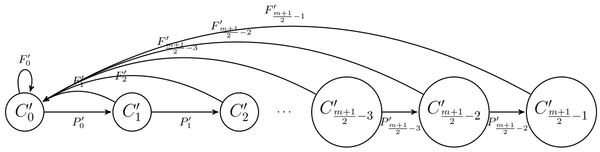

Figure 3: Graph G_{2c} that has as an adjacency matrix the Leslie matrix with $2c$-age classes equivalent to L_t, when m, the number of age classes, is an odd number.

4.1. m is even

Suppose that m is an even number. Notice that, to obtain $F'_0, F'_1, \ldots, F'_{\frac{m}{2}}$ and $P'_0, P'_1, \ldots, F'_{\frac{m}{2}}$ from a Leslie matrix with c-age-classes, these weights must be written in order of the number of females alive in each age group $(n_{x_0}, n_{x_1}, \ldots, n_{x_m})$, and in order of the coefficients of a Leslie matrix with c-age classes $(F_0, F_1, \ldots, F_{\frac{m}{2}}$ and $P_0, P_1, \ldots, F_{\frac{m}{2}})$.

4.1.1 Fertility Rates

A fertility rate F_i can be obtained by:

$$F_i = \frac{\#(\text{daughters given by women in age class i})}{\#(\text{female population in age-class i})}. \tag{13}$$

Therefore, the fertility rates for a Leslie matrix with $2c$ age classes can be obtained by:

$$F'_j = \frac{\dfrac{\#(\text{daughters given by women in age class 2j})+}{\#(\text{female population in age class 2j})+}}{\dfrac{+\#(\text{daughters given by women in age class 2j+1})}{+\#(\text{female population in age class 2j+1})}} \quad \text{if } 0 \leq j \leq \frac{m}{2} - 1 \tag{14}$$

and

$$F'_{\frac{m}{2}} = \frac{\#(\text{daughters given by women in age class m})}{\#(\text{female population in age class m})}. \tag{15}$$

That is, the weights of the graph G_{2c} are given by:

$$\begin{cases} F'_j = \dfrac{F_{2j} + F_{2j+1}}{n_{x_{2j}} + n_{x_{2j+1}}} \text{ if } 0 \leq j \leq \dfrac{m}{2} - 1 & (16) \\ F'_{\frac{m}{2}} = F_m. & (17) \end{cases}$$

4.1.2 Survival Rates

Considering that the system is closed, the survival rates are obtained as:

$$P_i = 1 - \frac{\#(\text{women that passed away in age-class i})}{\#(\text{midyear female population in age-class i})}. \tag{18}$$

Proceeding the same way as with the fertility rates, the Leslie matrix with $2c$ age classes can be translated into a graph with weights given as:

$$P'_j = \frac{P_{2j} + P_{2j+1}}{n_{x_{2j}} + n_{x_{2j+1}}} \text{ if } 0 \leq j \leq \frac{m}{2} - 1. \tag{19}$$

4.2. m is odd

Suppose that m is an odd number and let the graph in Figure 3 represent the incidence matrix of a Leslie matrix with $2c$ age classes.

4.2.1 Fertility Rates

The weights of the graph in Figure 3 representing the fertility rates are given by, for $0 \leq j \leq \frac{m+1}{2} - 1$:

$$F'_j = \frac{F_{2j} + F_{2j+1}}{n_{x_{2j}} + n_{x_{2j+1}}}. \tag{20}$$

4.2.2 Survival Rates

Mutatis mutantis, the weights that, for $0 \leq j \leq \frac{m+1}{2} - 2$, represent the survival rates are:

$$P'_j = \frac{P_{2j} + P_{2j+1}}{n_{x_{2j}} + n_{x_{2j+1}}}. \tag{21}$$

4.3. Re-weighting the edges of the graph $\mathbf{G_{2c}}$

The key idea is to perform a mean of the two edges, i and $i+1$, that are going to be joined with a correction factor.

