Barrier Option Pricing under the 2-Hypergeometric Stochastic Volatility Model

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Resumo

O objetivo desta dissertação é investigar o *pricing* de opções financeiras sob o modelo 2-hipergeométrico de volatilidade estocástica. Este é um modelo analiticamente tratável que foi recentemente introduzido com o intuito de superar uma das principais limitações do famoso modelo de Black e Scholes: este não reproduz os efeitos de *volatility smile* e de *volatility skew* que habitualmente estão presentes nos preços efetivamente observados no mercado de opções.

Depois de uma breve revisão da teoria básica de *pricing* de opções sob volatilidade estocástica, usamos o método de perturbação regular da análise assintótica de equações diferenciais parciais para deduzir uma fórmula explícita e facilmente calculável para a obtenção de preços aproximados para opções de barreira — uma das categorias mais populares de opções exóticas — sob o modelo 2-hipergeométrico de volatilidade estocástica. A convergência assintótica do método é provada sob condições de regularidade apropriadas, e um método multi-etapa para melhorar a qualidade da aproximação é também discutido.

Palavras-chave: Teoria de *pricing* de opções, Opções de barreira, Volatilidade estocástica, Análise assintótica, Método de perturbação regular.

Abstract

The purpose of this thesis is to investigate the pricing of financial options under the 2-hypergeometric stochastic volatility model. This is an analytically tractable model which has recently been introduced as an attempt to tackle one of the most serious shortcomings of the famous Black and Scholes option pricing model: the fact that it does not reproduce the volatility smile and skew effects which are commonly seen in observed price data from option markets.

After a review of the basic theory of option pricing under stochastic volatility, we employ the regular perturbation method from asymptotic analysis of partial differential equations to derive an explicit and easily computable approximate formula for the pricing of barrier options — one of the most popular types of exotic options — under the 2-hypergeometric stochastic volatility model. The asymptotic convergence of the method is proved under appropriate regularity conditions, and a multi-stage method for improving the quality of the approximation is also discussed.

Keywords: Option pricing theory, Barrier options, Stochastic volatility, Asymptotic analysis, Regular perturbation method.

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Chapter 1

Introduction

Barrier options, which are one of the oldest types of exotic options, have become increasingly popular in the financial derivative industry because they allow for much more flexible payoff schemes than plain vanilla options. It is thus important to construct good barrier option pricing models which are able to reproduce the features observed in real market data.

The simplest model for the pricing of barrier options, and financial derivatives in general, is the Black and Scholes model, in which the price of all the standard barrier call and put options can be written in closed form. However, it is widely known that the strong assumptions of this model do not hold true in the actual financial market. In particular, the constant volatility assumption is clearly incompatible with the so-called smile and skew patters which are generally present in empirical option prices.

A natural way to address this significant issue is to introduce randomness in the volatility. For this reason, option pricing under stochastic volatility has been the subject of a great deal of research in recent years. In this thesis we focus on the 2-hypergeometric stochastic volatility model, an analytically tractable model which was introduced by Da Fonseca and Martini [1] as a model which ensures that the volatility is strictly positive — this is an important property which is not present in some other well-established stochastic volatility models. In a very recent paper, Privault and She [2] demonstrated that, under this model, a closed-form asymptotic vanilla option pricing formula can be determined through a regular perturbation method. This is a notable result because their formulas are analytically very simple, which is rarely the case in models with stochastic volatility: as discussed by Zhu [3], the higher complexity of these models usually yields the need for rather sophisticated numerical implementations.

The pricing of exotic options under the 2-hypergeometric model has to our knowledge never been studied in the literature. Motivated by this, we intend to extend the regular perturbation approach of Privault and She in order to derive an asymptotic pricing formula for barrier-type options.

After going through some essential theoretical notions, we will explain that the key modification is to introduce a Dirichlet boundary condition in the Cauchy problems which define each of the terms in the asymptotic expansion. We will show that, for a given class of nonconstant barrier functions, an explicit asymptotic barrier option pricing formula can indeed be obtained and its asymptotic convergence can be proved with the help of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic partial differential equations (PDEs). As we will see, the latter will be a fundamental tool throughout this thesis. Given that in general our class of barrier functions does not include constant functions, the choice of a nonconstant barrier function which approximates a certain constant barrier level will also be discussed.

This thesis is structured as follows:

- Chapter 2 provides a brief introduction to option pricing theory, focusing on barrier options and on stochastic volatility models. The first section familiarizes the reader with both the martingale and the PDE approaches for the pricing of vanilla options. In Section 2.2 we define barrier options, we discuss how the martingale and PDE approaches can be extended to this class of options, and we formulate a version of the Feynman-Kac theorem which allows us to interchange between these two pricing approaches. Section 2.3 motivates the study of stochastic volatility models and indicates how the pricing formulas become modified under these models. Finally, Section 2.4 gives a brief review of asymptotic expansion techniques for option pricing under stochastic volatility, with emphasis on the singular and regular PDE perturbation methods.

- In Chapter 3 we address the problem of option pricing under the 2-hypergeometric stochastic volatility model. Section 3.1 quickly summarizes the asymptotic vanilla option pricing approach of Privault and She [2]. Then we turn our attention to the barrier option pricing problem which is the central subject of this work: our asymptotic pricing theory for barrier-type options is developed in Section 3.2.

- In Chapter 4 we summarize the main conclusions of our work and we indicate some possible directions for further research.

- Appendix A collects some auxiliary computations from the derivation of the explicit expression for our asymptotic pricing formula.

- In Appendix B we derive several growth estimates which are necessary for the proofs of our results.

Chapter 2

Background

2.1 Basic concepts in option pricing

In this section we will briefly present the fundamental theoretical concepts and results which lead to the mathematical formulation of the price of a financial option under a Markovian diffusion model. We closely follow Chapter 2 of Jeanblanc et al. [4], as well as Chapter 3 of Kwok [5].

We consider a model with d financial assets which are traded in continuous time, and whose prices $S^1, \dots, S^d$ are adapted Itô processes modeled by the diffusion equations

$$dS_t^i = b^i(t, S_t) S_t^i \, dt + \sum_{j=1}^{n} \sigma^{ij}(t, S_t) \, dW_t^j, \qquad i = 1, \dots, n \tag{2.1}$$

where $W^1, \dots, W^n$ are independent Brownian motions. The underlying filtered probability space is $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\mathcal{F}_t = \sigma\big(W_u^i,\ i = 1, \dots, n,\ 0 \leq u \leq t\big)$. Moreover, we will make the following standard assumptions on the financial market:

1. Trading takes place continuously in time;
2. The riskless interest rate $r(t)$ is a deterministic function of time;
3. There are neither transaction costs nor taxes in buying and selling the assets and the options;
4. The assets are perfectly divisible;
5. Short-selling of securities is allowed;
6. There are no riskless arbitrage opportunities.

A very important notion in option pricing theory is that of equivalent martingale measure, which we now define.

Definition 2.1. A probability measure $\mathbb{Q}$ on the space $(\Omega, \mathcal{F})$ is said to be an **equivalent martingale measure** if:

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$, i.e, for any $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$;

(ii) The discounted asset prices $\widetilde{S}_t^i := e^{-\int_0^t r(u)\, du} S_t^i$ are martingales under $\mathbb{Q}$.

Recall that a **portfolio** is a $(d+1)$-dimensional predictable process $\{\pi_t = (\pi_t^0, \dots, \pi_t^d)\}_{t \in [0,T]}$, where π_t^i is the number of shares of asset i held at time t, and T is the predetermined maturity date. Here the zeroth asset is the riskless asset, whose deterministic price is given by $S_t^0 = e^{\int_0^t r(u)\, du}$. The value of the portfolio π at time t is defined by $V_t(\pi) = \sum_{i=0}^d \pi_t^i S_t^i$. The portfolio is said to be **self-financing** if

$$V_t(\pi) = V_0(\pi) + \sum_{i=0}^{d} \int_0^t \pi_u^i dS_u^i$$

which means that, after the initial investment at time $t = 0$, no money is added or withdrawn from the savings account.

We are now ready to state the definition of a complete market.

Definition 2.2. We say that the market is **complete** if any contingent claim (i.e, any $\mathcal{F}_T$-measurable random variable) is the value at time T of some self-financing portfolio.

Now we state a theorem (Proposition 8.2.1 of Musiela and Rutkowski [6]) which gives us sufficient conditions for the completeness of the Markovian diffusion model.

Theorem 2.3. *Assume that the class of equivalent martingale measures is nonempty. Then, the following properties are equivalent:*

(i) The Markovian diffusion model (2.1) *is complete;*

(ii) The inequality $d \geq n$ holds and the volatility matrix $\sigma(t, S_t)$ has full rank for Lebesgue almost every $t \in [0, T]$, with probability 1;

(iii) A unique equivalent martingale measure $\mathbb{Q}$ exists.

The martingale approach for the pricing of European options is based on the following theorem, whose proof can be found in Subsection 2.2.2 of Jeanblanc et al. [4].

Theorem 2.4. *Assume that $d = n$, the assets pay no dividends, the matrix $\sigma(t, S_t) = \left[\sigma^{ij}(t, S_t)\right]_{i,j=1,\dots,n}$ is almost surely invertible for all t, and the vector $\theta(t, S_t) = \sigma^{-1}(t, S_t)(\mu(t, S_t) - r(t)\mathbf{1})$ (which is called the **risk premium**) is almost surely bounded. Then all the properties in the previous theorem hold. Furthermore, if $x = (x^1, \dots, x^d)$ are the observed asset prices at time $t \in [0, T]$, then the arbitrage-free price at time t of any contingent claim Y is given by*

$$f(t, x) = e^{-\int_t^T r(u)\, du}\, \mathbb{E}_{\mathbb{Q}}\left[Y \mid S_t = x\right].$$

The dynamics of S under the unique equivalent martingale measure $\mathbb{Q}$ are given by

$$dS_t^i = r(t) S_t^i \, dt + \sum_{j=1}^{n} \sigma^{ij}(t, S_t) S_t^i \, d\widehat{W}_t^j, \qquad i = 1, \dots, n$$

where $\widehat{W}^1, \dots, \widehat{W}^d$ are independent $\mathbb{Q}$-Brownian motions.

The next theorem, which is proved in Subsection 2.2.3 of Jeanblanc et al. [4], provides us an alternative method for the computation of the price of a path-independent European option (such as a plain vanilla call or put) which consists in finding the solution for a PDE which satisfies a suitable terminal condition:

Theorem 2.5. *In the conditions of Theorem 2.4, assume that* $Y = \phi(S_T)$*. If the observed asset price is* $S_t = x$*, then the price* $f(t,x)$ *of the contingent claim* Y *at time* $t \in [0,T]$ *solves the terminal value problem*

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}\sum_{k=1}^{d}\sigma^{ik}\sigma^{jk}(t,x) + r(t)\sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i} - r(t)f = 0 \tag{2.2}$$

$$f(T,x) = \phi(x). \tag{2.3}$$

Remark 2.6. Theorems 2.4 and 2.5, which are valid under the assumption that the assets pay no dividend, can be adapted without difficulties to the case where the assets have a continuously paid deterministic dividend yield rate. As shown in Subsection 3.2.1 of Musiela and Rutkowski [6], in the case of a single asset ($d = n = 1$) with dividend yield rate $q(t)$, the martingale formulation of the price of a contingent claim Y is still $f(t,x) = e^{-\int_t^T r(u)\,du}\,\mathbb{E}_{\mathbb{Q}}\big[Y \mid S_t = x\big]$, but the dynamics of S under the equivalent martingale measure $\mathbb{Q}$ are now given by

$$dS_t = \big(r(t) - q(t)\big)S_t\,dt + \sigma(t,S_t)S_t\,d\widehat{W}_t.$$

Correspondingly, the PDE formulation for the path-independent option is now given by the equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 f}{\partial x^2} + \big(r(t) - q(t)\big)x\frac{\partial f}{\partial x} - r(t)f = 0 \tag{2.4}$$

whose terminal condition is the same.

For the proof of the equivalence of the martingale and the PDE approaches to the pricing of contingent claims, we can rely on the Feynman-Kac theorem, which is a classical theorem on the stochastic representation of solutions of parabolic PDEs. For completeness, let us state a version of this theorem whose proof can be found in Section 6.5 of Friedman [7]:

Theorem 2.7. *Assume that:*

(i) There exists a constant $\theta > 0$ *such that*

$$\sum_{i,j=1}^{d} a^{ij}(t,x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \textit{for all } (t,x) \in [0,T]\times\mathbb{R}^d,\ \xi \in \mathbb{R}^d;$$

(ii) The functions $a^{ij}(t,x)$*,* $b^i(t,x)$ *are bounded in* $[0,T]\times\mathbb{R}^d$ *and uniformly Lipschitz continuous in* (t,x) *in compact subsets of* $[0,T]\times\mathbb{R}^d$*;*

(iii) The functions $a^{ij}(t,x)$ *are Hölder continuous in* x*, uniformly with respect to* (t,x) *in* $\mathbb{R}^n \times [0,T]$*;*

(iv) The function $c(t,x)$ *is bounded in* $[0,T]\times\mathbb{R}^d$ *and uniformly Hölder continuous in* (t,x) *in compact subsets of* $[0,T]\times\mathbb{R}^d$*;*

(v) The function $h(t,x)$ *is continuous in* $[0,T]\times\mathbb{R}^d$*, Hölder continuous in* x *uniformly with respect to* $(t,x) \in [0,T]\times\mathbb{R}^d$*, and* $|h(t,x)| \leq C(1+|x|^k)$ *in* $[0,T]\times\mathbb{R}^d$ *for some constants* $C, k > 0$*;*

(vi) The function $\phi(x)$ is continuous in $\mathbb{R}^d$ and $|\phi(x)| \leq C(1+|x|^k)$ in $\mathbb{R}^d$ for some constants $C, k > 0$.

Then, the unique solution u of the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u = h(t,x) \qquad \text{in } [0,T]\times\mathbb{R}^d$$

$$u(T,x) = \phi(x) \qquad \text{in } \mathbb{R}^d$$

satisfying

$$u(t,x) \leq C(1+|x|^k) \qquad \text{for some constants } C, k > 0$$

is given by

$$u(t,x) = \mathbb{E}\Big[e^{\int_t^T c(u,X_u)\,du}\phi(X_T)\;\Big|\; X_t = x\Big] - \mathbb{E}\Big[\int_t^T e^{\int_t^u c(s,X_s)\,ds} h(u,X_u)\,du \;\Big|\; X_t = x\Big].$$

Here X is the d-dimensional Markovian diffusion process with dynamics

$$dX_t^i = b^i(t,X_t)\,dt + \sum_{j=1}^{d}\sigma^{ij}(t,X_t)\,dW_t^j, \qquad i = 1,\ldots,d. \tag{2.5}$$

where the matrix $\sigma(t,x) = \big[\sigma^{ij}(t,x)\big]_{i,j=1,\ldots,d}$ is the nonnegative definite square root of the matrix $a(t,x) = \big[a^{ij}(t,x)\big]_{i,j=1,\ldots,d}$.

We point out that many popular models in mathematical finance do not comply with the strong assumptions of the above Feynman-Kac theorem. For instance, the boundedness condition in (ii) fails in the context of the famous Black and Scholes model (where $d = 1$, $a(t,x) = \sigma^2 x^2$, $\mu(t,x) = rx$ and $r, \sigma > 0$ are constants). However, in most cases such difficulties can be overcome if we use a generalized version of the Feynman-Kac theorem. (See e.g. Theorem 1 of Heath and Schweizer [8], as well as the discussion preceding it.)

2.2 Barrier options

2.2.1 What is a barrier option?

Single-asset barrier options are one of the simplest types of the so-called path-dependent options. The fundamental distinguishing characteristic between vanilla and barrier options is the fact that the payoff of the latter does not depend only on the value of the underlying asset at maturity, but also on whether the path of the asset's price touches a given barrier level during the lifetime of the option.

Options of the barrier type are nowadays quite popular in over-the-counter markets. As reported by Kwok [5], these options became popular because both buyers and sellers can benefit from the barrier feature: for instance, an up-and-out call (defined on Table 2.1 below) allows the buyer to reduce the premium of the option by not paying to cover the unlikely scenario of a large increase in the asset price, while it insures the seller against unlimited liabilities when the underlying asset's value rises strikingly.

Barrier options subdivide into two categories: **knock-out** barrier options, which become worthless if the asset price hits the barrier before maturity, and **knock-in** barrier options, which only come to existence if the asset price hits the barrier prior to the expiration date. Moreover, some barrier options include

Table 2.1: The eight types of standard European barrier options

European barrier option	Condition	Payoff value	Payoff time
Down-and-in call	$S_t \leq H$ for some $0 \leq t \leq T$	$(S_T - K)^+$	T (maturity)
	$S_t > H$ for all $0 \leq t \leq T$	R	T (maturity)
Up-and-in call	$S_t \geq H$ for some $0 \leq t \leq T$	$(S_T - K)^+$	T (maturity)
	$S_t < H$ for all $0 \leq t \leq T$	R	T (maturity)
Down-and-in put	$S_t \leq H$ for some $0 \leq t \leq T$	$(K - S_T)^+$	T (maturity)
	$S_t > H$ for all $0 \leq t \leq T$	R	T (maturity)
Up-and-in put	$S_t \geq H$ for some $0 \leq t \leq T$	$(K - S_T)^+$	T (maturity)
	$S_t < H$ for all $0 \leq t \leq T$	R	T (maturity)
Down-and-out call	$S_t > H$ for all $0 \leq t \leq T$	$(S_T - K)^+$	T (maturity)
	$S_t \leq H$ for some $0 \leq t \leq T$	$R(t)$	t (time of hit)
Up-and-out call	$S_t < H$ for all $0 \leq t \leq T$	$(S_T - K)^+$	T (maturity)
	$S_t \geq H$ for some $0 \leq t \leq T$	$R(t)$	t (time of hit)
Down-and-out put	$S_t > H$ for all $0 \leq t \leq T$	$(K - S_T)^+$	T (maturity)
	$S_t \leq H$ for some $0 \leq t \leq T$	$R(t)$	t (time of hit)
Up-and-out put	$S_t < H$ for all $0 \leq t \leq T$	$(K - S_T)^+$	T (maturity)
	$S_t \geq H$ for some $0 \leq t \leq T$	$R(t)$	t (time of hit)

a **rebate**, i.e, a compensation which is paid to the holder of a knock-out option when it is canceled, or to the holder of a knock-in option in case the barrier is not hit before maturity.

