Barrier Option Pricing under the 2-Hypergeometric Stochastic Volatility Model

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Abstract

The purpose of this work is to investigate the pricing of financial options under the 2-hypergeometric stochastic volatility model. This is an analytically tractable model which has recently been introduced as an attempt to tackle one of the most serious shortcomings of the famous Black and Scholes option pricing model: the fact that it does not reproduce the volatility smile and skew effects which are commonly seen in observed price data from option markets.

After a review of the basic theory of option pricing under stochastic volatility, we employ the regular perturbation method from asymptotic analysis of partial differential equations to derive an explicit and easily computable approximate formula for the pricing of barrier options – one of the most popular types of exotic options – under the 2-hypergeometric stochastic volatility model. The asymptotic convergence of the method is proved under appropriate regularity conditions, and a multi-stage method for improving the quality of the approximation is also discussed.

Keywords: Option pricing theory, Barrier options, Stochastic volatility, Asymptotic analysis, Regular perturbation method.

1 Introduction

Barrier options, which are one of the oldest types of exotic options, have become increasingly popular in the financial derivative industry because they allow for much more flexible payoff schemes than plain vanilla options. It is thus important to construct good barrier option pricing models which are able to reproduce the features observed in real market data.

The simplest model for the pricing of financial derivatives is the Black and Scholes model, in which the price of all the standard barrier call and put options can be written in closed form. However, it is widely known that the strong assumptions of this model are unrealistic. In particular, the constant volatility assumption is clearly incompatible with the so-called smile and skew patterns which are generally present in empirical option prices.

A natural way to address this significant issue is to introduce randomness in the volatility. For this reason, option pricing under stochastic volatility has been the subject of a great deal of research in recent years. Here we focus on the 2-hypergeometric stochastic volatility model, which was introduced by Da Fonseca and Martini [1] as a model which ensures that the volatility is strictly positive — this is an important property which is not present in some other well-established stochastic volatility models. In a very recent paper, Privault and She [2] demonstrated that, under this model, a closed-form asymptotic vanilla option pricing formula can be determined through a regular perturbation method. This is a notable result because their formulas are analytically very simple, which is rarely the case in models with stochastic volatility: as discussed by Zhu [3], the higher complexity of these models usually yields the need for rather sophisticated numerical implementations.

The pricing of exotic options under the 2-hypergeometric model has to our knowledge never been studied in the literature. Motivated by this, we extend the regular perturbation approach of Privault and She in order to derive an asymptotic pricing formula for barrier-type options. We show that, for a given class of nonconstant barrier functions, an explicit asymptotic formula can be obtained...
and its convergence can be proved with the help of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic partial differential equations (PDEs). Given that in general our class of barrier functions does not include constant functions, the choice of a nonconstant barrier function which approximates a certain constant barrier level is also discussed.

2 What is a barrier option?

In this subsection we give a very brief introduction to barrier options, where we restrict our attention to the standard knock-out European barrier options which will be addressed in Section 4. A short overview on other important barrier-type options can be found in [4].

Standard single-asset barrier options are one of the simplest types of the so-called path-dependent options. The fundamental distinguishing characteristic between (European) vanilla and barrier options is the fact that the payoff of the latter does not depend only on the value of the underlying asset at maturity, but also on whether the path of the asset’s price touches a given barrier level during the lifetime of the option.

Knock-out barrier options become worthless if the asset price hits the barrier before maturity. The two standard knock-out barrier call options are:

- The down-and-out call (DOC), whose payoff at maturity $T$ is
  \[(S_T - K)^+ 1_{\{S_t > H \text{ for all } 0 \leq t \leq T\}},\]
i.e., it has the usual vanilla call payoff if the asset price process $S$ does not go below the barrier $H$ during the lifetime of the option, and it is worthless otherwise;

- The up-and-out call, whose payoff at maturity is
  \[(S_T - K)^+ 1_{\{S_t < H \text{ for all } 0 \leq t \leq T\}}\]
meaning that the option becomes worthless if the price process goes above the barrier.

The definition of the two standard knock-out put options is analogous.

A straightforward generalization of these definitions consists in replacing the constant barrier $H$ by a time-dependent barrier $H(t)$ — such options are known as time-dependent barrier options.

According to Section 3.6 of Jeanblanc et al. [5], the knock-out options defined above can furthermore be classified as regular or reverse barrier options, depending on whether they are out or in the money when the barrier is reached, respectively. For instance, a down-and-out call option is a regular barrier option if $K \geq H$; otherwise, it is a reverse barrier option.

3 Barrier option pricing under stochastic volatility

We now present the mathematical formulation of the price of a barrier option under a Markovian stochastic volatility model whose dynamics under the physical measure $\mathbb{P}$ are given by
\[
dS_t = \mu(t, S_t) S_t dt + h(V_t) S_t dW^1_t \\
dV_t = a(t, V_t) dt + b(t, V_t) dW^*_t
\]
where $S$ is the asset price process, $V$ is the volatility process, $W^1$ and $W^*$ are Brownian motions with correlation $\rho \neq \pm 1$, and $h$ is a smooth, positive and increasing function. For brevity, we skip some details, referring the reader to [4].

It is worth stressing that, unlike the Black and Scholes model, this family of stochastic volatility models is able to reproduce the smile and skew effects in implied volatility structures (see [4] for an introduction to these concepts).

Under standard assumptions on the financial market, it is known that stochastic volatility models are incomplete (cf. Definition 2.2 in [4]) and, accordingly, there exist infinitely many risk-neutral measures under which arbitrage-free pricing can be performed. Indeed, if the asset pays no dividends and the riskless (deterministic and time-dependent) interest rate is $r(t)$, then for any sufficiently regular deterministic function $\eta(t, x, v)$ the formula
\[
f^{(n)}(t, x, v) = e^{-\int_0^t r(u) du} \mathbb{E}^{Q^{(n)}}[Y \mid S_t = x, V_t = v] \tag{1}
\]
allows us to compute the arbitrage-free price at time $t$ of all contingent claims $Y$ with maturity at future time $T$. Here $Q^{(n)}$ is a risk-neutral measure equivalent to $\mathbb{P}$ under which the dynamics of the process $(S, V)$ are given by
\[
dS_t = r(t) S_t dt + h(V_t) S_t d\hat{W}^1_t \\
dV_t = [a(t, V_t) - b(t, V_t) \Lambda_t] dt + b(t, V_t) d\hat{W}^*_t
\]
where $\Lambda_t \equiv \Lambda(t, S_t, V_t)$ is defined as
\[
\Lambda(t, S_t, V_t) := \rho \frac{\mu(t, S_t) - r(t)}{h(V_t)} + \sqrt{1 - \rho^2} \eta(t, S_t, V_t)
\]
and $\hat{W}^1_t, \hat{W}^*_t$ are $Q^{(\rho)}$-Brownian motions with correlation $\rho$.