To find this factor, g, and as consequence the weight of the corresponding edge w'_j of G_{2c}, we must solve the system:

$$\begin{cases} w_i = \dfrac{a_i}{n_{x_i}} & (22) \\ w_{i+1} = \dfrac{a_{i+1}}{n_{x_{i+1}}} & (23) \\ w'_j = \dfrac{w_i + w_{i+1}}{g}, & (24) \end{cases}$$

in order of w_i, w_{i+1}, n_{x_i} and $n_{x_{i+1}}$, that has as result:

$$g = \frac{(n_{x_i} + n_{x_{i+1}})(w_i + w_{i+1})}{n_{x_i} w_i + n_{x_{i+1}} w_{i+1}}. \quad (25)$$

Once again we obtain two different scenarios.

If m is even, the weight of the edge w'_j is given by:

$$\begin{cases} w'_j = \dfrac{w_{2j} + w_{2j+1}}{\dfrac{(n_{x_{2j}} + n_{x_{2j+1}})(w_{2j} + w_{2j+1})}{n_{x_{2j}} w_{2j} + n_{x_{2j+1}} w_{2j+1}}} & (26) \\ \text{if } 0 \leq j \leq \dfrac{m}{2} - 1 & (27) \\ w'_{\frac{m}{2}} = w_m & (28) \\ \textit{if such } w_m \textit{ exists.} & (29) \end{cases}$$

If m is odd, for $0 \leq j \leq \frac{m+1}{2} - 1$, the weight of the edge w'_j is given by, if such w_m exists:

$$w'_j = \frac{w_{2j} + w_{2j+1}}{\frac{(n_{x_{2j}} + n_{x_{2j+1}})(w_{2j} + w_{2j+1})}{n_{x_{2j}} w_{2j} + n_{x_{2j+1}} w_{2j+1}}}. \quad (30)$$

4.4. Proposed Algorithm

Suppose that the input Leslie matrix, lc, has dimensions $n \times n$ (a Leslie matrix is always a square matrix).

The complexity of the proposed algorithm, present on Appendix B, is $O(2\lceil \frac{n}{2} \rceil)$.

4.5. Application to open systems

An interesting remark is that, even if the studied Leslie matrix included migrations, the proposed algorithm would still apply, since the migrations (mig_x) are added to the sub-diagonal as follows:

$$P_j = 1 - mort_j - mig_j, \quad (31)$$

where the migration rate is the emigration rate (em_x) minus the immigration rate (imm_x):

$$mig_x = em_j - imm_j. \quad (32)$$

That is, considering an open system, the probability of survival of some class j to the next class $j+1$, is not only dependent on the mortality rate, but also on the migration flow (the proportion of inhabitants leaving the system and the proportion of individuals deciding to be a part of the system).

However, this information can be compressed the same way as P'_j in a closed system 21:

$$P'_j = \frac{P_{2j} + P_{2j+1}}{n_{x_{2j}} + n_{x_{2j+1}}}.$$

5. Real World Context

Since this work was developed alongside a project called *Conhecer Arroios*, an intercensus study in 2017, it allowed us to apply and refine the models, while having a real world context behind the analysed data.

The calculations of the several parameters were based upon Census data from 2001 and 2011. These estimates, needed to perform projections with the models exposed before, are suppressed, being only presented on a table. Next, the effect of changes to the parameters that constitute the population with and without migration will be studied.

5.1. Application of the models

In order to decrease the error, and given that the Leslie Matrix obtained for 2001 is supposed constant overtime, when we obtain any projection for a year greater than 2011, we start all iterations at $t = 2011$, considering 5 years age classes. That is,

$$\mathbf{n_{2011+5i}} = \mathbf{L_{2011}}^i \times \mathbf{n_{2011}}. \quad (33)$$

We follow the supposition that the migration flows follow an uniform distribution with time.

5.1.1 Model without migratory flows

Therefore, the parameters for Leslie Matrices that constitute this model for closed systems are displayed in the tables presented in the Appendix A.1.

The shift between the estimated female population using this model in 2011 and the actual female population encountered in Census 2011 is pictured in Figure 4, and the error estimate between these two populations is 94.068%.