Table 2.1 describes the eight types of standard European barrier options. Here S is the asset price process, K is the strike, H is the barrier level, and R is the (possibly time-dependent) rebate.

According to Section 3.6 of Jeanblanc et al. [4], the barrier options in Table 2.1 can furthermore be classified as **regular** or **reverse** barrier options, depending on whether they are out or in the money when the barrier is reached, respectively. For instance, a down-and-out call option is a regular barrier option if $K \geq H$; otherwise, it is a reverse barrier option.

A portfolio made of one (zero-rebate) European in-option and one European out-option, both with the same barrier, strike and maturity, is clearly equivalent to a European plain-vanilla option. Therefore, there is no loss of generality if only out-options are considered in barrier option pricing problems — by non-arbitrage arguments, the price of the corresponding in-option is then obtained as the difference between the price of the corresponding vanilla option and the price of the out-option. This is the so-called **in-out parity relation** for barrier options.

If in Table 2.1 we replace the payoffs $(S_T - K)^+$ or $(K - S_T)^+$ by some function $\phi(S_T)$, we obtain a general (down-and-in, up-and-in, down-and-out or up-and-out) European barrier option. Some important particular cases (e.g. for down-and-in options) are the **binary down-and-in call**, where $\phi(S_T) = \mathbb{1}_{\{S_T > K\}}$, the **binary down-and-in put**, where $\phi(S_T) = \mathbb{1}_{\{S_T < K\}}$, and the **down-and-in bond**, where $\phi(S_T) = 1$. Another possible generalization consists in replacing the constant barrier H by a time-dependent barrier $H(t)$ — such options are known as **time-dependent barrier options**.

Replacing the standard one-sided barriers by two-sided barriers, we obtain the so-called **double**

barrier options, which get knocked in or out when the price of the underlying asset hits either the down or the up barrier. For instance, a **double knock-out call** has a payoff $(S_T - K)^+$ if the price of S does not hit neither the upper nor the lower barrier; otherwise, a rebate is paid to the holder. The value of the rebate can depend on which of the boundaries was first hit by the underlying asset; moreover, as in the case of single barrier options, any payoff function $\phi(S_T)$ can be used to define a double barrier option.

2.2.2 Barrier option pricing theory

Under the Markovian diffusion model introduced in Section 2.1 (where we shall now assume, for simplicity, that $d = n = 1$), Theorem 2.4 allows us to define the price of, say, a down-and-out barrier option with barrier level H and payoff at maturity equal to $\phi(S_T)$ as

$$f_{\mathrm{do}}(t,x) = e^{-\int_t^T r(u)\,du}\,\mathbb{E}_{\mathbb{Q}}\left[\phi(S_T)\mathbb{1}_{\{\tau_H^t \geq T\}} | S_t = x\right] \tag{2.6}$$

where $\tau_H^t := \inf\{u \geq t : S_u \leq H\}$ and $dS_t = r(t)S_t\,dt + \sigma(t,S_t)S_t\,d\widehat{W}_t$ under the measure $\mathbb{Q}$. The **martingale approach** to barrier option pricing consists in computing the expectation in (2.6).

As in the case of path-independent options, the price of a barrier option can also be formulated through a **PDE approach**. Indeed, as in Theorem 2.5, the price of the down-and-out barrier option must satisfy the (one-dimensional version of) PDE (2.2) with the terminal condition (2.3), but now these should be combined with the boundary condition $f_{\mathrm{do}}(t,H) = 0,\ t \in [0,T]$ so as to take care of the knock-out barrier. That is, $f_{\mathrm{do}}(t,x)$ is the solution of the terminal and boundary value

$$\begin{aligned} \frac{\partial f_{\mathrm{do}}}{\partial t} + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 f_{\mathrm{do}}}{\partial x^2} + r(t)x\frac{\partial f_{\mathrm{do}}}{\partial x} - r(t)f_{\mathrm{do}} &= 0, \qquad && x > H, t \in [0,T] \\ f_{\mathrm{do}}(T,x) &= \phi(x), && x > H, \\ f_{\mathrm{do}}(t,H) &= 0, && t \in [0,T]. \end{aligned}$$

The above two formulations of the barrier option price are valid under the assumption that the asset pays no dividends. In the deterministic dividend yield scenario, the PDE for $f_{\mathrm{do}}(t,x)$ is instead given by (2.4), whereas the terminal and boundary conditions are not altered.

The adaptation to time-dependent barrier options is straightforward: we just need to redefine the stopping time in (2.6) as $\tau_H^t := \inf\{u \geq t : S_u \leq H(u)\}$ and to replace the boundary condition of the PDE problem by $f_{\mathrm{do}}(t,H(t)) = 0$.

In order to demonstrate the equivalence of the martingale and PDE barrier pricing approaches, we can employ the following theorem on the stochastic representation of solutions of Cauchy-Dirichlet problems for parabolic PDEs:

Theorem 2.8. *Let $D \subset \mathbb{R}^d$ be an open, connected and possibly unbounded set whose boundary ∂D has the outside strong sphere property, and let $\lambda \in (0,1)$. Assume that:*

(i) For all $n > 1$, the functions $\sigma^{ij}(t,x)$ and $b^i(t,x)$ are λ-Hölder continuous in t and Lipschitz continuous in x in the domain $\{(t,x) : 0 \leq t \leq T, |x| \leq n\}$;

(ii) There exists K_1 such that $\sum_{i,j=1}^d |\sigma^{ij}(t,x)|^2 + \sum_{i=1}^d |b^i(t,x)|^2 \leq K_1\big(1+|x|^2\big)$ for all $(t,x) \in [0,T]\times\mathbb{R}^d$;

(iii) Let $B \subset \overline{D}$ be any bounded, open, connected set. There exists $\theta(B) > 0$ such that

$$\sum_{i,j=1}^{d} a^{ij}(t,x)\xi_i\xi_j \geq \theta(B)\,|\xi|^2 \qquad \text{for all } (t,x) \in [0,T] \times \overline{B},\ \xi \in \mathbb{R}^d,$$

where $a(t,x) = \left[a^{ij}(t,x)\right]_{i,j=1,\dots,n} := \sigma\sigma'(t,x)$;

(iv) The functions $c(t,x)$ and $h(t,x)$ are λ-Hölder continuous in t and Lipschitz continuous in x in the domain $\{(t,x) : 0 \leq t \leq T, x \in \overline{D}, |x| \leq n\}$;

(v) There exists $c_0 \geq 0$ such that $c(t,x) \leq c_0$ for all $(t,x) \in [0,T] \times \overline{D}$;

(vi) There exist constants $K_2, k > 0$ such that $|h(t,x)| \leq K_2\big(1 + |x|^k\big)$ for all $(t,x) \in [0,T] \times \overline{D}$;

(vii) The functions $\phi(x)$ and $\varphi(t,x)$ are continuous and satisfy the consistency condition $\phi(x) = \varphi(T,x)$, $x \in \partial D$;

(viii) There exist constants $K_3, k > 0$ such that $|\phi(x)| + |\varphi(t,x)| \leq K_3\big(1 + |x|^k\big)$ for all $(t,x) \in [0,T] \times \overline{D}$.

Then, the unique solution $u \in C([0,T] \times \overline{D}) \cap C^{1,2,\lambda}_{\mathrm{loc}}((0,T) \times D)$ of the Cauchy-Dirichlet problem

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u = h(t,x) \qquad & (t,x) \in [0,T] \times D \\
u(T,x) = \phi(x) \qquad & x \in D \\
u(t,x) = \varphi(t,x) \qquad & (t,x) \in [0,T] \times \partial D
\end{aligned}$$

is given by

$$\begin{aligned}
u(t,x) =& \mathbb{E}\Big[e^{\int_t^{\tau_t} c(u,X_u)\,du}\varphi(\tau_t, X_{\tau_t})\mathbb{1}_{\{\tau_t < T\}} \,\Big|\, X_t = x\Big] \\
&+ \mathbb{E}\Big[e^{\int_t^{T} c(u,X_u)\,du}\phi(X_T)\mathbb{1}_{\{\tau_t \geq T\}} \,\Big|\, X_t = x\Big] \\
&- \mathbb{E}\Big[\int_t^{\tau_t} e^{\int_t^{u} c(s,X_s)\,ds} h(u,X_u)\,du \,\Big|\, X_t = x\Big].
\end{aligned}$$

Here $\tau_t = \inf\{u \geq t : X_u \notin D\}$, X is the diffusion process with dynamics (2.5)*, and $C^{1,2,\lambda}_{\mathrm{loc}}((0,T) \times D)$ is the space of all functions such that they and all their derivatives up to the second order in x and first order in t are λ-Hölder continuous. Furthermore, u satisfies the growth estimate*

$$\sup_{t \in [0,T]} |u(t,x)| \leq C(c_0, K_1, K_2, K_3, k)\big(1 + |x|^k\big), \qquad x \in \overline{D}.$$

This Feynman-Kac type theorem is proved in an article by Rubio [9]. (As a matter of fact, the theorem of Rubio is formulated for initial and boundary value problems, but the adaptation to problems with terminal instead of initial conditions is trivial: one only needs to perform the change of variables $u = T - t$.)

It is also worth mentioning that the **hedging approach** is another method which can also be used for determining the price of a barrier option. This approach consists in deducing the hedging strategy for the barrier option, because it can be used to obtain the price of the barrier option through non-arbitrage arguments. We refer the interested reader to Section 3.6 of Jeanblanc et al. [4].

Remark 2.9. Though our barrier pricing model assumes that the barrier is monitored continuously, in actual practical implementations the barrier monitoring procedures can be only performed at discrete times (e.g. hourly, daily or weekly) — which leads to the so-called **discrete barrier options**. The problem of discrete barrier option pricing is clearly less analytically tractable than its continuous counterpart and goes beyond the scope of this work. For a brief introduction to this topic, see Subsection 4.1.4 of Kwok [5] and the references therein.

Here we have restricted the discussion to barrier options of the European type, which can only be exercised at maturity. When one adds to these options an early exercise right, an **American barrier option** is obtained. An example is the American down-and-out call, which has payoff $(S_\tau - K)^+$ when it is (optimally) exercised at time $\tau \in [0, T]$, and loses its value if the down barrier H is touched by the asset price. A discussion of pricing models for American barrier options can be found in Subsection 5.2.4 of Kwok [5].

2.3 Stochastic volatility models

2.3.1 Smiles, skews and stochastic volatilities

We start this section with the definition of implied volatility, as formulated by Fouque et al. [10].

Definition 2.10. Let C^{obs} be the observed price, at time t, of a European call option with strike price K and maturity T. The **implied volatility** I is defined as the value of the volatility parameter for which the price of the option under the Black and Scholes framework matches the observed price of the option, i.e, I is the solution of

$$f_{\mathrm{BS}}(t, x; K, T; I) = C^{\mathrm{obs}}$$

where $f_{\mathrm{BS}}(t, x; K, T; \sigma)$ denotes the Black and Scholes price at time t and for an initial underlying asset value x of a European call with maturity T, strike K and volatility σ.

The implied volatility is well-defined due to the monotonicity of the Black and Scholes formula with respect to the volatility parameter.

If the Black and Scholes option pricing model were realistic, then I would be the same for all European option contracts traded in the market. But, according to Fouque et al. [10], the option price data observed from financial markets shows that in general $I = I(t, x; K, T)$. In particular, when market prices are used to construct a plot of $I(K)$ against the strike K for fixed t, x and T, the resulting curves usually show the so-called **smile** or **skew** effects. The first corresponds to the case where the curve is U-shaped with a minimum at $K \approx x$, whereas the skew effect refers to the situation where the curve is downward-sloping.

Such smile and skew patterns make clear that some deviation from the Black and Scholes framework must be introduced if one wants that the option pricing model reproduces the actual market prices. According to Lipton [11], the simplest approach is to switch to the **local volatility model**, which assumes that the underlying asset price is no longer a geometric Brownian motion with constant drift r and volatility

σ, but instead it follows the stochastic differential equation (SDE)

$$dS_t = rS_t\,dt + \sigma(t, S_t)S_t\,dW_t$$

where $\sigma(t, S_t)$ is the **local volatility function**, which should be chosen in a way that reproduces the implied volatilities which are observed in the market.

If all European option prices were available from the market, then there would be no difficulties in choosing the local volatility function because, as shown in Subsection 10.2.4 of Lipton [11], the latter can be written explicitly in terms of the European call price function and its derivatives. However, in practice the market prices of options are only known for a limited number of different maturities and strikes, and the construction of the local volatility function from the discrete data turns out to be very sensitive to the choice of interpolation scheme.

As mentioned by Kwok [5], the distribution of the returns of the underlying asset typically indicates a mixture of distributions with different variances. Hence, for the purpose of matching the market data, a natural alternative strategy is to model the volatility as a stochastic process, which leads to a **stochastic volatility model**. The most popular stochastic volatility models are of the form

$$\begin{aligned} dS_t &= \mu(t, S_t)S_t\,dt + h(V_t)S_t\,dW_t^1 \\ dV_t &= a(t, V_t)\,dt + b(t, V_t)\,dW_t^* \end{aligned} \tag{2.7}$$

where W^1 and W^* are Brownian motions with correlation $\rho \neq \pm 1$, and h is a smooth, positive and increasing function. (More generally, the dynamics for the volatility process V can also contain a jump component.)

Example 2.11. Some well-known examples of stochastic volatility models, which are presented in Section 6.7 of Jeanblanc et al. [4], are:

(1) **Hull and White's model**, with dynamics

$$\begin{aligned} dS_t &= \mu(t, S_t)S_t\,dt + V_tS_t\,dW_t^1 \\ dV_t &= aV_t\,dt + b\,V_t\,dW_t^2 \end{aligned}$$

where $a \in \mathbb{R}$, $b > 0$ and the Brownian motions W^1 and W^2 are uncorrelated. Note that the stochastic process for the volatility is simply a geometric Brownian motion with constant parameters.

(2) **Scott's model**, whose SDEs are

$$\begin{aligned} dS_t &= \mu(t, S_t)S_t\,dt + e^{V_t}S_t\,dW_t^1 \\ dV_t &= \kappa(a - V_t)\,dt + b\,dW_t^* \end{aligned}$$

where $\kappa, a \in \mathbb{R}$ and $b > 0$. (The Brownian motions may be correlated.) The process for the log-volatility is known as the **Ornstein-Uhlenbeck process**.

(3) **Heston's model**, with dynamics

$$\begin{aligned} dS_t &= \mu(t, S_t)S_t\,dt + \sqrt{V_t}S_t\,dW_t^1 \\ dV_t &= \kappa(a - V_t)\,dt + b\,\sqrt{V_t}\,dW_t^* \end{aligned}$$

where $\kappa, a \in \mathbb{R}$, $b > 0$ and $\kappa a > 0$. (The Brownian motions may be correlated.) The dynamics for V_t^* are the dynamics of the so-called **Cox-Ingerson-Ross process**.

A desirable feature for the volatility process V of a stochastic volatility model is the **mean-reverting property**: over time, the drift of V should tend to the long-term mean of the process. A more formal definition, given by Fouque et al. [12], is as follows: the stochastic volatility model (2.7) has the mean-reverting property if $a(t, V_t) = \alpha(m - V_t)$, where the constants m and α are the long-term mean and the rate of mean reversion, respectively. Observe that the drift is positive whenever $V_t < m$ and negative whenever $V_t > m$; consequently, V is a process that randomly fluctuates around its mean, thus replicating the empirical fact that the variances of price processes usually vary within some interval.

In Example 2.11, both Scott's model and Heston's model possess the mean-reversion property, while Hull and White's model is not mean-reverting.

2.3.2 Option pricing under stochastic volatility

The path-independent and barrier option pricing theory from Section 2.1 and Subsection 2.2.2 does not apply to stochastic volatility models. Indeed, the number of driving Brownian motions is $n = 2$ whereas the number of underlying assets is $d = 1$, thus Theorem 2.3 tells us that existence and uniqueness of the equivalent martingale measure does not hold.

In fact, it is generally known that stochastic volatility models are incomplete and, accordingly, infinitely many martingale measures exist. As shown in Section 2.5 of Fouque et al. [10], if the asset pays no dividends then for any adapted and suitably regular process $\{\eta_t\}_{t \geq 0}$ the probability measure $\mathbb{Q}^{(\eta)}$ defined by

$$\frac{d\mathbb{Q}^{(\eta)}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\int_0^T \left[\left(\frac{\mu(t, S_t) - r(t)}{h(V_t)}\right)^2 + \eta_t^2\right] dt - \int_0^T \frac{\mu(t, S_t) - r(t)}{h(V_t)}\, dW_t^1 - \int_0^T \eta_t dW_t^2\right\}$$

is an equivalent martingale measure. (Here W^2 is a Brownian motion independent of W^1 and such that $W_t^* = \rho W_t^1 + \sqrt{1 - \rho^2} W_t^2$.) Under $\mathbb{Q}^{(\eta)}$ the model (2.7) becomes

$$\begin{aligned} dS_t &= r(t) S_t\, dt + h(V_t) S_t\, d\widehat{W}_t^1 \\ dV_t &= \big[a(t, V_t) - b(t, V_t)\Lambda_t\big]\, dt + b(t, V_t)\, d\widehat{W}_t^* \end{aligned} \tag{2.8}$$

where $\Lambda_t = \rho\, \frac{\mu(t, S_t) - r(t)}{h(V_t)} + \sqrt{1 - \rho^2}\, \eta_t$, $\widehat{W}_t^* = \rho\, \widehat{W}_t^1 + \sqrt{1 - \rho^2}\, \widehat{W}_t^2$, and $\widehat{W}_t^1$, $\widehat{W}_t^2$ are independent $\mathbb{Q}^{(\eta)}$-Brownian motions.