The process $\eta(t,S_t,V_t)$, which is the so-called market price of volatility risk, cannot be identified within the stochastic volatility model, so it must be exogenously specified. But, unfortunately, there is no easy criterion for picking the right functional form for $\eta(t,S_t,V_t)$; consequently, a common practice is to judiciously choose $\eta(t,S_t,V_t)$ in order that the resulting pricing problem is analytically tractable. (See Section 2.7 of Fouque et al. [6] and Section 10.9 of Lipton [7].)

A barrier option with (say) a down barrier is just a contingent claim $Y = \phi(S_T)1_{(\tau_H^*)}$. This means that we can use Equation (1) for the pricing of barrier options under the stochastic volatility model (2) — this is known as the martingale approach to the pricing problem. Moreover, the barrier option price $f^{(\rho)}(t,x,v)$ can also be computed through a PDE approach: by virtue of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic PDEs (Theorem 2.8 in [4], which was proved by Rubio in [8]), $f^{(\rho)}$ is a solution of the two-space-dimensional terminal and boundary value problem

$$
\left(\frac{\partial}{\partial t} + \mathcal{L}^{(\rho)}\right) f^{(\rho)}(t,x,v) = 0, \quad t \in [0,T], \quad x > H
$$

$$
f^{(\rho)}(T,x,v) = \phi(x), \quad x > H
$$

$$
f^{(\rho)}(t,H,v) = 0, \quad t \in [0,T]
$$

where

$$
\mathcal{L}^{(\rho)} = \frac{1}{2}b^2(t,v)x^2 \frac{\partial^2}{\partial x^2} + \rho b(t,v)h(v) \frac{\partial^2}{\partial x \partial v} + \frac{1}{2}b^2(t,v) \frac{\partial^2}{\partial v^2} + r(t)x \frac{\partial}{\partial x} + \left[ a(t,v) - b(t,v)\Lambda(t,x,v) \right] \frac{\partial}{\partial v} - r(t) \Id.
$$

The adaptation of these two pricing approaches to time-dependent barrier options is simple: we just need to redefine the stopping time as $\tau_H := \inf\{u \geq t : S_u \leq H(u)\}$ and to replace the boundary condition of the PDE problem by $f^{(\rho)}(t,H(t),v) = 0$. It is also easy to generalize further to the case of options whose (down) barrier $H(t,v)$ depends both on the time and on the (random) volatility: the stopping time becomes $\tau_H := \inf\{u \geq t : S_u \leq H(u,V_u)\}$ and the boundary condition becomes $f^{(\rho)}(t,H(t),v) = 0$. We will be dealing with this more general class of barrier options in the next section because time and volatility-dependent barrier options will turn out to be very useful for the derivation of an approximate pricing formula for options with constant barriers.

4 An asymptotic expansion approach to barrier option pricing

In this section we will tackle the problem of pricing barrier options under the 2-hypergeometric stochastic volatility — a particular case of the $\alpha$-hypergeometric stochastic volatility model which was defined by Da Fonseca and Martini [1] as follows:

**Definition 4.1.** The $\alpha$-hypergeometric stochastic volatility model is the Markovian diffusion model with dynamics

$$
dS_t = r(t)S_t dt + e^{V_t} S_t dW^1_t
$$

$$
dV_t = \left(a - \frac{c}{2} e^{V_t}\right) dt + \theta dW^*_t
$$

where $W^1$ and $W^*$ are Brownian motions with correlation $\rho$, and $a, c, \theta > 0, a \in \mathbb{R}$ are constants.

Like Da Fonseca and Martini [1], we assume that the model is given directly under a risk-neutral measure $Q$. The deterministic function $r(t)$ represents the (possibly time-dependent) interest rate, while the parameters $a$ and $c$ can be used to set the market price of volatility risk.

It is important to emphasize that the formulation of the $\alpha$-hypergeometric stochastic volatility model given by Da Fonseca and Martini [1] and by Privault and She [2] does not include the drift term $r(t)S_t dt$. If, as in these two papers, the goal is to price vanilla options, then such a zero interest rate assumption does not entail any loss of generality because, as we show in [4], the general case of a nonzero interest rate can be reduced to the case $r(t) = 0$ by rewriting the pricing equation in forward terms. However, this argument breaks down when dealing with barrier options, so the model with nonzero drift is the one which must be considered for our barrier option pricing problem.

So as to lighten the notation, we will henceforth assume that the interest rate is constant, i.e., $r(t) \equiv r$ and therefore $\int_s^t r(s) ds = r(u-t)$). Otherwise, it suffices to replace the $r(u-t)$ terms by $\int_s^t r(s) ds$ in the forthcoming formulae. We will also assume that the asset pays no dividends; as discussed in [4], the extension to assets with a continuously paid deterministic dividend is straightforward.
4.1 The small vol of vol expansion

Our approach to the barrier option pricing problem is based on a PDE regular perturbation method — known as the small vol of vol asymptotic expansion — which consists in rewriting the model as a perturbed Black and Scholes model so as to derive a series expansion of the exact stochastic volatility price around the Black and Scholes price, which should converge when the perturbation parameter tends to zero. Our first step is thus to take the 2-hypergeometric model (4) and replace the constant \( \theta \) by \( \varepsilon \theta \), where \( \varepsilon \) is a small parameter:

\[
\frac{dS_t^\varepsilon}{S_t^\varepsilon} = rS_t^\varepsilon dt + e^{\varepsilon t} S_t^\varepsilon dW_t^1 + e^{\varepsilon \theta t} dW_t^2.
\]

\[
\frac{dV_t^\varepsilon}{V_t^\varepsilon} = \left( -\frac{\varepsilon}{2} e^{2\varepsilon t} \right) dt + \varepsilon \theta dW_t^1.
\]

(Without loss of generality we will take \( \theta = 1 \).)

It is worth pointing out that a somewhat more general approach consists in replacing \( \theta \) by a generic function \( \varepsilon \psi(t,v) \). As can be seen in [4], the more general case is handled in essentially the same way.