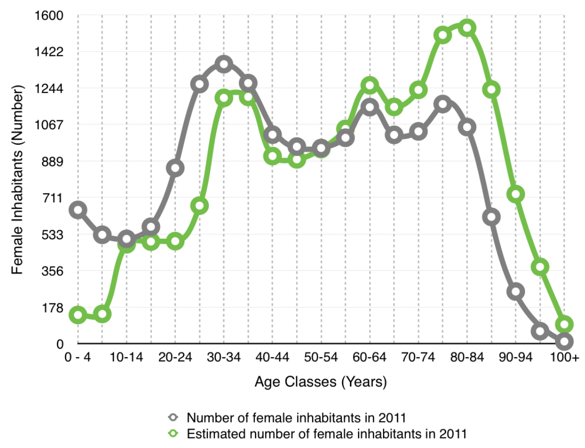

Figure 4: Distribution of female inhabitants by age classes in Arroios using 2011's Census data, and using the model without migrations.

This model predicts a decrease on the total female population along time.

The analysis of the projected distribution of female inhabitants by age classes shows that there is a shift in the most expressive class. Moreover, it is expected an increase in elderly people, and a decrease in younger people and in female birth, leading to the propagation of this effect throughout the years.

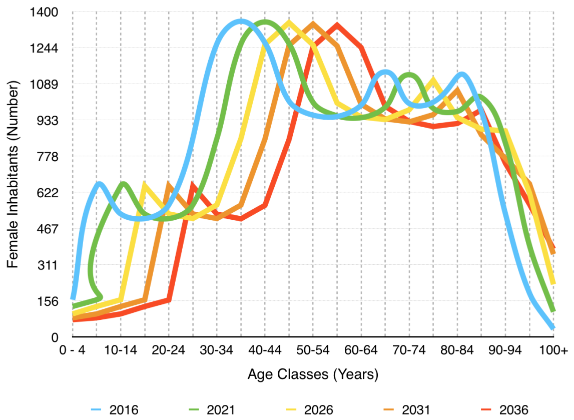

Figure 5: Projected distribution of female inhabitants by age classes in Arroios using the model without migrations in 2016, 2021, 2026, 2031 and 2036.

The model without migrations is clearly inadequate to a parish, since it lacks to explore migration flows of its inhabitants, which, in a long term, results in large errors in projections.

5.1.2 Model with migrations

The coefficients of the Leslie matrix presented in Appendix A.2 were obtained through the referred model. Afterwards, an iterative method, that diminishes the overall error between the calculated female population and the observed one in 2011, was applied.

The error for this Leslie matrix that includes migration for 2011, without performing any tuning, is 0.39237, and after using the iterative method is 0.0526008.

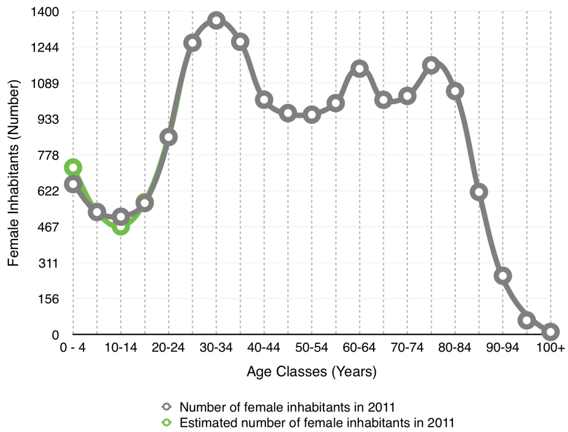

Figure 6: Distribution of female inhabitants by age classes in Arroios using 2011's Census data, and using the model with migrations.

This model predicts a slight decrease between 2001 and 2026, and a small increase from 2026 to 2036.

The analysis of the projected distribution of female inhabitants by age classes reveals that there is also a shift of the most expressive classes between 2016 and 2036, translating into a more mature predominant age class. This model projects a stable number of daughters and a decrease on the number of elderly women.