For simplicity, let us assume that the process η — which is known as the **market price of volatility risk** — is a deterministic function of the processes (meaning that $\eta_t = \eta(t, S_t, V_t)$ and $\Lambda_t = \Lambda(t, S_t, V_t)$), and therefore (2.8) is also a Markovian diffusion model. Then for any choice of the function $\eta(t, S_t, V_t)$, the formula

$$f^{(\eta)}(t, x, v) = e^{-\int_t^T r(u)\, du}\, \mathbb{E}_{\mathbb{Q}^{(\eta)}}\big[Y \mid S_t = x, V_t = v\big]$$

allows us to compute arbitrage-free prices for all contingent claims Y. Furthermore, under appropriate regularity assumptions, $f^{(\eta)}(t, x)$ can also be computed through a PDE approach: in the case of a path-independent claim $Y = \phi(S_T)$, the Feynman-Kac theorem (Theorem 2.7) assures that $f^{(\eta)}(t, x, v)$ is a solution of the two-space-dimensional parabolic Cauchy problem

$$\frac{\partial f^{(\eta)}}{\partial t} + \frac{1}{2}h^2(v)x^2\frac{\partial^2 f^{(\eta)}}{\partial x^2} + \rho\, b(t,v)xh(v)\frac{\partial^2 f^{(\eta)}}{\partial x \partial v} + \frac{1}{2}b^2(t,v)\frac{\partial^2 f^{(\eta)}}{\partial v^2} + \\ + r(t)x\frac{\partial f^{(\eta)}}{\partial x} + \left[a(t,v) - b(t,v)\Lambda(t,x,v)\right]\frac{\partial f^{(\eta)}}{\partial v} - r(t)f^{(\eta)} = 0, \\ f^{(\eta)}(T,x,v) = \phi(x).$$

For a barrier option whose barrier $H(t,v)$ may depend on time and also on the volatility, Theorem 2.8 yields that the price is the solution of the corresponding Cauchy-Dirichlet problem with zero boundary condition at $x = H(t,v)$. Like in the complete model scenario, the adaptation to a dividend-paying underlying asset is carried out by replacing $r(t)$ by $r(t) - q(t)$ in (2.8) and adapting the PDE accordingly.

Since the market price of volatility risk $\eta(t, S_t, V_t)$ cannot be identified within incomplete market models, one needs to exogenously specify the process $\eta(t, S_t, V_t)$ in order to price options and other derivatives under the stochastic volatility model (2.7). As stated by Fouque et al. [10], one possible interpretation for this is the following: the financial market selects a unique equivalent martingale measure under which derivative contracts are priced, so the consistent pricing of all contingent claims is fulfilled as long as we fix the correct market price of volatility risk.

Unfortunately, picking the right market price of volatility risk is nontrivial: according to Lipton [11], there is in general little guidance in choosing the proper functional form for $\eta(t, S_t, V_t)$, so a common practice is to judiciously choose $\eta(t, S_t, V_t)$ in order that the resulting pricing problem is analytically tractable. For instance, in the Heston model of Example 2.11(3), where the drift term of V_t in (2.8) becomes $\left[\kappa(a - V_t) - b\sqrt{V_t}\Lambda(t, S_t, V_t)\right] dt$, the usual strategy is to pick $\eta(t, S_t, V_t)$ such that $\Lambda(t, S_t, V_t) = \lambda\sqrt{V_t}$ for some constant $\lambda \in \mathbb{R}$, as this choice guarantees that the drift of V_t under the equivalent martingale measure $\mathbb{Q}^{(\eta)}$ is still an affine function of V_t.

2.4 Asymptotic expansion techniques

Given that the majority of stochastic volatility models do not admit exact closed-form analytical solutions, the literature on asymptotic methods for determining analytical approximations for option prices under stochastic volatility has grown tremendously in recent years. Basically, these methods consist in rewriting the model as a perturbed Black and Scholes model (or a perturbed version of some other simple model) so as to derive a series expansion of the exact stochastic volatility price around the Black and Scholes price, which should converge when the perturbation parameter tends to zero.

Though here we will focus on the application of asymptotic expansion techniques to stochastic volatility models, we remark that these methods have been successfully applied to a much broader class of valuation problems in mathematical finance — see Takahashi [13] and the references given there.

2.4.1 Path-independent options

One of the most popular asymptotic methods for the pricing of path-independent options under mean-reverting stochastic volatility models is that of Fouque et al. [10, 12], which is based on an asymptotic expansion of the pricing PDE around the invariant distribution of the volatility process. Let us consider the model (2.7) with $a(t, V_t) = \alpha(m - V_t)$. The **singular perturbation method** of Fouque et al., which is tailored for the case where the rate of mean-reversion α is large, starts by rewriting the risk-neutral model (2.8) as

$$\begin{aligned} dS_t^\varepsilon &= r(t) S_t^\varepsilon \, dt + h(V_t^\varepsilon) S_t^\varepsilon \, d\widehat{W}_t^1 \\ dV_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m - V_t^\varepsilon) - \frac{1}{\sqrt{\varepsilon}} b(V_t^\varepsilon)\Lambda(V_t^\varepsilon) \right] dt + \frac{1}{\sqrt{\varepsilon}} b(V_t^\varepsilon) \, d\widehat{W}_t^* \end{aligned} \tag{2.9}$$

where ε is a small parameter and the functions b and Λ are now being assumed to depend only on the volatility process. (We are writing the superscript ε in the processes S and V just to emphasize the dependence on the parameter.) Accordingly, the pricing PDE becomes $\frac{\partial f^\varepsilon}{\partial t}(t, x, v) + \mathcal{L}^\varepsilon f^\varepsilon(t, x, v) = 0$, where

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

$$\mathcal{L}_0 = \frac{1}{2} b^2(v) \frac{\partial^2}{\partial v^2} + (m - v)\frac{\partial}{\partial v}, \qquad \mathcal{L}_1 = \rho\, b(v) h(v) x \frac{\partial^2}{\partial x \partial v} - b(v)\Lambda(v)\frac{\partial}{\partial v},$$

$$\mathcal{L}_2 = \frac{1}{2} h^2(v) x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r\,\mathrm{Id}.$$

Then, we formally assume that $f^\varepsilon(t, x, v)$ can be asymptotically expanded as $f^\varepsilon = f_0 + \sqrt{\varepsilon} f_1 + \varepsilon f_2 + \ldots$ and we substitute this expansion into the pricing PDE and into its terminal condition $f^\varepsilon(T, x, v) = \phi(x)$. This yields

$$\begin{gathered} \frac{1}{\varepsilon}\mathcal{L}_0 f_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 f_1 + \mathcal{L}_1 f_0) + \Big(\frac{\partial f_0}{\partial t} + \mathcal{L}_0 f_2 + \mathcal{L}_1 f_1 + \mathcal{L}_2 f_0\Big) + \sqrt{\varepsilon}\Big(\frac{\partial f_1}{\partial t} + \mathcal{L}_0 f_3 + \mathcal{L}_1 f_2 + \mathcal{L}_2 f_1\Big) + \ldots = 0 \\ (f_0 + \sqrt{\varepsilon} f_1 + \varepsilon f_2 + \ldots)(T, x, v) = \phi(x) \end{gathered} \tag{2.10}$$

By equating the terms of order ε^{-1}, $\varepsilon^{-1/2}$, ε^0, … in (2.10), we obtain the system of PDEs

$$\mathcal{L}_0 f_0 = 0, \qquad \mathcal{L}_0 f_1 + \mathcal{L}_1 f_0 = 0, \qquad \frac{\partial f_0}{\partial t} + \mathcal{L}_0 f_2 + \mathcal{L}_1 f_1 + \mathcal{L}_2 f_0 = 0, \qquad \ldots \tag{2.11}$$

whose terminal conditions are $f_0(T, x, v) = \phi(x)$ and $f_j(T, x, v) = 0$ for $j = 1, 2, \ldots$. As shown in Subsection 4.2.1 of Fouque et al. [12], the functions $f_0, f_1, \ldots$ can be deduced by recursively solving this system of PDEs. In particular, the zero order term is shown to be equal to the Black and Scholes price where the volatility equals its average value, and the first order term is the solution of a nonhomogeneous Black and Scholes PDE with zero terminal condition. Even though the SDE for V_t in (2.9) diverges when ε tends to zero (and the mean reversion rate tends to infinity), the singular perturbation method converges in the sense that $f^\varepsilon(t, x, v) = f_0(t, x, v) + \sqrt{\varepsilon} f_1(t, x, v) + o(\sqrt{\varepsilon})$ for fixed (t, x, v) — see Theorem 4.11 in [12].

The singular perturbation method is clearly inadequate if the mean-reversion rate α is small. In that case, one may instead use the **regular perturbation method** of Fouque et al., where the risk-neutral dynamics (2.8) are rewritten as

$$\begin{aligned} dS_t^\delta &= r(t) S_t^\delta \, dt + h(V_t^\delta) S_t^\delta \, d\widehat{W}_t^1 \\ dV_t^\delta &= \left[\delta(m - V_t^\delta) - \sqrt{\delta} b(V_t^\delta)\Lambda(V_t) \right] dt + \sqrt{\delta} b(V_t^\delta) \, d\widehat{W}_t^* \end{aligned}$$

where δ is a small parameter. The pricing PDE is now $\frac{\partial f^\delta}{\partial t}(t,x,v) + \mathcal{L}^\delta f^\delta(t,x,v) = 0$ with

$$\mathcal{L}^\delta = \mathcal{L}_0 + \sqrt{\delta}\mathcal{L}_1 + \delta\mathcal{L}_2,$$
$$\mathcal{L}_0 = \frac{1}{2}h^2(v)x^2\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\,\mathrm{Id}, \qquad \mathcal{L}_1 = \rho\, b(v)h(v)x\frac{\partial^2}{\partial x \partial v} - b(v)\Lambda(v)\frac{\partial}{\partial v}$$
$$\mathcal{L}_2 = \frac{1}{2}b^2(v)\frac{\partial^2}{\partial v^2} + (m-v)\frac{\partial}{\partial v}.$$

Writing $f^\delta = f_0 + \sqrt{\delta}f_1 + \delta f_2 + \ldots$ and equating the terms of order δ^0, $\delta^{1/2}$, δ, $\ldots$ now gives the system of PDEs

$$\frac{\partial f_0}{\partial t} + \mathcal{L}_0 f_0 = 0, \qquad \frac{\partial f_1}{\partial t} + \mathcal{L}_0 f_1 + \mathcal{L}_1 f_0 = 0, \qquad \frac{\partial f_2}{\partial t} + \mathcal{L}_0 f_2 + \mathcal{L}_1 f_1 + \mathcal{L}_2 f_0 = 0, \qquad \ldots \tag{2.12}$$

with the same terminal conditions as above, which can also be solved in a recursive way. In particular, the zero-order term of the regular perturbation method is simply the price of the option in the model which corresponds to $\delta = 0$, which was not the case in the singular perturbation method. The first-order approximation which results from the regular perturbation technique satisfies $f^\delta(t,x,v) = f_0(t,x,v) + \sqrt{\delta}f_1(t,x,v) + o(\sqrt{\delta})$.

Actually, the singular and regular perturbation methods can be combined in the context of multi-factor stochastic volatility models, where the volatility of S_t is of the form $h(V_t, Z_t)$ and the volatility factors V and Z follow mean-reverting processes with large and small mean-reversion rates, respectively. We again refer the reader to Fouque et al. [12].

Asymptotic approaches for approximating option prices under stochastic volatility models are by no means restricted to PDE perturbation methods: purely probabilistic approaches based on the martingale formulation of the option pricing problem and on Malliavin calculus techniques can also be used to derive explicit asymptotic pricing formulas and to prove their convergence. See e.g. the review paper by Takahashi [13].

2.4.2 Barrier options

The asymptotic techniques discussed so far can also be extended not only to barrier options but also to other path-dependent contingent claims. In the context of the singular and regular perturbation theory of Fouque et al., as shown in [14], the adaptation to barrier options is particularly simple, since we just need to add the boundary conditions $f_j(t,H,v) = 0, j = 0,1,2,\ldots$ to the system of PDEs (2.11). The function f_0 becomes the Black and Scholes barrier option price for the average value of the volatility (or for the model with $\delta = 0$); the function f_1 is the solution of a nonhomogeneous PDE with zero terminal and boundary conditions; and suitable convergence results can also be derived.

Another important paper in barrier option pricing through asymptotic expansions is that of Kato et al. [15]. The authors generalize the PDE-based regular perturbation method of Fouque et al. to multidimensional Markovian diffusion processes driven by a perturbed SDE and show that the successive terms in the asymptotic expansion of the option price can be written through a semigroup representation formula. Moreover, the convergence of the expansion is proved under certain regularity assumptions.

Finally, it is also worth mentioning that techniques based on purely probabilistic methods can also be useful for approximate barrier option pricing: for instance, Shiraya et al. [16] used Malliavin calculus to provide an approximation of the price of a discrete barrier option under stochastic volatility.

Chapter 3

Option pricing under the 2-hypergeometric stochastic volatility model

In this chapter we will tackle the problem of pricing barrier options under the 2-hypergeometric stochastic volatility — a particular case of the α-hypergeometric stochastic volatility model which was defined by Da Fonseca and Martini [1] as follows:

Definition 3.1. The **α-hypergeometric stochastic volatility model** is the Markovian diffusion model with dynamics

$$\begin{aligned} dS_t &= r(t)S_t dt + e^{V_t} S_t dW_t^1 \\ dV_t &= \left(a - \frac{c}{2}e^{\alpha V_t}\right) dt + \theta\, dW_t^* \end{aligned} \tag{3.1}$$

where S is the asset price process, V is the log-volatility process, $W_t^* = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$ where W^1 and W^2 are independent Brownian motions, and $c, \alpha, \theta > 0$, $a \in \mathbb{R}$ are constants.

Remark 3.2. (a) The original formulation of the α-hypergeometric stochastic volatility model does not include the drift term $r(t)S_t\, dt$. We have added this term for reasons to be explained in Subsection 3.2.2.

(b) This model is of the form (2.7) with $a(t, V_t) = a - \frac{c}{2}e^{\alpha V_t}$, so the volatility process is not mean-reverting in the sense of the definition given in Subsection 2.3.1. Notwithstanding, it is mean-reverting in a generalized sense: the drift of V_t is positive whenever $V_t < \frac{1}{\alpha}\log\left(\frac{c}{2a}\right)$ and negative whenever $V_t > \frac{1}{\alpha}\log\left(\frac{c}{2a}\right)$, so V is also a volatility process that randomly fluctuates around a mean level.

(c) Like Da Fonseca and Martini [1], we will assume that the model is given directly under a risk-neutral measure $\mathbb{Q}$. If we make the natural assumption that under the original probability $\mathbb{P}$ the model is also α-hypergeometric,

$$\begin{aligned} dS_t &= \mu(t, S_t)S_t dt + e^{V_t} S_t d\overline{W}_t^1 \\ dV_t &= \left(\overline{a} - \frac{\overline{c}}{2}e^{\alpha V_t}\right) dt + \theta\, d\overline{W}_t^*, \end{aligned}$$

this means that the market price of volatility risk which we are implicitly choosing is $\eta(t, S_t, V_t)$ such that $\Lambda(t, S_t, V_t) = b_1 + b_2 e^{\alpha V_t}$ for some constants b_1 and b_2 (with $b_2 < \frac{\bar{c}}{2\theta}$). Accordingly, the parameters a and c in (3.1) can be used to set the market price of volatility risk.

As stated by Da Fonseca and Martini [1], the Heston model (defined in Example 2.11(3)) is highly analytically tractable, which makes it the most popular stochastic volatility model. Yet the Heston model is problematic because, as remarked in Subsection 6.5.2 of Henry-Labordère [17], its volatility process can reach zero in finite time unless the Feller condition $2a\kappa > b^2$ is imposed, whereas this condition is usually not satisfied when the model parameters are estimated from real financial data. The α-hypergeometric model was designed so as to preserve the analytical tractability properties of the Heston model while ensuring, by construction, that strict positivity of volatility always holds.

3.1 Vanilla option pricing

The good properties of the α-hypergeometric model spurred Privault and She [2] to investigate the problem of vanilla option pricing under this model. In this section we present a brief summary of their asymptotic pricing approach — an approach which we shall adapt to options with the barrier feature in Section 3.2.

3.1.1 The regular perturbation technique

The asymptotic approach of Privault and She is based on a PDE regular perturbation method where the drift of the asset price process is taken to be zero (see Remark 3.2(a)). Their first step is to replace the constant θ in the model (3.1) by $\varepsilon\psi(t, V_t)$, where ε is a small parameter and ψ is a generic function (to be specified later). We observe that this kind of regular perturbation approach where the parameter ε only influences the variance of V_t is known as the **small vol of vol expansion** and is commonly used in the literature — see for instance Chapter 3 of Lewis [18] or Section 10.10 of Lipton [11].

The risk-neutral dynamics of the model become

$$\begin{aligned} dS_t^\varepsilon &= e^{V_t^\varepsilon} S_t^\varepsilon dW_t^1 \\ dV_t^\varepsilon &= \left(a - \frac{c}{2} e^{\alpha V_t^\varepsilon}\right) dt + \varepsilon\psi(t, V_t^\varepsilon)\, dW_t^*. \end{aligned} \tag{3.2}$$

(As in Section 2.4, it is convenient to make explicit the dependence on ε through the superscripts in the stochastic processes.) The exact price of a plain vanilla call option under this model is defined by $f^\varepsilon(t, x, v) := \mathbb{E}\left[(S_T^\varepsilon - K)^+ | S_t^\varepsilon = x, V_t^\varepsilon = v\right]$, where (here and subsequently) the expected value is taken with respect to the fixed risk-free measure $\mathbb{Q}$; by the Feynman-Kac theorem, f^ε solves the PDE $\frac{\partial f^\varepsilon}{\partial t} + (\mathcal{L}_0 + \varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2) f^\varepsilon = 0$ with terminal condition $f^\varepsilon(T, x, v) = (x - K)^+$, where

$$\mathcal{L}_0 = \left(a - \frac{c}{2} e^{\alpha v}\right) \frac{\partial}{\partial v} + \frac{x^2}{2} e^{2v} \frac{\partial^2}{\partial x^2}, \qquad \mathcal{L}_1 = \rho x e^v \psi(t, v) \frac{\partial^2}{\partial x \partial v}, \qquad \mathcal{L}_2 = \psi^2(t, v) \frac{\partial^2}{\partial v^2}. \tag{3.3}$$

The series expansion $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$ yields the system (2.12) with the terminal conditions $f_0(T, x, v) = (x - K)^+$ and $f_j(T, x, v) = 0$ for $j = 1, 2, \ldots$, so like in the regular perturbation method described in Subsection 2.4.1 the zero-order term is just the price of the option in the model with $\varepsilon = 0$:

$$f_0(t,x,v) := \mathbb{E}\left[(S_T^{t,v} - K)^+ | S_t^{t,v} = x\right]$$

where the stochastic process $\{S_u^{t,v}\}_{u\in[t,T]}$ is driven by the equation $dS_u^{t,v} = S_u^{t,v} e^{V_u^{t,v}} dW_u^1$ and the degenerate log-volatility process $\{V_u^{t,v}\}_{u\in[t,T]}$ is the deterministic function of time which solves the ordinary differential equation $dV_u^{t,v} = (a - \frac{c}{2} e^{\alpha V_u^{t,v}})\, du$ with initial condition $V_t^{t,v} = v$; the explicit solution is

$$V_u^{t,v} = v + a(u-t) + \frac{1}{\alpha}\log\left(1 + \frac{c}{2a} e^{\alpha v}(e^{\alpha a(u-t)} - 1)\right). \tag{3.4}$$

In our notation we are replacing the (S_u^0, V_u^0) notation of Privault and She by $(S_u^{t,v}, V_u^{t,v})$, so as to emphasize the dependence of the deterministic function $V_u^{t,v}$ and the stochastic process $\{S_u^{t,v}\}_{u\in[t,T]}$ on the initial time t (i.e. on the time for which we are computing the price of the option) and on the initial value of the volatility. Unambiguously, we will keep using the superscript ε in our notation for the nondeterministic processes S^ε and V^ε corresponding to $\varepsilon > 0$.