Let \( \hat{h}(t,v) \) be some generic time and volatility-dependent barrier function, to be specified later. The (exact) price \( \hat{f}(t,x,v) \) of the DOC option with barrier function \( \hat{h}(t,v) \) under the model (5) is defined (in the PDE approach) as the solution of the terminal and boundary value problem

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^\varepsilon \right) \hat{f}^\varepsilon(t,x,v) = 0, \quad t \in [0,T], \ x > \hat{h}(t,v)
\]

\[
\hat{f}^\varepsilon(T,x,v) = (x-K)^+, \quad x > \hat{h}(T,v)
\]

\[
\hat{f}^\varepsilon(t,\hat{h}(t,v),v) = 0, \quad t \in [0,T]
\]

(6)

where

\[
\mathcal{L}^\varepsilon = \mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \varepsilon^2 \mathcal{L}_2
\]

\[
\mathcal{L}_0 = \left( a - \frac{\varepsilon}{2} e^{2\varepsilon t} \right) \frac{\partial}{\partial v} + \frac{x^2}{2} e^{2\varepsilon t} \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r \text{Id},
\]

\[
\mathcal{L}_1 = \rho x e^v \frac{\partial^2}{\partial x \partial v}, \quad \mathcal{L}_2 = \frac{1}{2} \frac{\partial^2}{\partial v^2}.
\]

(7)

Let us now formally assume that the price \( \hat{f}^\varepsilon(t,x,v) \) can be asymptotically expanded as \( \hat{f}^\varepsilon = \hat{f}_0 + \varepsilon \hat{f}_1 + \varepsilon^2 \hat{f}_2 + \ldots \). Substituting this expansion into the terminal and boundary value problem (6) and equating the terms of order \( \varepsilon^0, \varepsilon^1, \varepsilon^2, \ldots \), we obtain the system of PDEs

\[
\frac{\partial \hat{f}_0}{\partial t} + \mathcal{L}_0 \hat{f}_0 = 0, \quad \frac{\partial \hat{f}_1}{\partial t} + \mathcal{L}_0 \hat{f}_1 + \mathcal{L}_1 \hat{f}_0 = 0, \quad \frac{\partial \hat{f}_2}{\partial t} + \mathcal{L}_0 \hat{f}_2 + \mathcal{L}_1 \hat{f}_1 + \mathcal{L}_2 \hat{f}_0 = 0, \quad \ldots
\]

(8)

with terminal conditions \( \hat{f}_0(T,x,v) = (x-K)^+ \) and \( \hat{f}_j(T,x,v) = 0 \) for \( j = 1,2,\ldots \), and with boundary conditions \( \hat{f}_j(t,\hat{h}(t,v),v) = 0 \) for \( j = 0,1,2,\ldots \).

We intend to derive the first-order approximation for the option price and to demonstrate that (under suitable regularity conditions) it converges in the following sense:

\[
\hat{f}(t,x,v) = \hat{f}_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v) + o(\varepsilon).
\]

(9)

4.2 The zero-order term

The zero-order term \( \hat{f}_0(t,x,v) \) is defined as the solution of the terminal and boundary value problem

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_0 \right) \hat{f}_0(t,x,v) = 0, \quad t \in [0,T], \ x > \hat{h}(t,v)
\]

\[
\hat{f}_0(T,x,v) = (x-K)^+, \quad x > \hat{h}(T,v)
\]

\[
\hat{f}_0(t,\hat{h}(t,v),v) = 0, \quad t \in [0,T]
\]

(10)

In other words, \( \hat{f}_0 \) is simply the option price corresponding to the limiting case \( \varepsilon = 0 \). The equivalent definition of this option price under the martingale pricing framework is

\[
\hat{f}_0(t,x,v) = e^{-r(T-t)} \mathbb{E} \left[ (S^v_T - K)^+ \mathbb{1}_{\{r^*_T \geq T\}} \right] | S^v_t = x
\]

(11)

where \( r^*_T = \inf \{ u \geq t : S^v_u \leq \hat{h}(u,V^v_u) \} \) and \( \{(S^v_u, V^v_u)\}_{u \in [t,T]} \) denotes the diffusion process whose dynamics are given by the noiseless limit \( \varepsilon = 0 \) of the model (5). The (degenerate) log-volatility process \( V^v_t \) is therefore the deterministic function of time which solves the ordinary differential equation \( dV^v_t = (a - \frac{\varepsilon}{2} e^{2\varepsilon V^v_t}) \, dt \) with initial condition \( V^v_t = v \); the explicit solution is

\[
V^v_t = v + a(u-t) + \frac{1}{2} \log \left( 1 + \frac{c}{2u} e^{2\varepsilon (e^{2\varepsilon u} - 1)} \right).
\]

In turn, \( \{S^v_u\}_{u \in [t,T]} \) is simply a geometric Brownian motion with constant drift \( r \) and time-dependent deterministic volatility \( \varepsilon V^v_t \).

For a given (fixed) initial time \( t = t' \) and initial log-volatility \( v = v' \), by recalling the obvious semigroup property \( V^v_t V^{v'}_{t'} = V^{v+v'}_{t+t'} \) (\( t' \leq t \leq u \)) we see that

\[
\hat{f}_0(t,x,V^{v,v'}_{t+t'}) = e^{-r(T-t)} \mathbb{E} \left[ (S^{v,v'}_{T+t'} - K)^+ \mathbb{1}_{\{r^*_T \geq T\}} \right] | S_t^{v,v'} = x
\]

(12)

where \( r^*_T = \inf \{ u \geq t : S^{v,v'}_{u} \leq \hat{h}(u,V^{v,v'}_{u}) \} \). The function \( \hat{f}_0(t,x,V^{v,v'}_{t+t'}) \), which only depends on the variables \( t \) and \( x \), is clearly the definition of the price
of a DOC option under a Black and Scholes model where the interest rate is \( r \), the time-dependent deterministic volatility is \( e^{V_t'} \), \( t \in [t', T] \) and the time-dependent barrier function is \( \hat{h}(t, V_t') \), \( t \in [t', T] \).