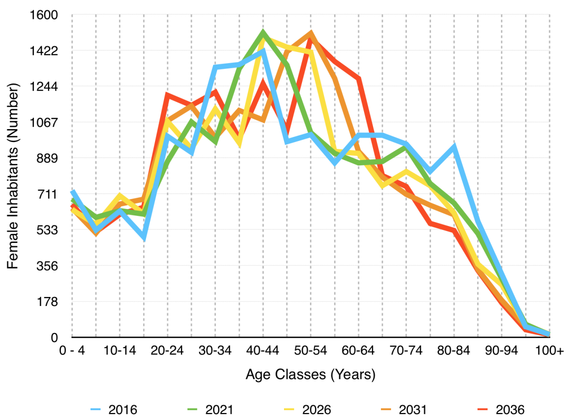

Figure 7: Projected distribution of female inhabitants by age classes in Arroios using the model with migrations in 2016, 2021, 2026, 2031 and 2036.

5.2. Application of the Growth Rate, the Reproductive Potential and the Evolutionary Entropy

The growth rate, the reproductive potential and the evolutionary entropy were found for the two previously explained models, being presented in the Appendix A.3.

In a general way, the results obtained for these measures align themselves to what was observed in our previously exposed projections.

In fact, the Leslie matrix without migrations has the smallest growth rate, suggesting that the decrease in Arroios' population is higher when only the inhabitants, without migrants, are considered. Furthermore, this population without migrations has the lowest reproductive potential, which indicates a low capacity of reproduction of the individuals and slow renovation of the population with time. The evolutionary entropy is also the lowest, suggesting that this population without migrations is more susceptible to changes in microscopic parameters that describe the system, being more exposed to extinction.

On the other hand, the model with migration has the highest growth rate and, even though it is close to zero, it is positive, leading to the conclusion that the population is slightly growing but tending to stability. Another important measure of population dynamics is the reproductive potential, which is the smallest but still negative, meaning that the renovation of individuals is not at a rate that compensates the exits and deaths verified in this society with migrations. Also, the evolutionary entropy value is the highest but yet corresponds to a population that is very vulnerable to perturbations. In fact, some endangered species have higher entropy values. For example, the Desert tortoise (*Gopherus agassizii*) has a higher evolutionary entropy value ($\mathbf{H} = 0.548$) [5].

5.3. Population Sample from 2017 and the Application of Models

This work is closely related to a project called *Conhecer Arroios*, a partnership between *Junta de Freguesia de Arroios* and *Instituto Superior TÃľcnico de Lisboa*, allowed us to obtain a population sample of 2344 inhabitants. Notice that the total population in 2011 was 32262 inhabitants. The interviews were made between 1^{st} April 2017 and 13^{th} May 2017. Therefore, these results will be compared to the projections obtained for 2016. For the sake of the simplification, only female population will be considered, even though the total population can be calculated if we multiply the number of women by 1.86, which is the observed proportion between female inhabitants and total population in Arroios in 2011.

The sampling error, since we assume that the population is infinite, is given by:

$$E \approx \frac{1.29}{\sqrt{2344}} \approx 2.66\%. \tag{34}$$

The sampling method is further detailed on [8].

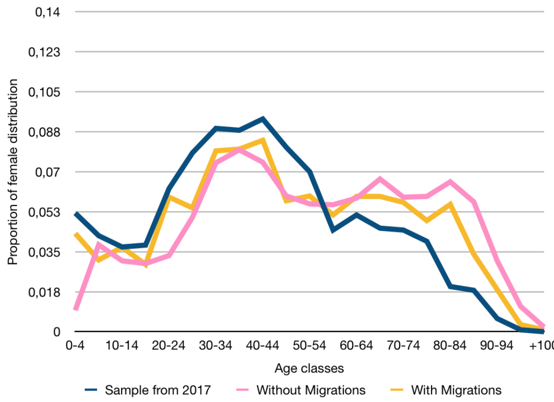

Figure 8: Projected distribution of female inhabitants using models without migration and with migration in 2016, and distribution of female inhabitants of the population sample taken from 2017, by age classes in Arroios.