The process $\{S_u^{t,v}\}_{u\in[t,T]}$ is simply a geometric Brownian motion with zero drift and time-dependent volatility $e^{V_u^{t,v}}$, so the conditional distribution of $S_T^{t,v}$ given $S_t^{t,v} = x$ is the lognormal distribution with parameters $\mu = \log x - \frac{1}{2}\gamma^2(t,T,v)$ and $\sigma^2 = \gamma^2(t,T,v)$, where $\gamma^2(t,T,v) = \int_t^T e^{2V_u^{t,v}} du$. Hence, the zero-order term f_0 is given by the "Black and Scholes style" option pricing formula

$$f_0(t,x,v) = x\mathcal{N}(d_+(t,x,v)) - K\mathcal{N}(d_-(t,x,v)) \tag{3.5}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of a standard normal random variable and $d_\pm(t,x,v) = \frac{1}{\gamma(t,T,v)}\big(\log\big(\frac{x}{K}\big) \pm \frac{1}{2}\gamma^2(t,T,v)\big)$, as can be seen in Proposition 1 of [2]. It is important to note that this is indeed just the usual pricing formula for the plain vanilla call option in the Black and Scholes model with time-dependent parameters (and with zero interest rate and dividends), as one can check by comparing with Equation (3.4.14) of Kwok [5].

In the case $\alpha = 2$ (i.e, under the 2-hypergeometric model) the integral $\gamma^2(t,T,v) = \int_t^T e^{2V_u^{t,v}} du$ can be given in closed form as

$$\gamma^2(t,T,v) = \frac{1}{c}\log\Big(1 + \frac{c}{2a} e^{2v}(e^{2a(T-t)} - 1)\Big) \tag{3.6}$$

which is convenient: the formula (3.5) for $f_0(t,x,v)$ becomes fully explicit, and the same is true for the first-order term given below. It is for this reason that Privault and She restrict their analysis to the 2-hypergeometric model; yet it is clear that this asymptotic expansion method applies to any α-hypergeometric stochastic volatility model.

The first-order term is the solution of $\frac{\partial f_1}{\partial t} + \mathcal{L}_0 f_1 = -\mathcal{L}_1 f_0$ with zero terminal condition. Here the nonhomogeneity term $\mathcal{L}_1 f_0$ is a known function, because it can be explicitly computed by differentiating (3.5). Using Heath and Schweizer's [8] generalized version of the Feynman-Kac theorem (Theorem 2.7), Privault and She deduce that the solution of the Cauchy problem can be written as

$$f_1(t,x,v) = \int_t^T \mathbb{E}\big[\mathcal{L}_1 f_0(u, S_u^{t,v}, V_u^{t,v}) \bigm| S_t^{t,v} = x\big] du.$$

where $V_u^{t,v}$ is the same deterministic function of time and $\{S_u^{t,v}\}_{u\in[t,T]}$ is the same geometric Brownian motion from above. The expected value inside the time integral can be computed using the lognormal distribution of $S_u^{t,v}$, and the time integral can also be computed in closed form provided that our choice for the function ψ in (3.2) is $\psi(t,v) = \kappa e^v$ for some constant κ. The resulting formula for the first-order term, which is given in Proposition 2 of [2], is

$$f_1(t,x,v) = -\kappa\rho K \frac{d_-(t,x,v)}{c^2\,\gamma^2(t,T,v)} n(d_-(t,x,v)) \left(e^{c\gamma^2(t,T,v)} + c\gamma^2(t,T,v) - 1\right)$$

where $n(\cdot)$ is the density of a standard normal random variable.

It is important to point out that, as mentioned by Privault and She, the choice $\psi(t,v) = \kappa e^v$ is analytically quite convenient, but its drawback is that the approximation $(S_t^\varepsilon, V_t^\varepsilon)$ no longer belongs to the class of 2-hypergeometric models. Nevertheless, the numerical results presented by Privault and She demonstrate the accuracy of their fully explicit approximate solution, thus legitimating this kind of approximation procedure.

Lastly, we observe that Privault and She have also computed a fully explicit second order expansion, as well as an asymptotic formula for the estimation of the implied volatility. The reader is referred to Sections 5 and 6 of [2].

3.1.2 The zero drift assumption

Throughout their paper, Privault and She [2] assume that the interest rate and the dividend rate are zero, which leads to an equation for S_t^ε where the drift term is absent. This simplification is due to the fact that, in the case of vanilla options, the standard model which allows for deterministic time-dependent interest and dividend rates can be reduced to the zero interest and dividend framework. Let us see why this is so.

Assume that the risk-neutral interest rate and dividend yield rate are given by the deterministic functions $r(t)$ and $q(t)$ respectively. The following reasoning applies to a very large class of option pricing models, but for concreteness we take the α-hypergeometric model (3.2) and add the drift term corresponding to nonzero interest and dividend yield rates (cf. Remark 2.6):

$$\begin{aligned} dS_t^\varepsilon &= \big(r(t) - q(t)\big) S_t^\varepsilon dt + e^{V_t^\varepsilon} S_t^\varepsilon dW_t^1 \\ dV_t^\varepsilon &= \left(a - \frac{c}{2} e^{\alpha V_t^\varepsilon}\right) dt + \varepsilon\psi(t, V_t^\varepsilon) dW_t^*. \end{aligned} \tag{3.7}$$

In these conditions, the price of a path-independent contingent claim is

$$f^\varepsilon(t,x,v) := e^{-\int_t^T r(u)\,du} \mathbb{E}\left[\phi(S_T^\varepsilon) | S_t^\varepsilon = x, V_t^\varepsilon = v\right]$$

which solves the PDE $\frac{\partial f^\varepsilon}{\partial t} + \mathcal{L}^\varepsilon f^\varepsilon = 0$ with terminal condition $f^\varepsilon(T,x,v) = \phi(x)$, where

$$\mathcal{L}^\varepsilon = \left(a - \frac{c}{2} e^{\alpha v}\right) \frac{\partial}{\partial v} + \frac{x^2}{2} e^{2v} \frac{\partial^2}{\partial x^2} + \varepsilon\rho x e^v \psi(t,v) \frac{\partial^2}{\partial x \partial v} + \frac{\varepsilon^2}{2} \psi^2(t,v) \frac{\partial^2}{\partial v^2} + \big(r(t) - q(t)\big) x \frac{\partial}{\partial x} - r(t)\,\mathrm{Id}. \tag{3.8}$$

Let us first take care of the nonzero dividend. If we let

$$f^*(t,x,v) = e^{\int_t^T q(u)\,du} f^\varepsilon(t,x,v) = e^{-\int_t^T r^*(u)\,du} \mathbb{E}\left[\phi(S_T^\varepsilon) | S_t^\varepsilon = x, V_t^\varepsilon = v\right],$$

where $r^*(t) = r(t) - q(t)$, then the PDE for $f^*(t,x,v)$ becomes

$$\frac{\partial f^*}{\partial t} + \left(a - \frac{c}{2} e^{\alpha v}\right) \frac{\partial f^*}{\partial v} + \frac{x^2}{2} e^{2v} \frac{\partial^2 f^*}{\partial x^2} + \varepsilon\rho x e^v \psi(t,v) \frac{\partial^2 f^*}{\partial x \partial v} + \frac{\varepsilon^2}{2} \psi^2(t,v) \frac{\partial^2 f^*}{\partial v^2} + r^*(t) x \frac{\partial f^*}{\partial x} - r^*(t) f^* = 0 \tag{3.9}$$

with the same terminal condition $f^*(T,x,v) = \phi(x)$, so f^* is the price of the option whose dividend rate is zero and whose deterministic interest rate is $r^*(t)$. In other words, the price of the option with continuously paid deterministic dividends is $f^\varepsilon(t,x,v) = e^{-\int_t^T q(u)\,du} f^*(t,x,v)$, where $f^*(t,x,v)$ is the price of the option with zero dividends and shifted interest rate $r^*(t)$. Hence there is no loss of generality when Privault and She consider that their underlying asset has zero dividend rate.

Now we take $q(t) = 0$ in (3.7)–(3.8) and deal with the reduction to the case $r(t) = 0$. Much like in Subsection 9.2.1 of Lipton [11], the strategy is to rewrite the pricing equation *in forward terms*. Define $\widetilde{f}^\varepsilon(t,z,v) = e^{\int_t^T r(u)\,du} f^\varepsilon(t, e^{-\int_t^T r(u)\,du} z, v)$. Then

$$\frac{\partial \widetilde{f}^\varepsilon}{\partial t} = e^{\int_t^T r(u)\,du}\left(\frac{\partial f^\varepsilon}{\partial t} + r(t)x\frac{\partial f^\varepsilon}{\partial x} - r(t) f^\varepsilon\right), \qquad \frac{\partial \widetilde{f}^\varepsilon}{\partial z} = \frac{\partial f^\varepsilon}{\partial x}, \qquad \frac{\partial^2 \widetilde{f}^\varepsilon}{\partial z^2} = e^{-\int_t^T r(u)\,du}\frac{\partial^2 f^\varepsilon}{\partial x^2}$$

where $x = e^{-\int_t^T r(u)\,du} z$, and therefore

$$\frac{\partial \widetilde{f}^\varepsilon}{\partial t} + \left(a - \frac{c}{2}e^{\alpha v}\right)\frac{\partial \widetilde{f}^\varepsilon}{\partial v} + \frac{z^2}{2}e^{2v}\frac{\partial^2 \widetilde{f}^\varepsilon}{\partial z^2} + \varepsilon\rho z e^v \psi(t,v)\frac{\partial^2 \widetilde{f}^\varepsilon}{\partial z \partial v} + \frac{\varepsilon^2}{2}\psi^2(t,v)\frac{\partial^2 \widetilde{f}^\varepsilon}{\partial v^2} = 0$$
$$\widetilde{f}^\varepsilon(T,z,v) = \phi(z),$$

which is the pricing PDE for the α-hypergeometric model with zero interest rate. This shows that the price $f^\varepsilon(t,x,v)$ of the vanilla option in the model with nonzero interest rate is given by

$$f^\varepsilon(t,x,v) = e^{-\int_t^T r(u)\,du}\widetilde{f}^\varepsilon(t, e^{\int_t^T r(u)\,du} x, v)$$

where $\widetilde{f}^\varepsilon$ is the price in the model with zero interest rate. Accordingly, Privault and She can also assume without loss of generality that the interest rate is zero in their vanilla option pricing model.

3.2 Barrier option pricing

We are finally ready to develop our asymptotic pricing method for the pricing of barrier options under the α-hypergeometric stochastic volatility model. We will restrict our attention to the 2-hypergeometric model and to the zero-rebate down-and-out call (DOC) option described in Table 2.1, but we note that essentially the same techniques may be used to price other knock-out options (with sufficiently smooth payoff) in other α-hypergeometric stochastic volatility models.

3.2.1 The regular perturbation technique and the zero-order term in the zero drift scenario

As we will see in Subsection 3.2.2, the zero drift assumption is too restrictive when the option is of the barrier type. In any case, it is instructive to start by briefly discussing the simplified case of zero drift.

Let $\hat{f}^\varepsilon(t,x,v)$ be price of the DOC option in the driftless model (3.2) with $\alpha = 2$. As explained in Subsection 2.4.2, the adaptation of the regular perturbation method from path-independent options to barrier options simply consists in adding to the PDE problems the usual boundary conditions $\hat{f}_j(t,H,v) = 0$, $j = 0,1,2,\ldots$. In particular, the zero-order term $\hat{f}_0$ in our series expansion $\hat{f}^\varepsilon = \hat{f}_0 + \varepsilon \hat{f}_1 + \varepsilon^2 \hat{f}_2 + \ldots$ is now the price of the DOC barrier option in the (Black and Scholes) model which is obtained by setting $\varepsilon = 0$:

$$\hat{f}_0(t,x,v) = \mathbb{E}\left[(S_T^{t,v} - K)^+ \mathbb{1}_{\{\tau_H^t \geq T\}} \middle| S_t^{t,v} = x\right], \qquad t \in [0,T] \tag{3.10}$$

where $\tau_H^t := \inf\{u \geq t : S_u^{t,v} \leq H\}$ and the stochastic process $\{S_u^{t,v}\}_{u\in[t,T]}$ is a driftless geometric Brownian motion with time-dependent deterministic volatility $e^{V_u^{t,v}}$, where $V_u^{t,v}$ is given explicitly by (3.4) with $\alpha = 2$, i.e,

$$V_u^{t,v} = v + a(u-t) + \frac{1}{2}\log\left(1 + \frac{c}{2a}e^{2v}(e^{2a(u-t)} - 1)\right). \tag{3.11}$$

According to Remark 0.3 of Rapisarda [19], the exact closed-form analytical expression for the price of options with constant barriers in the Black and Scholes model with time-dependent parameters is known provided that the drift is proportional to the squared volatility. Given that for now we are considering a driftless Black and Scholes model, we can indeed provide an explicit closed-form expression for the zero-order term $\hat{f}_0$. Equation (27) of Rapisarda [19] shows that, in the domain $t \in [0,T]$ and $x > H$, (3.10) is equivalent to

$$\hat{f}_0(t,x,v) = x\mathcal{N}(d_1(t,x,v)) - K\mathcal{N}(d_2(t,x,v)) - H\mathcal{N}(d_3(t,x,v)) + \frac{Kx}{H}\mathcal{N}(d_4(t,x,v))$$

where

$$d_1(t,x,v) = \frac{1}{\gamma(t,T,v)}\left(\log\left(\frac{x}{K\vee H}\right) + \frac{1}{2}\gamma^2(t,T,v)\right), \qquad d_2(t,x,v) = d_1(t,x,v) - \gamma(t,T,v),$$
$$d_3(t,x,v) = d_1(t,T,v) + \frac{2}{\gamma(t,T,v)}\log\left(\frac{H}{x}\right), \qquad d_4(t,x,v) = d_2(t,x,v) + \frac{2}{\gamma(t,T,v)}\log\left(\frac{H}{x}\right).$$

and $\gamma^2(t,T,v)$ is given in (3.6). (As usual, $K \vee H := \max\{K,H\}$.)

3.2.2 Barrier options and the zero drift assumption

Unlike in vanilla options, it turns out that we must consider models where the drift is nonzero when dealing with barrier options, since the zero drift assumption entails some loss of generality.

To explain why this is so, we start by recalling that under deterministic time-dependent interest and dividend yield rates, the price of the DOC option with constant barrier H is the solution $\hat{f}^\varepsilon(t,x,v)$ of the PDE $\frac{\partial \hat{f}^\varepsilon}{\partial t} + \mathcal{L}^\varepsilon \hat{f}^\varepsilon = 0$ with terminal condition $\hat{f}^\varepsilon(T,x,v) = (x-K)^+$ and boundary condition $\hat{f}^\varepsilon(t,H,v) = 0$, where $\mathcal{L}^\varepsilon$ is defined by (3.8).

The reduction to the zero dividend case from Subsection 3.1.2 can also be carried out without difficulties when dealing with barrier options. If we define $\hat{f}^*(t,x,v) = e^{\int_t^T q(u)\,du}\hat{f}^\varepsilon(t,x,v)$, then $\hat{f}^*$ satisfies the PDE (3.9) and its terminal and boundary conditions are the same as those of $\hat{f}^\varepsilon$, i.e. $\hat{f}^*(T,x,v) = (x-K)^+$ and $\hat{f}^*(t,H,v) = 0$; consequently, the aforementioned relation between the option price under nonzero dividends and the option price under zero dividends and a shifted interest rate is also valid for barrier-type options.

But unfortunately the method (described in Subsection 3.1.2) for reducing the model with zero dividend and nonzero interest rate to the model with zero interest and dividend rates does not comply with the existence of boundary conditions. If, mimicking what we did in the path-independent scenario, we write $\widetilde{f}^{\varepsilon}(t,z,v) = e^{\int_t^T r(u)\,du}\hat{f}^{\varepsilon}(t, e^{-\int_t^T r(u)\,du}z, v)$, we conclude that

$$\frac{\partial \widetilde{f}^{\varepsilon}}{\partial t} + \Big(a - \frac{c}{2}e^{\alpha v}\Big)\frac{\partial \widetilde{f}^{\varepsilon}}{\partial v} + \frac{z^2}{2}e^{2v}\frac{\partial^2 \widetilde{f}^{\varepsilon}}{\partial z^2} + \varepsilon\rho z e^{v}\psi(t,v)\frac{\partial^2 \widetilde{f}^{\varepsilon}}{\partial z \partial v} + \frac{\varepsilon^2}{2}\psi^2(t,v)\frac{\partial^2 \widetilde{f}^{\varepsilon}}{\partial v^2} = 0$$
$$\widetilde{f}^{\varepsilon}(T,z,v) = (z-K)^+$$
$$\widetilde{f}^{\varepsilon}(t, e^{\int_t^T r(u)\,du}H, v) = 0.$$

This is the PDE formulation of the price of a barrier option under the zero drift model, but the barrier is no longer constant — now we have a curved barrier $e^{\int_t^T r(u)\,du}H$. If we expand $\widetilde{f}^{\varepsilon}(t,z,v)$ (or equivalently $\hat{f}^{\varepsilon}(t,x,v)$) in powers of ε, the zero-order term is the price of a barrier option in the Black and Scholes model with time-dependent deterministic volatility and the curved barrier $e^{\int_t^T r(u)\,du}H$. But this curved barrier option pricing problem is not at all trivial, and no explicit representation for its solution is known from the literature.