The barrier option pricing problem under the Black and Scholes model has been studied in the literature. Rapisarda [9] and Dorfleitner et al. [10] showed that the conditional expectation (12) can be written in closed form provided the barrier function \( \hat{h}(t, V_t') \), \( t \in [t', T] \) is of the form

\[
H_1 \exp \left\{ -r(T - t) + \frac{1 + 2\beta}{2} \gamma^2(t, T, V_t') \right\}
\]

where \( \gamma^2(t, u, v) := \frac{1}{2} \log \left( 1 + \frac{u}{\gamma} e^{2v} (e^{2u} - 1) \right) \), while \( \beta \in \mathbb{R} \) and \( H_1 > 0 \) are parameters. Unfortunately, to the best of our knowledge, an explicit expression for \( \hat{f}_0(t, x, V_t') \) cannot be obtained unless \( \hat{h}(t, V_t') \) has this particular functional form. For this reason, until Subsection 4.4 we will assume that the barrier function \( \hat{h}(t, v) \) takes the specific form

\[
\hat{h}(t, v) := H_1 \exp \left\{ -r(T - t) + \frac{1 + 2\beta}{2} \gamma^2(t, T, V_t') \right\}
\]

in the domain \((t, v) \in [0, T] \times \mathbb{R}\). Given this choice of barrier function, Equation (27) of Rapisarda yields the following result:

**Proposition 4.2.** Let \( \hat{f}_0(t, x, v) \) be the zero-order term in the first-order expansion (9) for the price of a DOC option with barrier function \( \hat{h}(t, v) \) under the model (5). Then

\[
\hat{f}_0(t, x, v) = x \mathcal{N}(d_1(t, x, v)) - Ke^{-r(T-t)} \mathcal{N}(d_2(t, x, v)) - \left( \frac{\hat{h}(t, v)}{x} \right)^{2+2\beta} x \mathcal{N}(d_3(t, x, v)) + \left( \frac{\hat{h}(t, v)}{x} \right)^{2\beta} Ke^{-r(T-t)} \mathcal{N}(d_4(t, x, v))
\]

for \( t \in [0, T] \) and \( x > \hat{h}(t, v) \), where

\[
\begin{align*}
d_1(t, x, v) &= \frac{1}{\gamma}(t, T, x) \\
\times \left( \log \left( \frac{x}{K \sqrt{H_1}} \right) + r(T - t) + \frac{1}{2} \gamma^2(t, T, v) \right)
\end{align*}
\]

\[
\begin{align*}
d_2(t, x, v) &= d_1(t, x, v) - \gamma(t, T, v), \\
d_3(t, x, v) &= d_1(t, x, v) + \frac{2}{\gamma(t, T, v)} \log \left( \frac{\hat{h}(t, v)}{x} \right), \\
d_4(t, x, v) &= d_2(t, x, v) + \frac{2}{\gamma(t, T, v)} \log \left( \frac{\hat{h}(t, v)}{x} \right).
\end{align*}
\]

### 4.3 The first-order term

The first-order term solves

\[
\begin{align*}
\frac{\partial}{\partial t} + L_0 \hat{f}_1 &= -L_1 \hat{f}_0, & t \in [0, T], x > \hat{h}(t, v) \\
\hat{f}_1(T, x, v) &= 0, & x > \hat{h}(T, v) \\
\hat{f}_1(t, \hat{h}(t, v), v) &= 0, & t \in [0, T]
\end{align*}
\]

where the operators \( L_0 \) and \( L_1 \) were defined in (7).

The first step towards the computation of an explicit expression for the first order term is to give a stochastic representation formula for the solution of this terminal and boundary value problem:

**Lemma 4.3.** Assume that \( K \geq H_1 \). Then the function

\[
\hat{f}_1(t, x, v) = \mathbb{E} \left[ \int_t^{T \wedge \tau_k^1} e^{-r(u-t)} L_1 \hat{f}_0(u, S^{t,v}_u, V^{t,v}_u) \, du \bigg| S^{t,v}_t = x \right]
\]

(17)

(16)

where the process \( (S^{t,v}, V^{t,v}) \) and the stopping time \( \tau_k^1 \) are defined as in (11) is the unique classical solution of the terminal and boundary value problem (16).

**Proof.** The proof of this lemma requires a (nontrivial) generalization of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic PDEs (Theorem 2.8 in [4]) to a setting where the ellipticity assumption is not satisfied, as well as other usual assumptions of Feynman-Kac type theorems.

The details are given in [4], pp. 31-34. \( \square \)

Having established this result, the natural course of action would be for us to compute the expected value (17) using the joint law of \((S^{t,v}_u, \tau_k^1)\). However, to the best of our knowledge the analytical expression for the law of \((S^{t,v}_u, \tau_k^1)\) is not available in the literature. For this reason, we instead take an alternative approach where we will take advantage of the known results on the Black and Scholes equation with time-dependent coefficients so as to derive the explicit expression for the first-order term.

With this in mind, we fix an initial time \( t = t' \) and an initial log-volatility \( v = v' \). Then we can write

\[
\hat{f}_1(t, x, V^{t',v'}) = \mathbb{E} \left[ \int_t^{T \wedge \tau_k^1} e^{-r(u-t)} L_1 \hat{f}_0(u, S^{t',v'}_u, V^{t',v'}_u) \, du \bigg| S^{t',v'}_t = x \right]
\]
where \( \tau^t_h = \inf \{ u \geq t : S_t^{u,v'} \leq \hat{h}(u,V_t^{u,v'}) \} \). The next lemma provides the PDE problem associated to this function of the variables \( t \) and \( x \):

**Lemma 4.4.** Assume that \( K \geq H_1 \). Then the function \( \tilde{f}_1(t,x,V_t^{u,v'}) \) is the unique solution of the terminal and boundary value problem

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_t^{u,v'} \right) u(t,x) = -\mathcal{L}_1 \tilde{f}_0, \quad t \in [t', T], \quad x > \hat{H}(t)
\]

\[
u(t, x) = 0, \quad x > \hat{H}(T)
\]

\[
u(t, \hat{H}(t)) = 0, \quad t \in [t', T]
\]

where \( \mathcal{L}_1 \tilde{f}_0 \equiv \mathcal{L}_1 \tilde{f}_0(t,x,V_t^{u,v'}) \), \( \hat{H}(t) \equiv \hat{h}(t,V_t^{u,v'}) \), and \( \mathcal{L}_t^{u,v'} \) is the differential operator defined as

\[
\mathcal{L}_t^{u,v'} := \frac{x^2}{2} e^{2V_t^{u,v'}} \frac{\partial^2}{\partial x^2} + rx \frac{\partial}{\partial x} - r \text{Id}
\]

**Proof.** This lemma is an easy consequence of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic PDEs. (See [4], page 27.)