The model without migrations predicts a population more mature and with less young people. The projections for 2016 obtained using the model with migration describes better the observed female distribution than using the model without migrations. Consequently, we arrive to the conclusion that the impact of migration in Arroios is very high. Notice that only the model with migrations correctly predicted the most expressive age class in the female sample. However, this model underestimates the youngest age classes and overestimates the eldest age classes.

Since we assumed, in the construction of the models, that the migration flow was steady and it would have the same behaviour of the one observed between 2001 and 2011, the comparison between our projections and the real world results may lead us to the conclusion that more young couples, with and without children, entered this parish than between the same period ten years before. Furthermore, the emigration of people with more that 55 years old increased. Almost certainly, Arroios parish is even more rejuvenated than our projections.

6. Conclusions

In this work, we explored models for closed and open systems, using as basis the work of Demetrius [6], Leslie [11], Caswell [4], Lefkovitch [9], and L.A. Demetrius and H.M. Oliveira[5]. The model with migratory flows, which is as far as we know a new way of modelling an ecosystem, rises as a possible answer to the need of including migrations in a Leslie matrix. Therefore, it enables the calculation of measures that indicate how much the system changes from its steady state after a perturbation.

The evolutionary entropy is then explained, being followed by its characteristics combined with parameters such as generation time, variational principles and robustness. Moreover, we developed the theoretical analysis of the growth rate sensitivity.

The compression of Leslie matrices is detailed and is, as far as we know, new and original work. It consists in method based on graphs that reduces the number of nodes by half, with null error. In a future work, we pretend to implement the proposed algorithm in languages such as R, which can be very useful to researchers in this field.

Lastly, we applied and analysed the population models and measures (growth rate, reproductive potential and evolutionary entropy) in a real world context, namely in a parish in Lisbon called Arroios. The fact that we applied our theoretical models to a real population generated very interesting remarks such as the emigration of the elderly, due to the lack of nursing homes in this region, and high immigration of children, due to the high number of young couples with children that were immigrants.

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A. Appendix

A.1. Fertility and Survival Rates of the Model Without Migratory Flows

	F_3	F_4	F_5	F_6	F_7	F_8	F_9
Leslie Matrix I	0.0101563	0.0286868	0.0443657	0.0401565	0.0126653	0.00325448	0.0000779663

Table 1: Fertility rates obtained from 2001's data without considering migratory flows.

	P_0	P_1	P_2	P_3	P_4	P_5	P_6
Leslie Matrix I	0.998069	0.999676	0.999902	0.999466	0.999733	0.999054	0.998966

	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
Leslie Matrix I	0.998043	0.997554	0.995478	0.99625	0.995094	0.991306	0.989036

	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}
Leslie Matrix I	0.978306	0.960356	0.921155	0.859404	0.744179	0.573544

Table 2: Survival rates obtained from 2001's data without considering migratory flows.

A.2. Fertility and Survival Rates of the Model With Leslie Matrix that Includes Immigration and Emigration

	F_3	F_4	F_5	F_6	F_7	F_8	F_9
Leslie Matrix I	0.0608461	0.109119	0.168138	0.187409	0.0598817	0.0147132	0.000326433

Table 3: Fertility rates using the model with migrations and an iterative method.

	P_0	P_1	P_2	P_3	P_4	P_5	P_6
Leslie Matrix I	0.868069	1.00968	1.03572	1.08096	1.36505	0.83779	1.07718

	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}
Leslie Matrix I	1.03561	1.03418	0.969863	0.98868	0.924553	0.949489	0.870746

	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}
Leslie Matrix I	0.868008	0.751969	0.593521	0.4676	0.2264	0.225816

Table 4: Survival rates using the model with migrations and an iterative method.

A.3. Sensitivity Analysis of the Models with Migration and without Migration

	Model without migrations	**Model without migrations**
Growth Rate	-0.308654	0.0134648
Reproductive Potential	-0.497105	-0.210466
Evolutionary Entropy	0.188451	0.22393

Table 5: Sensitivity analysis of the models with migration and without migration.