The conclusion is that, while it is fine to assume zero dividends in the barrier option pricing problem, we cannot assume that the interest rate is zero as well, so the relevant pricing PDE is

$$\frac{\partial \hat{f}^{\varepsilon}}{\partial t} + \Big(a - \frac{c}{2}e^{2v}\Big)\frac{\partial \hat{f}^{\varepsilon}}{\partial v} + \frac{x^2}{2}e^{2v}\frac{\partial^2 \hat{f}^{\varepsilon}}{\partial x^2} + \varepsilon\rho x e^{v}\psi(t,v)\frac{\partial^2 \hat{f}^{\varepsilon}}{\partial x \partial v} + \frac{\varepsilon^2}{2}\psi^2(t,v)\frac{\partial^2 \hat{f}^{\varepsilon}}{\partial v^2} + r(t)x\frac{\partial \hat{f}^{\varepsilon}}{\partial x} - r(t)\hat{f}^{\varepsilon} = 0$$

with terminal condition $\hat{f}^{\varepsilon}(T,x,v) = (x-K)^+$ and boundary condition $\hat{f}^{\varepsilon}(t,H,v) = 0$. This is a more difficult pricing problem, because we will need to replace the constant barrier by a time-dependent barrier in order to derive explicit formulas for the various terms in the asymptotic expansion of the barrier option price.

Remark 3.3. So as to lighten the notation, we will henceforth assume that the interest rate is constant, i.e, $r(t) = r$ for all t and therefore $\int_t^u r(s)\,ds = r(u-t)$. Otherwise, it suffices to replace the $r(u-t)$ terms by $\int_t^u r(s)\,ds$ in the forthcoming formulae. Moreover, we will assume that the function $\psi(t,v)$ is smooth on the domain $[0,T]\times\mathbb{R}$.

3.2.3 The time-dependent barrier approximation and the asymptotic expansion in the nonzero drift scenario

We start by claiming that the price of the DOC option with constant barrier H under the 2-hypergeometric model

$$\begin{aligned} dS_t^{\varepsilon} &= rS_t^{\varepsilon}dt + e^{V_t^{\varepsilon}}S_t^{\varepsilon}dW_t^1 \\ dV_t^{\varepsilon} &= \Big(a - \frac{c}{2}e^{2V_t^{\varepsilon}}\Big)dt + \varepsilon\psi(t,V_t^{\varepsilon})dW_t^*. \end{aligned} \tag{3.12}$$

is approximated by the price of a DOC option with time and volatility-dependent barrier

$$\hat{h}(t,v) := H_1 \exp\Big\{-r(T-t) + \tfrac{1+2\beta}{2}\gamma^2(t,T,v)\Big\}, \qquad (t,v)\in[0,T]\times\mathbb{R}, \tag{3.13}$$

for a suitable choice of $\beta\in\mathbb{R}$ and $H_1 = \hat{h}(T,v)$. It should be clear that the time and volatility dependence of the barrier is in the following sense: the option becomes nullified whenever $S_t^{\varepsilon} \le \hat{h}(t,V_t^{\varepsilon})$. This choice of time and volatility-dependent barrier is motivated by the fact that, to the best of our knowledge, an explicit expression for the various terms in the asymptotic expansion of the barrier option price cannot be obtained unless the barrier function is given by this particular functional form.

This kind of approximation procedure is commonly used in the literature addressing the pricing of barrier options in the Black and Scholes model with time-dependent parameters (see e.g. Rapisarda [19], Dorfleitner et al. [20] and Lo et al. [21]). For a suitable choice of the parameters of the barrier function such as the one indicated in the next paragraph, the approximation is quite good for small maturities, and for large maturities it is possible to use a multi-stage procedure to improve the quality of the approximation, as we will demonstrate in Subsection 3.2.7.

Our pricing strategy is based on a small vol of vol expansion which is performed around the noiseless limit $V_t^{t',v'}$ of the log-volatility process V^ε. Therefore, if one wishes to compute the price of the option at time $t' \in [0,T]$ and the initial log-volatility (i.e, the log-volatility at time t') is equal to v', then the parameters H_1 and β should be chosen such that $\hat{h}(t, V_t^{t',v'})$ is as close to the constant function H as possible. The simplest choice is $H_1 = H$ and β such that $\hat{h}(t',v') = H$, i.e. $\beta = \frac{r(T-t')}{\gamma^2(t',T,v')} - \frac{1}{2}$, but this choice can be improved by choosing the parameters in some optimal way (cf. page 3 of Rapisarda [19]).

Note that we are considering a barrier function which depends not only on time but also on the volatility. A consequence of this is that the quality of the approximation will be better or worse depending on whether the initial log-volatility (i.e, the log-volatility at the time when the option price is computed) is respectively close or far from its invariant value $v' = \log\left(\frac{2a}{c}\right)$. In fact, the two cases where (by picking $\beta = \frac{r(T-t')}{\gamma^2(t',T,v')} - \frac{1}{2}$) the barrier function (3.13) can be chosen to be constant are precisely the zero interest rate case (discussed in Subsection 3.2.1) and the case where the initial volatility equals its invariant value (i.e, where the noiseless log-volatility function $V_t^{t',v'}$ is constant).

Next, we apply the usual regular perturbation technique to the time and volatility-dependent barrier option. The price $\hat{f}^\varepsilon(t,x,v)$ of the DOC option with barrier function $\hat{h}(t,v)$ is defined (in the PDE approach) as the solution of the terminal and boundary value problem

$$\begin{aligned} \frac{\partial \hat{f}^\varepsilon}{\partial t}(t,x,v) + \mathcal{L}^\varepsilon \hat{f}^\varepsilon(t,x,v) &= 0, & t \in [0,T],\ x > \hat{h}(t,v) \\ \hat{f}^\varepsilon(T,x,v) &= (x-K)^+, & x > \hat{h}(T,v) \\ \hat{f}^\varepsilon(t,\hat{h}(t,v),v) &= 0, & t \in [0,T] \end{aligned} \tag{3.14}$$

where, much like in (3.3),

$$\begin{aligned} \mathcal{L}^\varepsilon &= \mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \varepsilon^2 \mathcal{L}_2 \\ \mathcal{L}_0 &= \left(a - \frac{c}{2}e^{2v}\right)\frac{\partial}{\partial v} + \frac{x^2}{2}e^{2v}\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\,\mathrm{Id}, \\ \mathcal{L}_1 &= \rho x e^v \psi(t,v)\frac{\partial^2}{\partial x \partial v}, \qquad \mathcal{L}_2 = \frac{1}{2}\psi^2(t,v)\frac{\partial^2}{\partial v^2}. \end{aligned} \tag{3.15}$$

If we formally expand the price function $\hat{f}^\varepsilon$ as $\hat{f}^\varepsilon = \hat{f}_0 + \varepsilon \hat{f}_1 + \varepsilon^2 \hat{f}_2 + \ldots$, we end up with the system of PDEs

$$\frac{\partial \hat{f}_0}{\partial t} + \mathcal{L}_0 \hat{f}_0 = 0, \qquad \frac{\partial \hat{f}_1}{\partial t} + \mathcal{L}_0 \hat{f}_1 + \mathcal{L}_1 \hat{f}_0 = 0, \qquad \frac{\partial \hat{f}_2}{\partial t} + \mathcal{L}_0 \hat{f}_2 + \mathcal{L}_1 \hat{f}_1 + \mathcal{L}_2 \hat{f}_0 = 0, \qquad \ldots \tag{3.16}$$

with the typical terminal conditions $\hat{f}_0(T,x,v) = (x-K)^+$ and $\hat{f}_j(T,x,v) = 0$ for $j = 1,2\ldots$, and with the nonconstant boundary conditions $\hat{f}_j(t,\hat{h}(t,v),v) = 0$ for $j = 0,1,2,\ldots$.

Our goal is to derive the first-order approximation for the option price and to demonstrate that (under suitable regularity conditions) it converges in the following sense:

$$\hat{f}^{\varepsilon}(t,x,v) = \hat{f}_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v) + o(\varepsilon). \tag{3.17}$$

An analogous convergence result was conjectured, without proof, by Privault and She in [2] when dealing with plain vanilla options. We will give the detailed proof of (3.17) in Theorem 3.12.

3.2.4 The zero-order term in the nonzero interest rate scenario

The zero-order term $\hat{f}_0(t,x,v)$, which, as seen above, is defined as the solution of the terminal and boundary value problem

$$\begin{aligned} \frac{\partial \hat{f}_0}{\partial t}(t,x,v) + \mathcal{L}_0 \hat{f}_0(t,x,v) &= 0, && t \in [0,T],\ x > \hat{h}(t,v) \\ \hat{f}_0(T,x,v) &= (x-K)^+, && x > \hat{h}(T,v) \\ \hat{f}_0(t,\hat{h}(t,v),v) &= 0, && t \in [0,T], \end{aligned} \tag{3.18}$$

is nothing but the option price corresponding to the limiting case $\varepsilon = 0$. The equivalent definition of this option price under the martingale pricing framework is similar to (3.10): here we have

$$\hat{f}_0(t,x,v) = e^{-r(T-t)} \mathbb{E}\left[(S_T^{t,v} - K)^+ \mathbb{1}_{\{\tau_{\hat{h}}^t \geq T\}} \middle| S_t^{t,v} = x \right], \qquad t \in [0,T] \tag{3.19}$$

where $\tau_{\hat{h}}^t := \inf\{u \geq t : S_u^{t,v} \leq \hat{h}(u, V_u^{t,v})\}$ and the process $\{S_u^{t,v}\}_{u\in[t,T]}$ follows a geometric Brownian motion with constant drift r and time-dependent deterministic volatility $e^{V_u^{t,v}}$ (where $V_u^{t,v}$ is given by (3.11)).

For a given (fixed) initial time t' and initial log-volatility v', by recalling the obvious semigroup property $V_u^{t,V_t^{t',v'}} = V_u^{t',v'}$ ($t' \leq t \leq u$) we see that

$$\hat{f}_0(t,x,V_t^{t',v'}) = e^{-r(T-t)} \mathbb{E}\Big[(S_T^{t',v'} - K)^+ \mathbb{1}_{\{\tau_{\hat{h}}^t \geq T\}} \ \Big| \ S_t^{t',v'} = x \Big], \qquad t \in [t',T] \tag{3.20}$$

where $\tau_{\hat{h}}^t = \inf\{u \geq t : S_u^{t',v'} \leq \hat{h}(u, V_u^{t',v'})\}$. The function $\hat{f}_0(t,x,V_t^{t',v'})$, which only depends on the variables t and x, is clearly the definition (in the martingale approach) of the price of a DOC option under a Black and Scholes model where the constant interest rate is r, the dividend rate is zero, the volatility is given by the (smooth) deterministic function of time $e^{V_t^{t',v'}}$, $t \in [t',T]$ (which depends on the initial value $v' = V_{t'}^{t',v'}$), and the time-dependent barrier function is $\hat{h}(t, V_t^{t',v'})$, $t \in [t',T]$.

We will tackle this problem by resorting to the closed-form analytical solutions which are available on the literature. Rapisarda [19] and Dorfleitner et al. [20] obtained the exact closed-form solution of the following terminal and boundary value problem, which defines the price of a DOC time-dependent barrier option under the Black and Scholes model with constant drift r and time-dependent volatility $e^{V_t^{t',v'}}$:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\mathrm{bs}}^{t',v'} u(t,x) &= 0, && t \in [t',T],\ x > \hat{h}^{t',v'}(t) \\ u(T,x) &= (x-K)^+, && x > \hat{h}^{t',v'}(T) \\ u(t,\hat{h}^{t',v'}(t)) &= 0, && t \in [t',T] \end{aligned} \tag{3.21}$$

where

$$\mathcal{L}_{\mathrm{bs}}^{t',v'} := \frac{x^2}{2} e^{2V_t^{t',v'}} \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r\,\mathrm{Id}, \tag{3.22}$$

$$\hat{h}^{t',v'}(t) := \exp\left\{ -r\left(\frac{\gamma^2(t',t,v')}{\gamma^2(t',T,v')} T + \frac{\gamma^2(t,T,V_t^{t',v'})}{\gamma^2(t',T,v')} t' - t \right) \right\} \left(\frac{H_1}{H_0} \right)^{\frac{\gamma^2(t',t,v')}{\gamma^2(t',T,v')}} H_0, \qquad t \in [t',T], \tag{3.23}$$

$H_0 = \hat{h}^{t',v'}(t')$ and $H_1 = \hat{h}^{t',v'}(T)$ are parameters, and the functions $V_t^{t',v'}$, $\gamma^2(t,u,v)$ are defined by (3.11) and (3.6) respectively. We emphasize that the constant v' which defines the initial log-volatility $v' = V_{t'}^{t',v'}$ should be interpreted as a fixed parameter, as the above PDE problem only involves the variables t and x.

Notice that the barrier function (3.23) can be also written in the reparameterized version

$$\hat{h}^{(\beta)}(t) := H_1 \exp\left\{ -r(T-t) + \frac{1+2\beta}{2}\gamma^2(t,T,V_t^{t',v'}) \right\}, \qquad t \in [t',T] \tag{3.24}$$

provided that $\beta \equiv \beta(t',v')$ is defined through the equality $\frac{1+2\beta}{2} = \frac{1}{\gamma^2(t',T,v')}\left(\log\left(\frac{H_0}{H_1}\right) + r(T-t')\right)$. (This parameterization is the one of Rapisarda [19], while the parameterization (3.23) is that of Dorfleitner et al. [20].) Alternatively, we may simply assume that the parameter β is fixed and use the parameterization (3.24), under which the initial values $H_0 \equiv H_0(v') \equiv \hat{h}^{(\beta)}(t')$ depend on the values of $t' \in [0,T]$ and $v' \in \mathbb{R}$.

It is now easy to see why we have chosen the functional form (3.13) for the barrier function $\hat{h}(t,v)$: that choice is the one which ensures that $\hat{h}(t,V_t^{t',v'})$ is equal to the function $\hat{h}^{(\beta)}(t)$ from (3.24). Therefore the terminal and boundary value problem (3.21) is simply the PDE formulation for the option price $\hat{f}_0(t,x,V_t^{t',v'})$ from (3.20).

Therefore $\hat{f}_0(t',x,v')$ can be obtained through Equation (27) of Rapisarda [19]. Replacing (t',v') by (t,v), we obtain the following result:

Proposition 3.4. *Let $\hat{f}_0(t,x,v)$ be the zero-order term in the first-order expansion* (3.17) *for the price of a DOC option with barrier function $\hat{h}(t,v)$ under the model* (3.12)*. Then*

$$\begin{aligned} \hat{f}_0(t,x,v) =& x\mathcal{N}(d_1(t,x,v)) - Ke^{-r(T-t)}\mathcal{N}(d_2(t,x,v)) \\ &- \left(\frac{\hat{h}(t,v)}{x}\right)^{2+2\beta} x\mathcal{N}(d_3(t,x,v)) + \left(\frac{\hat{h}(t,v)}{x}\right)^{2\beta} Ke^{-r(T-t)}\mathcal{N}(d_4(t,x,v)) \end{aligned} \tag{3.25}$$

for $t \in [0,T]$ and $x > \hat{h}(t,v)$, where

$$\begin{aligned} d_1(t,x,v) &= \frac{1}{\gamma(t,T,v)}\left(\log\left(\frac{x}{K \vee H_1}\right) + r(T-t) + \frac{1}{2}\gamma^2(t,T,v)\right), \\ d_2(t,x,v) &= d_1(t,x,v) - \gamma(t,T,v), \\ d_3(t,x,v) &= d_1(t,x,v) + \frac{2}{\gamma(t,T,v)}\log\left(\frac{\hat{h}(t,v)}{x}\right), \\ d_4(t,x,v) &= d_2(t,x,v) + \frac{2}{\gamma(t,T,v)}\log\left(\frac{\hat{h}(t,v)}{x}\right). \end{aligned} \tag{3.26}$$

3.2.5 The first-order term

The first-order term solves

$$\begin{aligned} \frac{\partial \hat{f}_1}{\partial t}(t,x,v) + \mathcal{L}_0 \hat{f}_1(t,x,v) &= -\mathcal{L}_1 \hat{f}_0(t,x,v), && t \in [0,T],\ x > \hat{h}(t,v) \\ \hat{f}_1(T,x,v) &= 0, && x > \hat{h}(T,v) \\ \hat{f}_1(t,\hat{h}(t,v),v) &= 0, && t \in [0,T] \end{aligned} \tag{3.27}$$

where the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ were defined in (3.15).

As in the vanilla option framework of Subsection 3.1.1, here the first step towards the computation of an explicit expression for the first order term is to give a stochastic representation formula for the solution of this terminal and boundary value problem. In view of the Feynman-Kac theorem for Cauchy-Dirichlet problems for parabolic PDEs (Theorem 2.8), our candidate solution for (3.27) is

$$\widetilde{f}_1(t,x,v) = \mathbb{E}\left[\int_t^{T \wedge \tau_{\hat{h}}^t} e^{-r(u-t)} \mathcal{L}_1 \hat{f}_0(u, S_u^{t,v}, V_u^{t,v})\, du \;\middle|\; S_t^{t,v} = x \right] \tag{3.28}$$

where the process $(S^{t,v}, V^{t,v})$ and the stopping time $\tau_{\hat{h}}^t$ are defined as in (3.19). Notice that we cannot simply invoke Theorem 2.8 to justify that $\widetilde{f}_1(t,x,v)$ solves (3.27), because not all of its hypothesis are satisfied: for instance, the ellipticity condition (iii) fails because the differential operator $\mathcal{L}_0$ has no term in $\frac{\partial^2}{\partial v^2}$. Notwithstanding, it is indeed possible to prove the desired result through a generalization of Theorem 2.8:

Lemma 3.5. *Assume that $K \geq H_1$. Then $\hat{f}_1(t,x,v) = \widetilde{f}_1(t,x,v)$, i.e, the function $\widetilde{f}_1(t,x,v)$ is the unique classical solution of the terminal and boundary value problem* (3.27)*.*

The proof will be provided in the end of this subsection.

Remark 3.6. The condition $K \geq H_1$, which for simplicity we will assume throughout the proof of this and other theoretical results, means that the DOC option is a regular barrier option and guarantees that the payoff of the option is continuous at the intersection of the terminal and boundary conditions. It is worth mentioning here that it is in fact well-known that there are additional difficulties in the pricing of reverse barrier options, and this has been addressed in a number of articles (see e.g. Schmock et al. [22]).