The proofs of Lemmas 4.3 and 4.4 given in [4] only cover the case \( K \geq H_1 \), i.e., the case of a regular DOC option. If the DOC option is reverse (that is, \( K < H_1 \)), the discontinuity of the payoff makes the problem more difficult, but it is natural to conjecture that it should be possible to generalize these lemmas to the case of a reverse option through some kind of regularization argument where the reverse option is approximated by some sequence of options with smooth payoffs. Therefore, we will not assume that \( K \geq H_1 \) in the subsequent computations, so that it will also be possible to use our first-order expansion for the (approximate) pricing of reverse DOC options.

Next, we want to eliminate the nonhomogeneity in the problem (18) so that it becomes possible to obtain an explicit expression for \( \tilde{f}_1(t,x,V_t^{u,v'}) \) using the known results on barrier option pricing under the Black and Scholes model with time-dependent coefficients. With that in mind, we decompose

\[
\tilde{f}_1(t,x,v) = \tilde{f}_1^{(A)}(t,x,v) - \tilde{f}_1^{(B)}(t,x,v)
\]

where \( \tilde{f}_1^{(A)}(t,x,V_t^{u,v'}) \) is the solution of the nonhomogeneous problem with no boundary conditions

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_t^{u,v'} \right) u(t,x) = -\mathcal{L}_1 \tilde{f}_0, \quad t \in [t', T], \quad x \in \mathbb{R}
\]

\[
u(t, x) = 0, \quad x \in \mathbb{R}
\]

and \( \tilde{f}_1^{(B)}(t,x,V_t^{u,v'}) \) is the solution of the homogeneous terminal and boundary value problem

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_t^{u,v'} \right) u(t,x) = 0, \quad t \in [t', T], \quad x > \hat{H}(t)
\]

\[
u(T,x) = 0, \quad x > \hat{H}(T)
\]

\[
u(t, \hat{H}(t)) = 0, \quad t \in [t', T]
\]

where \( \tilde{f}_1^{(A)}(t,x,V_t^{u,v'}) \equiv \tilde{f}_1^{(A)}(t,h(t),V_t^{u,v'}) \). The existence and uniqueness of solution for each of the PDE problems (19) and (20) is assured by Theorem 1 of Heath and Schweizer [11] and by Theorem 1 of Dorfleitner et al. [10], respectively. Consequently, the difference between these solutions must be equal to the unique solution of (18).

By virtue of the Feynman-Kac theorem of Heath and Schweizer (Theorem 1 in [11]), the stochastic representation formula for the unique solution of the terminal value problem (19) is

\[
\tilde{f}_1^{(A)}(t,x,V_t^{u,v'}) = \exp \left( \int_t^T e^{-r(u-t)} \mathcal{E} \left[ \mathcal{L}_1 f_0(u,S_t^{u,v'},V_t^{u,v'}) \right] \right) du
\]

where \( S_t^{u,v'} \) is a geometric Brownian motion, it follows that \( S_t^{u,v'} \) is lognormally distributed for any \( u \in [t', T] \), and this makes it possible to explicitly compute the expected value inside the time integral. We can therefore write

\[
\tilde{f}_1(t',x,v') = \int_t^{T} e^{-r(u-t)} \mathcal{E} \left[ \mathcal{L}_1 f_0(u,S_t^{u,v'},V_t^{u,v'}) \right] dt
\]

Since \( S_t^{u,v'} \) is a geometric Brownian motion, it follows that \( S_t^{u,v'} \) is lognormally distributed for any \( u \in [t', T] \), and this makes it possible to explicitly compute the expected value inside the time integral. We can therefore write

\[
\tilde{f}_1(t',x,v') = \int_t^{T} e^{-r(u-t)} \mathcal{E} \left[ \mathcal{L}_1 f_0(u,S_t^{u,v'},V_t^{u,v'}) \right] dt
\]

where \( E \) is the (known) Green function associated to the PDE problem; according to our computations in page 30 of [4],

\[
\frac{\partial G^+}{\partial u} (H(u), u, x, t') = \frac{2}{\sqrt{2\pi\gamma^3}(t', u, v')} \hat{H}^2(u) \times
\]

\[
\times \log \left( \frac{x}{\hat{H}(u)} \right) \exp \left\{ - \frac{1}{2\gamma^2(t', u, v')} \left( \log \hat{H}(u) \right)^2 \right\}
\]

\[
-r(u-t') + \frac{1}{2} \gamma^2(t', u, v') \right)^2.
\]
We are thus led to the following explicit expression for the first-order term:

**Proposition 4.5.** Let \( \hat{f}_1(t,x,v) \) be the first-order term in the first-order expansion (9) for the price of a DOC option with barrier function \( h(t,v) \) under the model (5). Assume that \( K \geq H_1 \). Then

\[
\hat{f}_1(t,x,v) = \int_t^T \left[ E_h^\varepsilon(t,u,x,v) - \frac{1}{2} e^{-r(u-t)} e^{2\varepsilon t + \varepsilon \tilde{H}} \frac{\partial \tilde{G}}{\partial w} (\tilde{H}(u),u,x,t) \right] \times \int_u^T E_h^\varepsilon(u,s,\tilde{H}(u),V_{u,s}) \, ds \, du 
\]

for \( t \in [0,T] \) and \( x > \hat{h}(t,v) \), where \( \hat{h}(u) \equiv h(u,V_{u,T}) \).

We observe that the numerical computation of the integral in (22) is much easier than solving numerically the associated PDE problem (16) or computing the expectation (17) via Monte Carlo simulation. We additionally recall that if \( K < H_1 \) then, as discussed above, the integral formula (22) can also be used to compute the first-order term of an (in this case formal) asymptotic expansion for the price of the option.

### 4.4 Proving the convergence of the asymptotic expansion

Now that we have already derived an explicit expression for our first-order approximation (9), it is time to demonstrate that it converges in the limit \( \varepsilon \to 0 \).