B. Appendix: Algorithm 1 Compression of the Leslie Matrix

procedure LESLIECOMPRESSED(lc,p)

Input:lc, Leslie matrix subdivided into c age-classes in the considered initial year t, and p, the corresponding column vector containing the number of females

Output:$l2c$, Leslie matrix subdivided into 2c age-classes

$\textbf{Solve}\left[\frac{(a_{2j-1}+a_{2j})}{(n_{x_{2j-1}}+n_{x_{2j}})} == \frac{(w_{2j-1}+w_{2j})}{g},\ w_{2j-1} == \frac{a_{2j-1}}{n_{x_{2j-1}}},\ w_{2j} == \frac{a_{2j}}{n_{x_{2j}}},\ a_{2j-1}, a_{2j}, n_{x_{2j-1}}, n_{x_{2j}}, g\right]$

$g[n_{x_{2j-1}}, n_{x_{2j}}, w_{2j-1}, w_{2j}] = \frac{(n_{x_{2j-1}}+n_{x_{2j}})(w_{2j-1}+w_{2j})}{(n_{x_{2j-1}} w_{2j-1})+(n_{x_{2j}} w_{2j})};$

$q = 1;$

$r = \{\}$

$l = \{lc_1, \textbf{Table}[lc_{j,j-1}, j, 2, \textbf{Dimensions}[lc]_1]$

while $q \leq \textbf{Dimensions}[l]_1$ **do**

$i = 1$

$m1 = \{\}$

$w = l_q$

while $i \leq \lceil \frac{\textbf{Dimensions}[\mathrm{w}]_1}{2} \rceil$ **do**

if $i \neq \lceil \frac{\textbf{Dimensions}[\mathrm{w}]_1}{2} \rceil$ **then**

if $w_{2i-1} \neq 0 \&\& w_{2i} \neq 0$ **then**

$m1 = \textbf{Append}\left[m1, \frac{w_{2i-1}+w_{2i}}{g[p_{2i-1,1}, p_{2i,1}, w_{2i-1}, w_{2i}]}\right]$

else

if $w_{2i-1} == 0 \&\& w_{2i} \neq 0$ **then**

$m1 = \textbf{Append}\left[m1, w_{2i}\right]$

else

if $w_{2i-1} \neq 0 \&\& w_{2i} == 0$ **then**

$m1 = \textbf{Append}\left[m1, w_{2i-1}\right]$

else

$m1 = \textbf{Append}\left[m1, 0\right]$

end if

end if

end if

else

if isEven$\left[\textbf{Dimensions}[\mathrm{w}]_1\right]$ **then**

$m1 = \textbf{Append}\left[m1, \frac{w_{2i-1}+w_{2i}}{g[p_{2i-1,1}, p_{2i,1}, w_{2i-1}, w_{2i}]}\right]$

else

$m1 = \textbf{Append}\left[m1, w_{2i-1}\right]$

end if

end if

$i++$

end while

$r = \textbf{Append}[r, m1]$

$q++$

end while

if isEven$[\textbf{Dimensions}[w]_1]$ **then**

$lc = \textbf{Table}[\textbf{PadLeft}[\textbf{PadRight}[\{r_{2,u+1}\}, \frac{\textbf{Ceiling}[\textbf{Dimensions}[lc]]_1}{2} - u], \frac{\textbf{Ceiling}[\textbf{Dimensions}[lc]]_1}{2}], \{u, 0, \textbf{Dimensions}[r_2]_1 - 1\}]$

else

$lc = \textbf{Table}[\textbf{PadLeft}[\textbf{PadRight}[\{r_{2,u+1}\}, \frac{\textbf{Ceiling}[\textbf{Dimensions}[lc]]_1}{2} - u], \frac{\textbf{Ceiling}[\textbf{Dimensions}[lc]]_1}{2}], \{u, 0, \textbf{Dimensions}[r_2]_1 - 2\}]$

end if

$lc = \textbf{Prepend}[lc, r_1]$

end procedure