It is natural to conjecture that it should be possible to generalize our theoretical results to the case of reverse barrier options through some kind of regularization argument where the reverse option is approximated by some sequence of barrier options with smooth payoffs. Therefore, we will not be assuming that $K \geq H_1$ in the explicit computations of the expression for the first-order term, so that it will also be possible to use our first-order expansion for the (approximate) pricing of reverse DOC options.

Having established this result, the natural course of action would be for us to compute the expected value (3.28) using the joint law of $(S_u^{t,v}, \tau_{\hat{h}}^t)$. However, to the best of our knowledge the analytical expression for the joint law of the random variables $(S_u^{t,v}, \tau_{\hat{h}}^t)$ is not available in the literature. For this reason, we instead take an alternative approach where we will take advantage of the known results on the Black and Scholes equation with time-dependent coefficients so as to derive the explicit expression for the first-order term.

With this in mind, we fix an initial time t' and an initial log-volatility v'. Then we can write

$$\hat{f}_1(t, x, V_t^{t',v'}) = \mathbb{E}\left[\int_t^{T\wedge\tau_{\hat{h}}^t} e^{-r(u-t)}\mathcal{L}_1\hat{f}_0(u, S_u^{t',v'}, V_u^{t',v'})\,du\middle|S_t^{t',v'} = x\right]$$

where $\tau_{\hat{h}}^t = \inf\{u \geq t : S_u^{t',v'} \leq \hat{h}(u, V_u^{t',v'})\}$. We can now invoke again the Feynman-Kac theorem to obtain the PDE problem associated to this function of the variables t and x:

Lemma 3.7. *Assume that $K \geq H_1$. Then the function $\hat{f}_1(t, x, V_t^{t',v'})$ is the unique solution of the terminal and boundary value problem*

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) + \mathcal{L}_{\text{bs}}^{t',v'}u(t, x) &= -\mathcal{L}_1\hat{f}_0(t, x, V_t^{t',v'}), & t \in [t', T],\ x > \hat{h}(t, V_t^{t',v'}) \\ u(T, x) &= 0, & x > \hat{h}(T, V_T^{t',v'}) \\ u(t, \hat{h}(t, V_t^{t',v'})) &= 0, & t \in [t', T] \end{aligned} \tag{3.29}$$

where $\mathcal{L}_{\text{bs}}^{t',v'}$ is the differential operator defined in (3.22).

Proof. Combining the closed-form expression (B.29)–(B.33) with the estimates (B.1)–(B.28) in Appendix B it is easily seen that, for fixed (t', v'), the function $\mathcal{L}_1\hat{f}_0(t, x, V_t^{t',v'})$ has no singularities and satisfies the continuity assumption (iv) of Theorem 2.8, and the estimate (B.34) in Appendix B ensures that $\mathcal{L}_1\hat{f}_0(t, x, V_t^{t',v'})$ satisfies the polynomial growth assumption (vi). In addition, the change of variable $y = x - \hat{h}(t, V_t^{t',v'})$ can be used to take care of the time-dependence of the domain. (The details of this change of variable argument can be found in the proof of Theorem 3.5, given below.) Hence the result follows from Theorem 2.8. □

Note that our twofold use of the Feynman-Kac theorem yields a result which is interesting from a strictly theoretical point of view: we have shown that in order to eliminate the space variable v from the original PDE problem (3.27), we just have to replace it by the solution of the ordinary differential equation $\frac{dV}{dt} = a - \frac{c}{2}e^{2V}$, i.e, by the differential equation corresponding to the noiseless limit of the process V^ε around which the small vol of vol expansion is being performed.

Next, we want to eliminate the nonhomogeneity in the problem (3.29) so that it becomes possible to obtain an explicit expression for the solution of the PDE problem through Theorem 1 of Dorfleitner et al. [20]. (This theorem provides an integral representation formula for the solution of a general terminal and boundary value problem for the homogeneous Black and Scholes equation with time-dependent coefficients, provided that the boundary follows a specific form determined by the coefficients.) With that in mind, we decompose

$$\hat{f}_1(t, x, v) = \hat{f}_1^{(A)}(t, x, v) - \hat{f}_1^{(B)}(t, x, v)$$

where $\hat{f}_1^{(A)}(t, x, V_t^{t',v'})$ is the solution of the nonhomogeneous terminal value problem with no boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) + \mathcal{L}_{\text{bs}}^{t',v'}u(t, x) &= -\mathcal{L}_1\hat{f}_0(t, x, V_t^{t',v'}), & t \in [t', T],\ x \in \mathbb{R} \\ u(T, x) &= 0, & x \in \mathbb{R} \end{aligned} \tag{3.30}$$

and $\hat{f}_1^{(B)}(t,x,V_t^{t',v'})$ is the solution of the homogeneous terminal and boundary value problem

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\mathrm{bs}}^{t',v'} u(t,x) &= 0, && t\in[t',T],\ x>\hat{h}(t,V_t^{t',v'})\\
u(T,x) &= 0, && x>\hat{h}(T,V_T^{t',v'})\\
u(t,\hat{h}(t,V_t^{t',v'})) &= \hat{f}_1^{(A)}(t,\hat{h}(t,V_t^{t',v'})), && t\in[t',T].
\end{aligned} \tag{3.31}$$

The existence and uniqueness of a classical solution for each of the PDE problems (3.30) and (3.31) is assured by Theorem 1 of Heath and Schweizer [8] and by Theorem 1 of Dorfleitner et al. [20], respectively. Consequently, the difference between these solutions must be equal to the unique solution of (3.29).

As in Privault and She [2], we can use the Feynman-Kac formula of Heath and Schweizer [8] to write the stochastic representation formula for $\hat{f}_1^{(A)}$ and, as we will see, it is possible to use the law of the process $\{S_t^{t',v'}\}_{t\in[t',T]}$ in order to express the conditional expectation in closed form. Then, Theorem 1 of Dorfleitner et al. [20] allows us to write an explicit integral representation for the function $f_1^{(B)}$.

We start by computing $\mathcal{L}_1\hat{f}_0$, where $\hat{f}_0$ is the first order term whose closed form was given in Proposition 3.4. Differentiating (3.26) we get

$$\frac{\partial d_1}{\partial x}(t,x,v) = \frac{\partial d_2}{\partial x}(t,x,v) = \frac{1}{x\gamma(t,T,v)}, \qquad \frac{\partial d_3}{\partial x}(t,x,v) = \frac{\partial d_4}{\partial x}(t,x,v) = -\frac{1}{x\gamma(t,T,v)} \tag{3.32}$$

and also

$$\begin{aligned}
\frac{\partial d_1}{\partial v}(t,x,v) &= \frac{\partial \gamma}{\partial v}(t,T,v)\Big(1-\frac{d_1(t,x,v)}{\gamma(t,T,v)}\Big),\\
\frac{\partial d_2}{\partial v}(t,x,v) &= -\frac{\partial \gamma}{\partial v}(t,T,v)\Big(1+\frac{d_2(t,x,v)}{\gamma(t,T,v)}\Big),\\
\frac{\partial d_3}{\partial v}(t,x,v) &= \frac{\partial \gamma}{\partial v}(t,T,v)\Big(1-\frac{d_3(t,x,v)}{\gamma(t,T,v)}\Big) + \frac{2}{\gamma(t,T,v)}\frac{\partial \log\hat{h}}{\partial v}(t,v),\\
\frac{\partial d_4}{\partial v}(t,x,v) &= -\frac{\partial \gamma}{\partial v}(t,T,v)\Big(1+\frac{d_4(t,x,v)}{\gamma(t,T,v)}\Big) + \frac{2}{\gamma(t,T,v)}\frac{\partial \log\hat{h}}{\partial v}(t,v).
\end{aligned} \tag{3.33}$$

But

$$\begin{aligned}
n\big(d_1(t,x,v)\big) &= \frac{1}{\sqrt{2\pi}}\exp\bigg\{-\frac{1}{2\gamma^2(t,T,v)}\bigg(\log\Big(\frac{x}{K\vee H_1}\Big)+r(T-t)+\frac{1}{2}\gamma^2(t,T,v)\bigg)^2\bigg\}\\
&= \frac{1}{\sqrt{2\pi}}\exp\bigg\{\log\Big(\frac{K\vee H_1}{x}\Big)-r(T-t)\bigg\}\exp\bigg\{-\frac{1}{2\gamma^2(t,T,v)}\bigg(\log\Big(\frac{x}{K\vee H_1}\Big)+r(T-t)-\frac{1}{2}\gamma^2(t,T,v)\bigg)^2\bigg\}\\
&= \frac{K\vee H_1}{x}e^{-r(T-t)}n\big(d_2(t,x,v)\big)
\end{aligned}$$

and similarly $n\big(d_3(t,x,v)\big) = \frac{x(K\vee H_1)}{\hat{h}^2(t,v)}e^{-r(T-t)}n\big(d_4(t,x,v)\big)$, so (3.32) yields

$$\begin{aligned}
\frac{\partial \hat{f}_0}{\partial x}(t,x,v) =& \mathcal{N}\big(d_1(t,x,v)\big) + (1+2\beta)\Big(\frac{\hat{h}(t,v)}{x}\Big)^{2+2\beta}\mathcal{N}\big(d_3(t,x,v)\big)\\
&- 2\beta\Big(\frac{\hat{h}(t,v)}{x}\Big)^{1+2\beta}\frac{K}{\hat{h}(t,v)}e^{-r(T-t)}\mathcal{N}\big(d_4(t,x,v)\big)\\
&+ \frac{A}{\gamma(t,T,v)}\bigg[n\big(d_1(t,x,v)\big) + \Big(\frac{\hat{h}(t,v)}{x}\Big)^{2+2\beta}n\big(d_3(t,x,v)\big)\bigg]
\end{aligned}$$

where $A := 1 - \frac{K}{K \vee H_1}$. Differentiating with respect to v and recalling (3.33),

$$
\begin{aligned}
\frac{\partial^2 \hat{f}_0}{\partial x \partial v}(t,x,v) &= \frac{\partial \gamma}{\partial v}(t,T,v)\left[\left(1 - \frac{d_1(t,x,v)}{\gamma(t,T,v)}\right)\left(1 - A\,\frac{d_1(t,x,v)}{\gamma(t,T,v)}\right) - \frac{A}{\gamma^2(t,T,v)}\right] n(d_1(t,x,v)) \\
&+ \frac{1+2\beta}{x^{2+2\beta}}\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t,v)\,\mathcal{N}(d_3(t,x,v)) \\
&+ \frac{1}{x^{2+2\beta}}\Bigg[-\frac{A\,\hat{h}^{2+2\beta}(t,v)}{\gamma^2(t,T,v)}\frac{\partial \gamma}{\partial v}(t,T,v) \\
&\qquad + \hat{h}^{2+2\beta}(t,v)\left\{\frac{\partial \gamma}{\partial v}(t,T,v)\left(1 - \frac{d_3(t,x,v)}{\gamma(t,T,v)}\right) + \frac{2}{\gamma(t,T,v)}\frac{\partial \log \hat{h}}{\partial v}(t,v)\right\}\left(1 + 2\beta - \frac{A\,d_3(t,x,v)}{\gamma(t,T,v)}\right) \\
&\qquad + \frac{A}{\gamma(t,T,v)}\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t,v)\Bigg] n(d_3(t,x,v)) \\
&+ K e^{-r(T-t)}\frac{2\beta}{x^{1+2\beta}}\Bigg[-\frac{\partial \hat{h}^{2\beta}}{\partial v}(t,v)\,\mathcal{N}(d_4(t,x,v)) \\
&\qquad + \hat{h}^{2\beta}(t,v)\left\{\frac{\partial \gamma}{\partial v}(t,T,v)\left(1 + \frac{d_4(t,x,v)}{\gamma(t,T,v)}\right) - \frac{2}{\gamma(t,T,v)}\frac{\partial \log \hat{h}}{\partial v}(t,v)\right\} n(d_4(t,x,v))\Bigg].
\end{aligned}
\tag{3.34}
$$

Now, according to the Feynman-Kac theorem of Heath and Schweizer [8], the stochastic representation formula for the unique solution of the terminal value problem (3.30) is

$$
\begin{aligned}
\hat{f}_1^{(A)}(t,x,V_t^{t',v'}) &= \int_t^T e^{-r(u-t)}\mathbb{E}\left[\mathcal{L}_1 \hat{f}_0(u, S_u^{t',v'}, V_u^{t',v'})\Big| S_t^{t',v'} = x\right] du \\
&= \int_t^T e^{-r(u-t)}\rho\, e^{V_u^{t',v'}}\psi(u, V_u^{t',v'})\,\mathbb{E}\left[S_u^{t',v'}\frac{\partial^2 \hat{f}_0}{\partial x \partial v}(u, S_u^{t',v'}, V_u^{t',v'}) \;\Big|\; S_t^{t',v'} = x\right] du
\end{aligned}
\tag{3.35}
$$

where, as usual, the process $\{S_u^{t',v'}\}_{u\in[t',T]}$ follows a geometric Brownian motion with constant drift r and time-dependent deterministic volatility $e^{V_u^{t',v'}}$; in particular, for each $u \in [t,T]$ the conditional distribution of $S_u^{t',v'}$ given $S_t^{t',v'} = x$ is the lognormal distribution with parameters $\mu = \log x + r(u-t) - \frac{1}{2}\gamma^2(t,u,V_t^{t',v'})$ and $\sigma^2 = \gamma^2(t,u,V_t^{t',v'})$.

Let us now compute the closed-form expression for the expectation inside the time integral in (3.35). Since the choice of initial time and log-volatility (t', v') is arbitrary, we can assume without loss of generality that $t = t'$ (i.e, that $\hat{f}_1^{(A)}$ is being computed at initial time t'). In Appendix A we compute the expected values of the type

$$
\mathbb{E}\left[\left(S_u^{t',v'}\right)^{C(\beta)} d_i^k(u, S_u^{t',v'}, V_u^{t',v'})\, n\big(d_i(u, S_u^{t',v'}, V_u^{t',v'})\big) \;\Big|\; S_{t'}^{t',v'} = x\right]
$$

and

$$
\mathbb{E}\left[\left(S_u^{t',v'}\right)^{C(\beta)} \mathcal{N}\big(d_i(t, S_u^{t',v'}, V_u^{t',v'})\big) \;\Big|\; S_{t'}^{t',v'} = x\right]
$$

which arise when we substitute (3.34) into (3.35). The conclusion is that

$$
\begin{aligned}
\hat{f}_1^{(A)}(t',x,v') &= \int_{t'}^T e^{-r(u-t')}\rho\, e^{V_u^{t',v'}}\psi(u, V_u^{t',v'}) \\
&\times\Bigg[\frac{\partial \gamma}{\partial v}\left\{\left(1 - \frac{A}{\gamma^2(u,T,V_u^{t',v'})}\right)E_{1,0} - \frac{1+A}{\gamma(u,T,V_u^{t',v'})}E_{1,1} + \frac{A}{\gamma^2(u,T,V_u^{t',v'})}E_{1,2}\right\} \\
&\quad + (1+2\beta)\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(u, V_u^{t',v'})\, E_2
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg\{-A\,\frac{\partial\gamma}{\partial v}\,\frac{\hat h^{2+2\beta}(u,V_u^{t',v'})}{\gamma^2(u,T,V_u^{t',v'})}+(1+2\beta)\,\hat h^{2+2\beta}(u,V_u^{t',v'})\Big(\frac{\partial\gamma}{\partial v}+\frac{2}{\gamma(u,T,V_u^{t',v'})}\frac{\partial\log\hat h}{\partial v}(u,V_u^{t',v'})\Big)\\
&\qquad\qquad+\frac{A}{\gamma(u,T,V_u^{t',v'})}\frac{\partial\hat h^{2+2\beta}}{\partial v}(u,V_u^{t',v'})\bigg\}E_{3,0}\\
&-\frac{\hat h^{2+2\beta}(u,V_u^{t',v'})}{\gamma(u,T,V_u^{t',v'})}\bigg\{A\Big(\frac{\partial\gamma}{\partial v}+\frac{2}{\gamma(u,T,V_u^{t',v'})}\frac{\partial\log\hat h}{\partial v}(u,V_u^{t',v'})\Big)+(1+2\beta)\frac{\partial\gamma}{\partial v}\bigg\}E_{3,1}\\
&+\hat h^{2+2\beta}(u,V_u^{t',v'})\frac{\partial\gamma}{\partial v}\frac{A}{\gamma^2(u,T,V_u^{t',v'})}\,E_{3,2}\\
&+2\beta Ke^{-r(T-u)}\bigg[-\frac{\partial\hat h^{2\beta}}{\partial v}(u,V_u^{t',v'})\,E_4\\
&\quad+\hat h^{2\beta}(u,V_u^{t',v'})\bigg\{\Big(\frac{\partial\gamma}{\partial v}-\frac{2}{\gamma(u,T,V_u^{t',v'})}\frac{\partial\log\hat h}{\partial v}(u,V_u^{t',v'})\Big)E_{5,0}+\frac{\partial\gamma}{\partial v}\frac{1}{\gamma(u,T,V_u^{t',v'})}\,E_{5,1}\bigg\}\bigg]\bigg]du.
\end{aligned}
\tag{3.36}
$$

where we abbreviated $\frac{\partial\gamma}{\partial v}\equiv\frac{\partial\gamma}{\partial v}(u,T,V_u^{t',v'})$, and the functions $E_i\equiv E_i(t',u,x,v)$ are given in Equations (A.4)–(A.13) in Appendix A. The above expression is a one-dimensional integral which is easy to solve numerically.