Let us start by looking into the PDE problem which is satisfied by the remainder term of the first-order approximation. For \( \varepsilon > 0 \), we define the remainder term as

\[
\hat{f}_2(t,x,v) := \frac{1}{\varepsilon} \left[ \hat{f}(t,x,v) - \left( f_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v) \right) \right].
\]

Then, \( \hat{f}_2 \) satisfies the terminal and boundary value problem

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^\varepsilon \right) u(t,x,v) = -g^\varepsilon_2, \quad t \in [0,T], \quad x > \hat{h}(t,v)
\]

\[
\begin{align*}
& u(T,x,v) = 0, \quad x > \hat{h}(T,v) \\
& u(t,\hat{h}(t,v),v) = 0, \quad t \in [0,T].
\end{align*}
\]

where \( \mathcal{L}^\varepsilon \) is the partial differential operator from (7), and the nonhomogeneity term is

\[
g^\varepsilon_2 \equiv g^\varepsilon_2(t,x,v) := \mathcal{L}^\varepsilon f_0(t,x,v) + (L_1 + \varepsilon L_2) \hat{f}_1(t,x,v).
\]

This is easily seen to be true by recalling that the functions \( \hat{f}^\varepsilon, f_0, \) and \( \hat{f}_1 \) are the unique solutions of the terminal and boundary value problems (6), (10) and (16), respectively.

Next, we use a stochastic representation formula to define a candidate solution \( \hat{f}^{\varepsilon}_2 \) for the PDE problem (24):

\[
\hat{f}^{\varepsilon}_2(t,x,v) := \mathbb{E} \left[ \int_t^T e^{-r(t-s)} g^\varepsilon_2(u,S^\varepsilon_t,V^\varepsilon_t) \, du \right] \bigg| \left. S^\varepsilon_t = x, V^\varepsilon_t = v \right|
\]

where \( r^{\varepsilon,\varepsilon}_k := \inf\{u \geq t : S^\varepsilon_k \leq \hat{h}(u,V^\varepsilon_k)\} \). We emphasize that the process \( (S^\varepsilon_t,V^\varepsilon_t) \) in (26) follows the 2-hypergeometric model (5) with \( \varepsilon > 0 \); in particular, here \( V^\varepsilon_t \) is a nondeterministic process.

We intend to establish a growth estimate for our candidate solution \( \hat{f}^{\varepsilon}_2 \). As a preliminary step, let us first obtain an upper bound for the growth of the function \( g^\varepsilon_2 \) defined in (25):

**Lemma 4.6.** Assume that \( K \geq H_1 \). Then, the function \( g^\varepsilon_2 \) satisfies the following growth condition: for any \( \varepsilon \geq 0 \), there exist constants \( C, k > 0 \) such that

\[
|g^\varepsilon_2(t,x,v)| \leq C (1 + |x|^{2k} + e^{2kv})
\]

for all \( t \in [0,T], v \in \mathbb{R} \) and \( x \geq \hat{h}(t,v) \).

**Proof.** We can obtain an explicit expression for the function \( g^\varepsilon_2 \) by differentiating the expressions (15) and (22) of the zero and first-order terms respectively. After a tedious estimation procedure, the lemma follows. (See Appendix B in [4].)

The next lemma provides the tool for transforming our growth estimate for \( g^\varepsilon_2 \) into a growth estimate for the candidate solution \( \hat{f}^{\varepsilon}_2 \):

**Lemma 4.7.** Let \((S^\varepsilon_t,V^\varepsilon_t)\) be the diffusion process with dynamics (5). Then, for any \( \varepsilon \geq 0 \), there exist constants \( C, m > 0 \) (which may depend on \( k \)) such that

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} \left( S^\varepsilon_s \right)^{2k} + e^{2kv_s} \right] \bigg| \left. S^\varepsilon_t = x, V^\varepsilon_t = v \right| \leq C (1 + |x|^{2m} + e^{2mv})
\]

for all \( t \in [0,T], x > 0 \) and \( v \in \mathbb{R} \).
Proof. The estimate for $\sup_{t \leq u \leq T} e^{2kV_u^\varepsilon}$ is obtained by using Itô’s formula to derive the dynamics of $Z^\varepsilon = e^{2V^\varepsilon}$ and then estimating the moments of the process $Z^\varepsilon$ through a comparison with a geometric Brownian motion. Then, the estimate for $\sup_{t \leq u \leq T} |S_u^\varepsilon|^{2k}$ can be derived from the closed-form expression

$$S_u^\varepsilon = x \exp \left( r(u-t) - \frac{1}{2} \int_t^u e^{2V^\varepsilon} ds + \int_t^u e^{V^\varepsilon} dW_s^1 \right).$$

(The full proof is in [4], pp. 36-37).

Let us now use the results from Lemmas 4.6 and 4.7 to derive the desired upper bound on the growth of the function $\tilde{f}_2$ defined in (26): for any $\varepsilon \geq 0$, there exist constants $C, m > 0$ which do not depend on $(t, x, v)$ such that

$$|\tilde{f}_2(t, x, v)| \leq \int_t^T e^{-r(u-t)} \mathbb{E} \left[ g_2^\varepsilon(u, S_u^\varepsilon, V_u^\varepsilon) | (S_u^\varepsilon \geq h(u, V_u^\varepsilon)) \right] du \leq C \int_t^T \left( 1 + \mathbb{E} \left[(S_u^\varepsilon)^{2k} + e^{2kV_u^\varepsilon} \right] | (S_0^\varepsilon = x, V_0^\varepsilon = v) \right) du \leq C(1 + |x|^{2m} + e^{2mv}) \leq C \left( 1 + |x|^{2m} + e^{2mv} \right)$$

for all $t \in [0, T]$, $x > 0$, $v \in \mathbb{R}$.

The only thing that remains to be proved is that the function $\tilde{f}_2$, which we defined as a candidate solution for the PDE problem (24), is indeed its unique solution. In fact, if we prove this, then it will follow that $\tilde{f}_2$ equals the remainder term $\hat{f}_2$ defined in (23), and the estimate (27) will assure the convergence of the first-order expansion.

Lemma 4.8. Assume that $K \geq H_1$ and fix $\varepsilon > 0$. Then, the function $\tilde{f}_2(t, x, v)$ defined in (26) is the unique solution of the terminal and boundary value problem (24).

Proof. The key ingredient of the proof is to perform the change of variables $z = e^{2v}$ and $y = x - h(t, v)$. It is then straightforward to show that the restated version of the problem is a consequence of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic PDEs. (See [4], pp. 38-41.)