As for the function $\hat f_1^{(B)}$ which we defined as the unique solution of (3.31), formula (A15) in Theorem 1 of Dorfleitner et al. [20] assures that it is given by

$$
\hat f_1^{(B)}(t',x,v')=\frac12\int_{t'}^{T}e^{r(u-t')}e^{2V_u^{t',v'}}\hat h^2(u,V_u^{t',v'})\,\frac{\partial G^+}{\partial w}\big(\hat h(u,V_u^{t',v'}),u,x,t';v'\big)\hat f_1^{(A)}(u,\hat h(u,V_u^{t',v'}),V_u^{t',v'})\,du
$$

where $\frac{\partial G^+}{\partial w}$ is the partial derivative with respect to w of the Green function

$$
\begin{aligned}
G^+(w,u,x,t;v)=&\frac{1}{\sqrt{2\pi}\gamma(t,u,v)w}\exp\bigg\{-\frac{1}{2\gamma^2(t,u,v)}\Big(\log\Big(\frac{w}{x}\Big)-r(u-t)+\frac12\gamma^2(t,u,v)\Big)^2\bigg\}\\
&\times\bigg(1-\exp\bigg\{-\frac{2}{\gamma^2(t,u,v)}\log\Big(\frac{\hat h(u,V_u^{t,v})}{w}\Big)\log\Big(\frac{\hat h(t,v)}{x}\Big)\bigg\}\bigg).
\end{aligned}
\tag{3.37}
$$

The explicit expression for this partial derivative is

$$
\begin{aligned}
\frac{\partial G^+}{\partial w}(w,u,x,t;v)=&\frac{1}{\sqrt{2\pi}\gamma^3(t,u,v)w^2}\exp\bigg\{-\frac{1}{2\gamma^2(t,u,v)}\Big(\log\Big(\frac{w}{x}\Big)-r(u-t)+\frac12\gamma^2(t,u,v)\Big)^2\bigg\}\\
&\times\bigg[\log\Big(\frac{x}{w}\Big)+r(u-t)-\frac32\gamma^2(t,u,v)\\
&\quad+\bigg(\log\Big(\frac{wx}{\hat h^2(t,v)}\Big)-r(u-t)+\frac32\gamma^2(t,u,v)\bigg)\exp\bigg\{-\frac{2}{\gamma^2(t,u,v)}\log\Big(\frac{\hat h(u,V_u^{t,v})}{w}\Big)\log\Big(\frac{\hat h(t,v)}{x}\Big)\bigg\}\bigg].
\end{aligned}
$$

In particular, if $w=\hat h(u,V_u^{t',v'})$ and $(t,v)=(t',v')$ then

$$
\begin{aligned}
\frac{\partial G^+}{\partial w}\big(\hat h(u,V_u^{t',v'}),u,x,t';v'\big)=&\frac{2}{\sqrt{2\pi}\gamma^3(t',u,v')\hat h^2(u,V_u^{t',v'})}\log\Big(\frac{x}{\hat h(t',v')}\Big)\\
&\times\exp\bigg\{-\frac{1}{2\gamma^2(t',u,v')}\Big(\log\Big(\frac{\hat h(u,V_u^{t',v'})}{x}\Big)-r(u-t')+\frac12\gamma^2(t',u,v')\Big)^2\bigg\}.
\end{aligned}
$$

Recalling the decomposition $\hat f_1=\hat f_1^{(A)}-\hat f_1^{(B)}$ and using the arbitrariness of the initial time and volatility so as to replace (t',v') by (t,v), we finally obtain the desired explicit expression for the first-order term:

Proposition 3.8. *Let $\hat{f}_1(t,x,v)$ be the first-order term in the first-order expansion* (3.17) *for the price of a DOC option with barrier function $\hat{h}(t,v)$ under the model* (3.12)*. Assume that $K \geq H_1$. Then*

$$\hat{f}_1(t,x,v) = \int_t^T \Bigg[E_\beta^{(A)}(t,u,x,v) - \frac{1}{2} e^{-r(u-t)} e^{2V_u^{t,v}} \hat{h}^2(u, V_u^{t,v}) \frac{\partial G^+}{\partial w}\big(\hat{h}(u,V_u^{t,v}), u, x, t; v\big) \times \\ \times \int_u^T E_\beta^{(A)}(u,s,\hat{h}(u,V_u^{t,v}), V_u^{t,v}) \Bigg] ds\, du \tag{3.38}$$

for $t \in [0,T]$ and $x > \hat{h}(t,v)$, where $E_\beta^{(A)}(t',u,x,v')$ is the integrand of (3.36)*.*

We observe that the numerical computation of the integral in (3.38) is much easier than solving numerically the associated PDE problem (3.27) or computing the expectation (3.28) via Monte Carlo simulation. We additionally recall from Remark 3.6 that if $K < H_1$ then (3.38) can also be used to compute the first-order term of an (in this case formal) asymptotic expansion for the price of the option.

We finish this subsection with the proof of Lemma 3.5:

Proof of Lemma 3.5. To prove the lemma, we need to generalize Theorem 2.8 so as to assure that the terminal and boundary value problem (3.27) has a unique (classical) solution $\hat{f}_1(t,x,v)$ whose stochastic representation is given by the function $\widetilde{f}_1(t,x,v)$ defined in (3.28).

The hypotheses of Theorem 2.8 which are not satisfied by our problem are the following:

- *Time-independence of the domain.* In our setting, instead of a domain of the form $[0,T] \times D$ with $D \subset \mathbb{R}^2$ the domain for the spatial variables, we have a domain

  $$\hat{D} = \Big\{(t,x,v) \subset [0,T] \times \mathbb{R}^2 : x > \hat{h}(t,v)\Big\} \subset [0,T] \times \mathbb{R}^2$$

  where $\hat{h}(t,v)$ is a nonconstant smooth function of time and volatility, so that the spatial domain $D_t = \{(x,v) \in \mathbb{R}^2 : (t,x,v) \in \hat{D}\}$ depends on time t.

  The workaround for this is to make the change of variables $y = x - \hat{h}(t,v)$: $\widetilde{f}_1$ satisfies (3.27) if and only if the function $\widetilde{f}_1^*$ defined as

  $$\widetilde{f}_1^*(t,y,v) := \widetilde{f}_1(t, y + \hat{h}(t,v), v) = \mathbb{E}\left[\int_t^{T \wedge \tau_{\hat{h}}^t} e^{-r(u-t)} \mathcal{L}_1 \hat{f}_0\big(u, Y_u^{t,v} + \hat{h}(u,V_u^{t,v}), V_u^{t,v}\big)\, du \,\middle|\, Y_t^{t,v} = y \right] \tag{3.39}$$

  (where $Y_u^{t,v} := S_u^{t,v} - \hat{h}(u,V_u^{t,v})$ and $\tau_{\hat{h}}^t = \inf\{u \geq t : Y_u^{t,v} \leq 0\}$) satisfies the terminal and boundary value problem with constant boundary

  $$\begin{aligned} \frac{\partial \hat{f}_1^*}{\partial t}(t,y,v) + \mathcal{L}_0^* \hat{f}_1^*(t,y,v) &= -\mathcal{L}_1 \hat{f}_0(t, y + \hat{h}(t,v), v), \qquad & t \in [0,T], y > 0 \\ \hat{f}_1^*(T,y,v) &= 0, & y > 0 \\ \hat{f}_1^*(t,0,v) &= 0, & t \in [0,T] \end{aligned} \tag{3.40}$$

  where $\mathcal{L}_0^* = \mathcal{L}_0 - \Big(\frac{\partial \hat{h}}{\partial t}(t,v) + \big(a - \frac{c}{2}e^{2v}\big)\frac{\partial \hat{h}}{\partial v}(t,v)\Big)\frac{\partial}{\partial x}$. The spatial domain for this PDE problem is $D = (0,\infty) \times \mathbb{R}$. Hence, in what follows we will instead prove the equivalent statement that the terminal and boundary value problem (3.40) has a unique classical solution $\hat{f}_1^*(t,y,v)$ whose stochastic representation is given by the function $\widetilde{f}_1^*$ from (3.39).

- *Linear growth and ellipticity assumptions on b and σ.* The coefficients of the partial differential operator $\mathcal{L}_0^*$ are the functions

$$b(t,y,v) = \begin{pmatrix} r(y+\hat{h}(t,v)) - \left(\frac{\partial \hat{h}}{\partial t}(t,v) + (a - \frac{c}{2}e^{2v})\frac{\partial \hat{h}}{\partial v}(t,v)\right) \\ a - \frac{c}{2}e^{2v} \end{pmatrix}, \qquad \sigma(t,x,v) = \begin{pmatrix} (y+\hat{h}(t,v))e^v & 0 \\ 0 & 0 \end{pmatrix}$$

which clearly satisfy the continuity assumption (i), but do not satisfy neither the linear growth assumption (ii) neither the ellipticity assumption (iii) of Theorem 2.8.

- *Polynomial growth assumption on h.* The nonhomogeneity term on our PDE problem is

$$h(t,y,v) \equiv -\mathcal{L}_1 \hat{f}_0\big(t, y+\hat{h}(t,v), v\big) = -\rho\big(y+\hat{h}(t,v)\big)e^v \psi(t,v)\frac{\partial^2 \hat{f}_0}{\partial x \partial v}\big(t, y+\hat{h}(t,v), v\big)$$

where $\frac{\partial^2 \hat{f}_0}{\partial x \partial v}$ is given by (3.34) with $A = 0$ (as we are assuming that $K \geq H_1$).

From the closed-form expression (B.29)–(B.33) given in Appendix B, together with the estimates (B.1)–(B.28) and the smoothness of the function $\psi(t,v)$, it follows that $h(t,y,v)$ satisfies the continuity assumption (iv). Furthermore, even though $\mathcal{L}_1 \hat{f}_0(t,x,v)$ does not satisfy the polynomial growth assumption (vi) (due to the presence of terms which grow exponentially in v), our estimate from Equation (B.34) in Appendix B implies that $\mathcal{L}_1 \hat{f}_0(t,x,v)$ satisfies the following polynomial growth condition in the variable x: for every $M_0 < M_1$, there exists $k > 0$ and a constant K_2 such that

$$\big|\mathcal{L}_1 \hat{f}_0(t,x,v)\big| \leq K_2(M_0,M_1)\big(1+|x|^k\big) \qquad \text{for all } t\in[0,T],\ v\in[M_0,M_1],\ x>\hat{h}(t,v) \tag{3.41}$$

and therefore

$$\big|h(t,y,v)\big| \leq K_2^*(M_0,M_1)\big(1+|y|^k\big) \qquad \text{for all } t\in[0,T],\ v\in[M_0,M_1],\ y>0$$

for some constant K_2^*.

The somewhat long proof of Theorem 2.8 which was presented by Rubio in [9] is organized into a sequence of auxiliary results from which the main theorem can be deduced. As we will show, it is possible to show that the theorem is valid despite the failure of the hypotheses mentioned above through some adaptations to the proofs of the various auxiliary results on Rubio [9]. Therefore, for the sake of brevity, instead of presenting the full proof, we will merely point out which changes must be considered in the proofs presented in [9] in order to demonstrate our desired result.

- *Remark 2.4 of [9].* In our setting, the stopping time can be written exclusively in terms of the process $\{Y_u^{t,v}\}$: indeed, in the notation of Theorem 2.8, $\tau_t = \inf\{u \geq t : Y_u^{t,v} \leq 0\}$. So if we interpret $\{Y_u^{t,v}\}$ as a one-dimensional diffusion process which (unlike the two-dimensional process $\{(Y_u^{t,v}, V_u^{t,v})\}$) satisfies the local ellipticity condition, we can use the arguments from [9] to conclude that the statement holds true.

- *Proposition 2.5 of [9].* In our setting, the (degenerate) process $\{V_u^{t,v}\}$ is given in closed form in (3.11) and the process $\{Y_u^{t,v}\}$ can also be written in closed form as

$$Y_u^{t,v} = (y+\hat{h}(t,v))\exp\left\{r(u-t) - \frac{1}{2}\gamma^2(t,u,v) + \int_t^u e^{V_s^{t,v}}dW_s^1\right\} - \hat{h}(u, V_u^{t,v}).$$

where $y = Y_t^{t,v}$. (This closed-form formula is trivially obtained from the well-known closed form for the geometric Brownian motion with deterministic time-dependent coefficients.) From these closed-form expressions it is clear that (i), (ii) and (iii) hold. In addition, we can obtain a weaker version of (iv) by invoking a classical result on estimates of the moments of solutions of SDEs which can be found in Corollary 2.5.12 of Krylov [23]: for fixed v, the process $\{Y_u^{t,v}\}$, interpreted as a one-dimensional diffusion process, satisfies the hypothesis of Krylov's corollary, so we can conclude that

$$\sup_{v\in[M_0,M_1]} \mathbb{E}\left[\sup_{t\le u\le T} \left|Y_u^{t,v}\right|^{2r} \,\middle|\, Y_t^{t,v} = y\right] \le C(M_0, M_1, r)\left(1 + |y|^{2r}\right)$$

for all $t \in [0,T]$, $y > 0$, $M_0 < M_1$ and $r \ge 1$. As we will see, this weaker form of property (iv) is enough for the purpose of proving our lemma.

- *Lemma 3.3 of [9].* Since for every fixed v the process $\{Y_u^{t,v}\}$, seen as a one-dimensional diffusion, satisfies all the hypothesis of Theorem 2.8, we can prove exactly like in [9] that the stochastic representation formula satisfies the PDE and the boundary condition. As for the second statement of the theorem, taking into account our weaker version of Proposition 2.5-(iv) of [9], we obtain instead the estimate

$$\sup_{\substack{t\in[0,T]\\ v\in[M_0,M_1]}} \left|\widetilde{f}_1^*(t,y,v)\right| \le C(M_0,M_1)\left(1+|y|^k\right) \qquad \text{for all } y>0,\ M_0<M_1.$$

- *Lemma 3.4 of [9].* We need to show that the lemma is valid under the following weaker growth condition: there exists C and a constant $\mu > 0$ such that

$$\sup_{\substack{t\in[0,T]\\ v\in[M_0,M_1]}} \left|u(t,y,v)\right| \le C(M_0,M_1)\left(1+|y|^\mu\right) \qquad \text{for all } y>0,\ M_0<M_1.$$

  The proof can be carried out using the same argument from [9]: in our framework, the upper bounds in Equation (3.23) of [9] are instead of the form $e^{c_0 t}C(M_0,M_1)\left(1+\sup_{0\le s\le t}|Y_s|^\mu\right)$ and $e^{c_0 t}tK_2(M_0,M_1)\left(1+\sup_{0\le s\le t}|Y_s|^k\right)$; as these estimates also ensure the applicability of the dominated convergence theorem, the proof can be finished similarly.

- *Theorem 4.1 of [9].* (In our problem it coincides with Lemma 4.2, because the terminal and boundary conditions are zero.) The proof can be performed without changes, with the exception of the sequences of inequalities in Equations (4.9) and (4.36) of [9]. These must be suitably adapted taking into account our weaker polynomial growth assumption on the nonhomogeneity term and our weaker version of Proposition 2.5-(iv) of [9]. In the case of Equation (4.9), a straightforward adaptation leads us to replace the upper bound by

$$C\big(1 + K(1 + \left(|y|+\alpha\right)^{2k})\big) + C\big(1 + K\big(1 + |y|^{2k}\big)\big) < \infty$$

  where the constants also depend on the initial value v for the log-volatility process. This estimate also assures the required uniform integrability. The adaptations in Equation (4.36) are analogous.

- *Theorems 5.1 and 5.3 of [9].* No generalization of these theorems is necessary for our purposes.

- *Theorem 5.4 of [9].* In our setting, we cannot simply invoke Theorem 1.5 in Chapter V of Krylov [24] to demonstrate the result, because Krylov's theorem requires the linear growth assumption on the SDE coefficients b and σ, which does not hold in our problem. However, we can overcome this difficulty through a localization argument. Let $\{(t_n, x_n, v_n)\}$ be a sequence such that $(t_n, x_n, v_n) \to (t, x, v)$. Then, the associated sequence of (degenerate) log-volatility processes is given by

$$V_u^{t_n,v_n} = v_n + a(u - t_n) + \frac{1}{2}\log\left(1 + \frac{c}{2a}e^{2v_n}(e^{2a(u-t_n)} - 1)\right), \qquad u \in [t_n, T].$$

  Since $|v_n| \le M$ for some constant M, we easily see that

$$\sup_{\substack{n\in\mathbb{N}\\ u\in[t_n,T]}} \left|V_u^{t_n,v_n}\right| = C < \infty \qquad \text{and} \qquad \sup_{u\in[t,T]} \left|V_u^{t,v}\right| \le C < \infty.$$

  Hence the two-dimensional processes $(Y^{t_n,v_n}, V^{t_n,v_n})$ associated to the initial values (t_n, y_n, v_n), as well as the two-dimensional process $(Y^{t,v}, V^{t,v})$ associated to the initial value (t, y, v), are also the solutions of the SDE where the original coefficients b and σ are replaced by smooth functions $\widetilde{b}$, $\widetilde{\sigma}$ such that

$$\widetilde{b}(t,y,v) = \begin{cases} b(t,y,v), & \text{if } |v| \le C \\ 0, & \text{if } |v| \ge 2C \end{cases} \qquad \widetilde{\sigma}(t,y,v) = \begin{cases} \sigma(t,x,v), & \text{if } |v| \le C \\ 0, & \text{if } |v| \ge 2C \end{cases}.$$

  The new coefficients $\widetilde{b}$ and $\widetilde{\sigma}$ satisfy the linear growth assumption. Therefore, Theorem 1.5 in Chapter V of Krylov [24] applied to the SDE with coefficients $\widetilde{b}$ and $\widetilde{\sigma}$ allows us to reach the desired conclusion.

Notice that we have not shown that Theorem 4.4 of [9] is valid in our framework. (Nor Theorem 5.5, which in [9] is just used as a lemma for the proof of the former.) The proofs of these two theorems strongly depend on the ellipticity condition, which is not satisfied by our coefficient function $\sigma(t, x, v)$.

The role of Theorem 4.4 in the proof given by Rubio in [9] is to ensure the differentiability of the stochastic representation formula which defines the candidate solution for the terminal and boundary value problem. But in our problem we have an alternative way to study the differentiability of the candidate solution $\widetilde{f}_1^*(t, y, v)$ which consists in taking advantage of the computations presented above. Indeed, if we fix $v = v'$, the Feynman-Kac theorem in one space dimension assures that $\widetilde{f}_1^*(t, y, v)$ has one Hölder continuous derivative in t and two Hölder continuous derivatives in y. (See the proof of Lemma 3.7.) Furthermore, from Lemma 3.7 and the remainder of the proof of Proposition 3.8 — which do not depend on Lemma 3.5 — we know that $\widetilde{f}_1^*(t,y,v) = \widetilde{f}_1(t, y+\hat{h}(t,v), v)$ where $\widetilde{f}_1(t,x,v)$ is explicitly given by the right-hand side of (3.38), and it is not hard to show that $\widetilde{f}_1^*(t, y, v)$ has two Hölder continuous derivatives in v by explicitly computing the derivative of this expression with respect to v. (See also Appendix B.5.)