Summarizing, we have established the following convergence theorem:

**Theorem 4.9.** Let $\tilde{f}_0(t, x, v)$ and $\tilde{f}_1(t, x, v)$ be, respectively, the zero and first-order term in the first-order expansion (9) for the price $f^\varepsilon(t, x, v)$ of a DOC with barrier function $h(t, v)$ under the model (5). Assume that $K \geq H_1$. Then, there exist positive constants $C$ and $m$ which are independent of $\varepsilon \in [0, 1]$ such that

$$|f^\varepsilon(t, x, v) - (\tilde{f}_0(t, x, v) + \varepsilon \tilde{f}_1(t, x, v))| \leq C \left( 1 + |x|^{2m} + e^{2mv} \right) \varepsilon^2$$

for all $t \in [0, T]$, $v \in \mathbb{R}$ and $x \geq h(t, v)$.

4.5 Single and multi-stage approximations to constant barriers

Recall that we have been assuming that the non-constant barrier function is of the form (14). For this reason, we have (in general) not been covering the case with greater practical interest, which is that of a barrier option with constant barrier $H$. Notwithstanding, an approximating formula for an option with constant barrier can be obtained if the parameters of (14) are chosen in order that the time and volatility-dependent barrier function is as constant as possible.

Such choice of parameters should take into account the fact that our pricing strategy is based on a small vol of vol expansion which is performed around the noiseless limit $V^c_{t', v'}$ of the log-volatility process $V^c_t$. Therefore, if one wishes to compute the price of the option at time $t' \in [0, T]$ and the initial log-volatility is equal to $v'$, then the parameters $H_1$ and $\beta$ should be chosen such that $h(t', v')$ is close to the constant function $H$ as possible. The simplest choice is $H_1 = H$ and $\beta$ such that $h(t', v') = H$, i.e. $\beta = \frac{v'(T-t')}{(T-t')^2} - \frac{1}{2}$, but this choice can be improved by choosing the parameters in some optimal way (cf. page 3 of Rapisarda [9]).

It should be noted that the two cases where the barrier function (14) can be chosen to be constant are the zero interest rate case (i.e., $r = 0$) and the case where the initial volatility equals its invariant value (i.e., $v' = \log \left( \frac{2m}{\pi} \right)$). Otherwise, the choice $\beta = \frac{v'(T-t')}{(T-t')^2} - \frac{1}{2}$ yields an approximation which is quite good for small maturities. For large maturities it is possible to improve the quality of the approximation through the multi-stage procedure which we describe next.

The idea of the multi-stage method is to resort to a stepwise PDE approach so as to generalize our
pricing technique to the case of a piecewise-smooth barrier function which is of the form (14) in each subinterval of time. Specifically, in analogy with Section 3 of Dorfleitner et al. [10], we now subdivide the interval \([t', T]\) into \(n\) subintervals defined by \(t' = T_0 < T_1 < \ldots < T_n = T\) and consider the continuous barrier function defined by

\[
\hat{h}^{(n)}(t, v) := H_{1,n} \exp \left\{ -r(T - t) + \sum_{i=1}^{n} \frac{1 + 2\beta_i}{2} I(t < T_{i-1}) \gamma^2(t \wedge T_{i-1}, T_{i}, V_{t,v}^{T_{i-1},T_i}) \right\}. \tag{28}
\]

Notice that if we set \(\beta_i = \beta\) for all \(i = 1, \ldots, n\) we obtain (14). But the idea here is to pick \(\beta_1, \ldots, \beta_n\) so that \(\hat{h}^{(n)}(t, v)\) is closer to \(H\) than the single-stage barrier function \(\hat{h}(t, v)\); our choice of \(\beta_i\) should ensure that the barrier function is as constant as possible in the interval \([T_{i-1}, T_i]\). Much like in the single-stage approximation, the simplest choice is \(H_{1,n} = H\) and \(\beta_i = \frac{\gamma^2(T_{i-1}, T_{i}, V_{t,v}^{T_{i-1},T_i})}{\gamma^2(T_{i-1}, T_{i-1}, V_{t,v}^{T_{i-1},T_{i-1}})} - \frac{2}{\gamma^2(T_{i-1}, T_{i-1}, V_{t,v}^{T_{i-1},T_{i-1}})}\).

In order to derive an explicit asymptotic pricing formula for the option with barrier function (28), we take the exact price \(\hat{f}^{(n)}(t, x, v)\), i.e., the solution of the PDE problem (6) with \(\hat{h}(t, v)\) replaced by \(\hat{h}^{(n)}(t, v)\), and formally expand it as

\[
\hat{f}^{(n)}(t, x, v) = \hat{f}_0^{(n)}(t, x, v) + \varepsilon \hat{f}_1^{(n)}(t, x, v) + o(\varepsilon)
\]

where the functions \(\hat{f}_0^{(n)}\) and \(\hat{f}_1^{(n)}\) satisfy (8). Naturally, the nonconstant boundary conditions are now

\[
\hat{f}_0^{(n)}(t, h^{(n)}(t, v), v) = 0 \quad \text{for } j = 0, 1.
\]

The same argument from the single-stage framework shows that for our fixed initial time \(t'\) and initial log-volatility \(v'\), the zero-order term is again the price, under the same Black and Scholes model, of a DOC option with barrier \(\hat{h}^{(n)}(t, V_t^{t,v'});\) that is, \(\hat{f}_0^{(n)}(t, x, V_t^{t,v'})\) is the solution of

\[
\left(\frac{\partial}{\partial t} + L_{bs}^{t,v'}\right) u(t, x, v) = 0, \quad t \in [t', T], \quad x > \hat{h}^{(n)}(t)
\]

\[
u(T, x) = (x - K)^+, \quad x > \hat{H}^{(n)}(T)
\]

\[
u(t, \hat{H}^{(n)}(t)) = 0, \quad t \in [t', T]
\]

where \(\hat{H}^{(n)}(t) \equiv \hat{h}^{(n)}(t, V_t^{t,v'})\). Since this barrier function is of the form (13) in each subinterval \([T_{i-1}, T_i]\), we can obtain an explicit expression for the zero-order term \(\hat{f}_0^{(n)}(t', x, v')\) as follows:

1. We compute \(\hat{f}_0^{(n)}(T_{n-1}, x, V_{T_{n-1}}^{t,v'})\) via the closed-form expression (15), where \(\hat{h}(t, v)\) becomes \(\hat{h}^{(n)}(T_{n-1}, V_{T_{n-1}}^{T_{n-1},T_n})\) and \(\beta\) is replaced by \(\beta_n\).

2. For \(i = n - 2, \ldots, 0\), we use \(\hat{f}_0^{(n)}(T_{i+1}, x, V_{T_{i+1}}^{t,v'})\) as the terminal condition for the PDE problem in the interval \([T_i, T_{i+1}]\), and we compute the explicit expression for \(\hat{f}_0^{(n)}(T_i, x, V_{T_i}^{T_{i-1},T_i})\) using the integral representation formula (A11) in Theorem 1 of Dorfleitner et al. [10].

We note that the resulting explicit formula for \(\hat{f}_0^{(n)}(t', x, v')\) can actually be written in closed form in terms of the cumulative distribution function of the \(n\)-dimensional normal distribution. (See Subsection 3.2.7 in [4].)

As for the first-order term, the strategy to obtain an explicit expression for \(\hat{f}_1^{(n)}(t', x, v')\) is similar to that from the single-stage framework: a twofold appeal to the Feynman-Kac yields that, for fixed \(v = v'\), \(\hat{f}_1^{(n)}(t, x, V_t^{t,v'})\) solves the boundary value problem (18) (with \(h(t, V_t^{t,v'})\) replaced by \(\hat{h}(t, V_t^{t,v'})\)); one can then decompose \(\hat{f}_1^{(n)} = \hat{f}_1^{(A,n)} - \hat{f}_1^{(B,n)}\) as in (19)–(20). We can obtain an explicit expression for \(\hat{f}_1^{(A,n)}(t', x, v')\) through a multistage procedure:

1. In the interval \([T_{n-1}, T]\) the function \(\hat{f}_1^{(A,n)}(t, x, V_t^{t,v'})\) satisfies

\[
\left(\frac{\partial}{\partial t} + L_{bs}^{t,v'}\right) u(t, x) = -L_{A,n}^{t,v'}(t', x, v'), \quad t \in [T_{n-1}, T], \quad x \in \mathbb{R}
\]

\[
u(T, x) = 0, \quad x \in \mathbb{R}
\]

where \(L_{A,n}^{t,v'} \equiv L_{A,n}^{t,v'}(t, x, V_t^{t,v'})\), so we can compute \(\hat{f}_1^{(A,n)}(T_{n-1}, x, V_{T_{n-1}}^{t,v'})\) via the explicit expression (21).

2. For \(i = n - 2, \ldots, 0\), \(\hat{f}_1^{(A,n)}(t, x, V_t^{t,v'})\) satisfies

\[
\left(\frac{\partial}{\partial t} + L_{bs}^{t,v'}\right) u(t, x) = -L_{A,n}^{t,v'}(t, x, V_t^{t,v'}), \quad t \in [T_i, T_{i+1}], \quad x \in \mathbb{R}
\]

\[
u(T_{i+1}, x) = \hat{f}_1^{(A,n)}(T_{i+1}), \quad x \in \mathbb{R}
\]

where the terminal condition \(\hat{f}_1^{(A,n)}(T_{i+1}) \equiv \hat{f}_1^{(A,n)}(T_{i+1}, x, V_{T_{i+1}}^{T_{i},T_{i+1}})\) is the function which has been derived in the previous stage. Hence we can compute the explicit expression for \(\hat{f}_1^{(A,n)}(T_i, x, V_{T_i}^{T_{i-1},T_i})\) by combining the Feynman-Kac theorem of Heath and Schweizer [11] with the known law of the process \(S^{t,v'}\).

After this, the explicit expression for \(\hat{f}_1^{(B,n)}(t', x, v')\) is deduced in a similar fashion: it is easy to see that \(\hat{f}_1^{(B,n)}(T_{n-1}, x, V_{T_{n-1}}^{T_{n-1},T_n})\) can be computed as in the single-stage method; then, for \(i = n - 2, \ldots, 0\), the
function \( \hat{f}_1^{(B,n)}(t, x, V_t^F, \nu') \) satisfies
\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_{\nu'}^{(B,n)} \right) u = 0, \quad t \in \left[ T_i, T_{i+1} \right], x > H^{(n)}(t)
\]
\[
u(T_{i+1}, x) = \hat{f}_1^{(B,n)}(T_{i+1}), \quad x > H^{(n)}(T_{i+1})
\]
\[
u(t, H^{(n)}(t)) = \hat{f}_1^{(A,n)}(t, H^{(n)}(t)), \quad t \in \left[ T_i, T_{i+1} \right]
\]
where \( \hat{f}_1^{(B,n)}(T_{i+1}) \equiv \hat{f}_1^{(B,n)}(T_{i+1}, x, V_{T_{i+1}}^F, \nu') \) and
\( \hat{f}_1^{(A,n)}(t, H^{(n)}(t)) \equiv \hat{f}_1^{(A,n)}(t, H^{(n)}(t), V_t^F, \nu'), \) so we can obtain an explicit representation for \( \hat{f}_1^{(B,n)}(T_i, x, V_T^F, \nu') \) by resorting to formula (A15) in Theorem 1 of Dorfleitner et al. [10].

It is worth pointing out that the justification of the validity of the Feynman-Kac theorems is somewhat more delicate in this multi-stage setting. We will not deal with the technicalities here, but we do note that the natural strategy to deal with the lack of global smoothness consists in applying the Feynman-Kac theorems sequentially in each interval \([T_{n-1}, T_i], \ldots, [t', T_1] \).

As a final remark, let us mention that the choice of \( n \) — and in particular the choice between the single and the multi-stage methods — should be a compromise between computational speed and numerical accuracy, depending on the practical problem at hand.

5 Conclusions

In this article we established an asymptotic pricing formula for barrier options under the 2-hypergeometric stochastic volatility model. Moreover, we showed that our asymptotic technique is not just formal, as it converges when the perturbation parameter tends to zero.

An important feature of our method is that our explicit pricing formula only requires the numerical evaluation of a double definite integral. This is much simpler than the computationally intensive methods which are commonly used for numerically computing option prices under stochastic volatility.

The only drawback of our barrier option pricing technique is the fact that, in general, it requires two approximation steps. However, this shortcoming is partly offset by the fact that the multi-stage method can be employed whenever one needs to improve the quality of the approximation.

It would be interesting to better examine the accuracy of our single and multi-stage approximations through a numerical comparison with (numerically)

exact values obtained e.g. through Monte Carlo simulation or a finite difference scheme. We leave this task for future research.

Bibliography