These considerations show that $\widetilde{f}_1^* \in C^{1,2,\lambda}_{\text{loc}}((0,T) \times (0,\infty) \times \mathbb{R})$. (This set was defined in Theorem 2.8.) Combining these results with our generalized versions of Theorems 3.3, 3.4 and 4.1 of [9], it follows that $\widetilde{f}_1$ is the unique classical solution of the terminal and boundary value problem (3.27), as we wanted to show. $\square$

3.2.6 Proving the convergence of the asymptotic expansion

Now that we have already derived an explicit expression for our first-order approximation (3.17), it is time to demonstrate that it converges in the limit $\varepsilon \to 0$.

Throughout this subsection we will assume that the generic function $\psi(t, v)$ in the small vol of vol model (3.12) is a constant function, $\psi(t, v) \equiv \theta > 0$. In other words, we will be addressing the case where the approximations belong to the class of 2-hypergeometric models, which is clearly the natural choice for the approximations because, unlike in the vanilla option framework of Privault and She [2], the choice of another function $\psi(t, v)$ is not more analytically convenient for the computation of the first-order term of the barrier option price. (Without loss of generality we will additionally take $\theta = 1$.)

The technique we will use in the proof is inspired by the proof of Theorem 4 of Kato et al. [15]. In the context of their regular perturbation method for barrier option pricing (which we outlined in Subsection 2.4.2), these authors were able to derive an upper bound of the type

$$\left|\hat{f}^{\varepsilon}(t,x,v) - \left(\hat{f}_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v)\right)\right| \leq C\big(1 + |(x,v)|^{2m}\big)\varepsilon^2, \qquad t \in [0,T],\ v \in \mathbb{R},\ x \geq \hat{h}(t,v).$$

The fundamental step of their method consisted in associating, by virtue of the Feynman-Kac theorem, a stochastic representation formula to the remainder term of the first-order expansion.

To obtain this kind of estimate which grows polynomially on the two space variables, Kato et al. [15] make use of polynomial growth assumptions on some of the parameters of their problem. In our framework, it will be necessary to replace the polynomial growth assumptions on v by exponential growth assumptions, and this will naturally lead us to an upper bound for the remainder term which grows exponentially in v. In other words, it is unsurprising that our convergence result will turn out to be

$$\left|\hat{f}^{\varepsilon}(t,x,v) - \left(\hat{f}_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v)\right)\right| \leq C\left(1 + |x|^{2m} + e^{2mv}\right)\varepsilon^2, \qquad t \in [0,T],\ v \in \mathbb{R},\ x \geq \hat{h}(t,v).$$

(Clearly, this upper bound is enough for assuring that the $o(\varepsilon)$ convergence in (3.17) holds true.)

Let us start by looking into the PDE problem which is satisfied by the remainder term of the first-order approximation. For $\varepsilon > 0$, we define the remainder term as

$$\hat{f}_2^{\varepsilon}(t,x,v) := \frac{1}{\varepsilon^2}\left[\hat{f}^{\varepsilon}(t,x,v) - \left(\hat{f}_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v)\right)\right]. \tag{3.42}$$

Our claim is that $\hat{f}_2^{\varepsilon}$ satisfies the terminal and boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x,v) + \mathcal{L}^{\varepsilon}u(t,x,v) &= -g_2^{\varepsilon}(t,x,v), && t \in [0,T],\ x > \hat{h}(t,v) \\ u(T,x,v) &= 0, && x > \hat{h}(T,v) \\ u(t,\hat{h}(t,v),v) &= 0, && t \in [0,T]. \end{aligned} \tag{3.43}$$

where $\mathcal{L}^{\varepsilon}$ is the partial differential operator from (3.15), and the nonhomogeneity term is the function

$$g_2^{\varepsilon}(t,x,v) := \mathcal{L}_2 \hat{f}_0(t,x,v) + \mathcal{L}_1 \hat{f}_1(t,x,v) + \varepsilon \mathcal{L}_2 \hat{f}_1(t,x,v). \tag{3.44}$$

This claim is easily seen to be true by recalling that the functions $\hat{f}^{\varepsilon}$, $\hat{f}_0$ and $\hat{f}_1$ are the unique solutions of the terminal and boundary value problems (3.14), (3.18) and (3.27), respectively. It should be noticed that the nonhomogeneity term is given by $-g_2^{\varepsilon}(t,x,v)$ precisely because we have defined g_2^{ε} as the sum of the terms which do not cancel out when we take the difference of the PDEs satisfied by the functions $\hat{f}^\varepsilon$, $\hat{f}_0$ and $\hat{f}_1$.

Next, we use a stochastic representation formula to define a candidate solution $\widetilde{f}_2^\varepsilon$ for the PDE problem (3.43):

$$\widetilde{f}_2^\varepsilon(t,x,v) := \mathbb{E}\left[\int_t^{T\wedge\tau_{\hat{h}}^{t,\varepsilon}} e^{-r(u-t)} g_2^\varepsilon(u, S_u^\varepsilon, V_u^\varepsilon)\, du \,\middle|\, S_t^\varepsilon = x, V_t^\varepsilon = v\right] \tag{3.45}$$

where $\tau_{\hat{h}}^{t,\varepsilon} := \inf\{u \geq t : S_u^\varepsilon \leq \hat{h}(u, V_u^\varepsilon)\}$. We emphasize that the process $(S^\varepsilon, V^\varepsilon)$ underlying this stochastic representation formula follows the 2-hypergeometric model (3.12) with $\varepsilon > 0$; in particular, here V_t^ε is not a deterministic function of time.

We intend to establish a growth estimate for our candidate solution $\widetilde{f}_2^\varepsilon$. As a preliminary step, let us first obtain an upper bound for the growth of the function g_2^ε defined in (3.44):

Lemma 3.9. *Assume that $K \geq H_1$ and that $\psi(t,v) \equiv 1$. Then, the function g_2^ε satisfies the following growth condition: for any $\varepsilon \geq 0$, there exist constants C, $k > 0$ such that*

$$|g_2^\varepsilon(t,x,v)| \leq C\left(1 + |x|^{2k} + e^{2kv}\right), \qquad t \in [0,T],\ v \in \mathbb{R},\ x \geq \hat{h}(t,v). \tag{3.46}$$

Proof. This estimate follows from the inequalities (B.43), (B.53) and (B.58), whose proof is given in Appendix B. □

The next lemma provides the tool for transforming our growth estimate for g_2^ε into a growth estimate for the candidate solution $\widetilde{f}_2^\varepsilon$:

Lemma 3.10. *Let $(S^\varepsilon, V^\varepsilon)$ be the diffusion process with dynamics* (3.12) *and assume that $\psi(t,v) \equiv 1$. Then, for any $\varepsilon \geq 0$, there exist constants C, $m > 0$ (which may depend on k) such that*

$$\mathbb{E}\left[\sup_{t\leq u\leq T}\left(\left|S_u^\varepsilon\right|^{2k} + e^{2kV_u^\varepsilon}\right) \,\middle|\, S_t^\varepsilon = x, V_t^\varepsilon = v\right] \leq C\big(1 + |x|^{2m} + e^{2mv}\big), \qquad t \in [0,T],\ x > 0,\ v \in \mathbb{R}.$$

Proof. We first turn our attention to the log-volatility process V^ε. If we let $Z_t^\varepsilon = e^{2V_t^\varepsilon}$, then Itô's formula applied to the log-volatility process in (3.12) shows that the dynamics of the two-dimensional diffusion process $(S^\varepsilon, Z^\varepsilon)$ are

$$\begin{aligned} dS_t^\varepsilon &= rS_t^\varepsilon\, dt + S_t^\varepsilon \sqrt{Z_t^\varepsilon}\, dW_t^1 \\ dZ_t^\varepsilon &= \big(2(a+\varepsilon^2)Z_t^\varepsilon - c\,(Z_t^\varepsilon)^2\big)\, dt + 2\varepsilon\, Z_t^\varepsilon\, dW_t^*. \end{aligned} \tag{3.47}$$

If the coefficients of the diffusion process Z^ε satisfied the usual linear growth assumption, then we could resort to standard theorems on estimates of the moments of solutions of SDEs (e.g. Corollary 2.5.12 of Krylov [23]) to argue that

$$\mathbb{E}\left[\sup_{t\leq u\leq T}\left|Z_u^\varepsilon\right|^k \,\middle|\, Z_t^\varepsilon = z\right] \leq C\big(1 + z^k\big)$$

where $z = e^{2v}$. The quadratic growth of the drift coefficient of Z^ε obviously hinders a direct application of this argument but, as we will see, it is still possible to take advantage of this classical result in an indirect fashion.

Observe that the term $-c(Z_t^\varepsilon)^2 dt$ in the expression for dZ_t has a negative effect on the value of the drift, because $c > 0$. In these conditions, Theorem IX.3.7 of Revuz and Yor [25] assures the validity of the following rather intuitive statement: the process $\widetilde{Z}^\varepsilon$ defined by the SDE

$$d\widetilde{Z}_t^\varepsilon = 2(a+\varepsilon^2)\,\widetilde{Z}_t^\varepsilon\,dt + 2\varepsilon\,\widetilde{Z}_t^\varepsilon\,dW_t^*$$

— whose volatility coefficient is equal to that of Z^ε and whose drift coefficient function $\widetilde{b}_2(t,z) = 2(a+\varepsilon^2)z$ is always greater than or equal to the drift coefficient function $b_2(t,z) = 2(a+\varepsilon^2)z - cz^2$ of the process Z^ε — is almost surely greater than the process Z^ε, that is, for any $t \in [0,T]$ and $z > 0$,

$$\mathbb{Q}\Big[0 \le Z_u^\varepsilon \le \widetilde{Z}_u^\varepsilon \text{ for all } u \in [t,T] \;\Big|\; Z_t^\varepsilon = \widetilde{Z}_t^\varepsilon = z\Big] = 1.$$

(Recall that throughout all this chapter we are working under the risk-neutral probability $\mathbb{Q}$. The positivity of the process $\widetilde{Z}^\varepsilon$ follows from the fact that it is a geometric Brownian motion.)

Unlike Z^ε, the process $\widetilde{Z}^\varepsilon$ satisfies the assumption on the linear growth of the coefficients, so Corollary 2.5.12 of Krylov [23] allows us to conclude that, for $z > 0$,

$$\mathbb{E}\Big[\sup_{t\le u\le T} \big|Z_u^\varepsilon\big|^k \;\Big|\; Z_t^\varepsilon = z\Big] \le \mathbb{E}\Big[\sup_{t\le u\le T} \big|\widetilde{Z}_u^\varepsilon\big|^k \;\Big|\; \widetilde{Z}_t^\varepsilon = z\Big] \le C(1+z^k). \tag{3.48}$$

Now we will derive the upper bound for the price process S^ε. As pointed out in Subsection 2.2.1 of Da Fonseca and Martini's paper [1], conditionally on $x = S_t^\varepsilon$, the process S^ε can be written in closed form as

$$S_u^\varepsilon = x\exp\bigg(r(u-t) - \frac{1}{2}\int_t^u e^{2V_s^\varepsilon}\,ds + \int_t^u e^{V_s^\varepsilon}\,dW_s^1\bigg) = xe^{r(u-t)}M_u^\varepsilon, \qquad u \in [t,T]$$

where $M_u^\varepsilon := \exp\Big(-\frac{1}{2}\int_t^u e^{2V_s^\varepsilon}\,ds + \int_t^u e^{V_s^\varepsilon}\,dW_s^1\Big)$ is a uniformly integrable martingale. Hence

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\le u\le T}\big|S_u^\varepsilon\big|^{2k} \;\Big|\; S_t^\varepsilon = x, V_t^\varepsilon = v\Big] &\le |x|^{2k}e^{2k|r|T}\,\mathbb{E}\Big[\sup_{t\le u\le T}\big|M_u^\varepsilon\big|^{2k} \;\Big|\; S_t^\varepsilon = x, V_t^\varepsilon = v\Big] \\
&\le C_1|x|^{2k}\,\mathbb{E}\Big[\big|M_T^\varepsilon\big|^{2k} \;\Big|\; S_t^\varepsilon = x, V_t^\varepsilon = v\Big] \\
&\le C_1|x|^{2k}\bigg(\mathbb{E}\bigg[\exp\bigg\{C_2\int_t^T e^{2V_u^\varepsilon}\,du\bigg\} \;\bigg|\; S_t^\varepsilon = x, V_t^\varepsilon = v\bigg]\bigg)^{C_3} \\
&\le C_4\,|x|^{2k}(1+e^{2C_5 v})^{C_6} \\
&\le C\big(1+|x|^{2m}+e^{2mv}\big)
\end{aligned} \tag{3.49}$$

where the constants C_i, C and m do not depend on x nor on v. In the second inequality we have used the well-known Doob's inequality (see e.g. Theorem 3.12 of Seppäläinen [26]); the third inequality uses Theorem 1 of Grigelionis and Mackevičius [27], which is a result on the moments of a stochastic exponential; and the fourth inequality follows from a straightforward adaptation of the estimation procedure from page 13 of Da Fonseca and Martini [1].

Combining (3.48) and (3.49), the lemma follows. $\square$

Let us now use the results from Lemmas 3.9 and 3.10 to derive our desired upper bound on the growth of the function $\widetilde{f}_2^\varepsilon$ defined in (3.45): for any $\varepsilon \ge 0$, there exist constants C, $m > 0$ which do not depend on (t,x,v) such that

$$
\begin{aligned}
|\widetilde{f}_2^\varepsilon(t,x,v)| &\le \int_t^T e^{-r(u-t)}\mathbb{E}\left[|g_2^\varepsilon(u,S_u^\varepsilon,V_u^\varepsilon)|\mathbb{1}_{\{S_u^\varepsilon\ge\hat{h}(u,V_u^\varepsilon)\}}\Big|S_t^\varepsilon=x,V_t^\varepsilon=v\right]\,du \\
&\le C_1e^{|r|T}\int_t^T\left(1+\mathbb{E}\left[(S_u^\varepsilon)^{2k}+e^{2kV_u^\varepsilon}\Big|S_t^\varepsilon=x,V_t^\varepsilon=v\right]\right)du \\
&\le C_2\left(1+\mathbb{E}\left[\sup_{t\le u\le T}\left(\left|S_u^\varepsilon\right|^{2k}+e^{2kV_u^\varepsilon}\right)\Big|S_t^\varepsilon=x,V_t^\varepsilon=v\right]\right) \\
&\le C\big(1+|x|^{2m}+e^{2mv}\big)
\end{aligned}
\tag{3.50}
$$

for all $t\in[0,T]$, $x>0$, $v\in\mathbb{R}$. It is important here that the estimate is valid for all $\varepsilon\ge0$ and not only for all $\varepsilon>0$, because that means that we can actually take C and m to be independent of $\varepsilon\in[0,1]$, which is essential for assuring the convergence when $\varepsilon\to0$.

Now that we have established this upper bound, the only thing that remains to be proved is that the function $\widetilde{f}_2^\varepsilon$, which we defined as a candidate solution for the terminal and boundary value problem (3.43), is indeed the unique solution of (3.43). In fact, if we prove this, then it will follow that $\widetilde{f}_2^\varepsilon$ equals the remainder term $\hat{f}_2^\varepsilon$ defined in (3.42), and the estimate (3.50) for $|\hat{f}_2^\varepsilon(t,x,v)|=|\widetilde{f}_2^\varepsilon(t,x,v)|$ will assure the convergence of the first-order expansion.

Lemma 3.11. *Assume that $K\ge H_1$ and $\psi(t,v)\equiv1$, and fix $\varepsilon>0$. Then, the function $\widetilde{f}_2^\varepsilon(t,x,v)$ defined in* (3.45) *is the unique solution of the terminal and boundary value problem* (3.43)*.*

Proof. Like in the proof of Lemma 3.5, here we intend to obtain the result through the Feynman-Kac theorem for parabolic Cauchy-Dirichlet problems (Theorem 2.8).

Let us first perform the change of variable $z=e^{2v}$ and $y=x-\hat{h}(t,v)$, which will reduce our problem to one where we need not worry neither about the exponential growth in the variable v nor about the dependence of the barrier function on time and on the volatility. If we define

$$u^*(t,y,z):=u\big(t,y+\hat{h}(t,\tfrac12\log z),\tfrac12\log z\big)$$

then $u(t,x,v)$ satisfies (3.43) if and only if $u^*(t,y,z)$ satisfies

$$
\begin{aligned}
\frac{\partial u^*}{\partial t}(t,y,z)+\mathcal{L}^{*,\varepsilon}u^*(t,y,z)&=-g_2^\varepsilon\big(t,y+\hat{h}(t,\tfrac12\log z),\tfrac12\log z\big), && t\in[0,T],\ y>0 \\
u^*(T,y,z)&=0, && y>0 \\
u^*(t,0,z)&=0, && t\in[0,T].
\end{aligned}
\tag{3.51}
$$

where, for $v=\frac12\log z$,

$$
\begin{aligned}
\mathcal{L}^{*,\varepsilon}&=-\frac{\partial\hat{h}}{\partial t}(t,v)\frac{\partial}{\partial y}+\left(a-\frac{c}{2}z\right)\left[2z\frac{\partial}{\partial z}-\frac{\partial\hat{h}}{\partial v}(t,v)\frac{\partial}{\partial y}\right]+\frac12\big(y+\hat{h}(t,v)\big)^2z\frac{\partial^2}{\partial y^2}+r\big(y+\hat{h}(t,v)\big)\frac{\partial}{\partial y}-r\,\mathrm{Id} \\
&\quad+\varepsilon\rho\big(y+\hat{h}(t,v)\big)\sqrt{z}\left[2z\frac{\partial}{\partial y\partial z}-\frac{\partial\hat{h}}{\partial v}(t,v)\frac{\partial^2}{\partial y^2}\right] \\
&\quad+\varepsilon^2\frac12\left[-\frac{\partial^2\hat{h}}{\partial v^2}(t,v)\frac{\partial}{\partial y}+4z\frac{\partial}{\partial z}+\left(\frac{\partial\hat{h}}{\partial v}(t,v)\right)^2\frac{\partial^2}{\partial y^2}-4z\frac{\partial\hat{h}}{\partial v}(t,v)\frac{\partial^2}{\partial y\partial z}+4z^2\frac{\partial^2}{\partial z^2}\right] \\
&=\left[r\big(y+\hat{h}(t,v)\big)-\frac{\partial\hat{h}}{\partial t}(t,v)-\left(a-\frac{c}{2}z\right)\frac{\partial\hat{h}}{\partial t}(t,v)-\frac{\varepsilon^2}{2}\frac{\partial^2\hat{h}}{\partial v^2}(t,v)\right]\frac{\partial}{\partial y}+\big(2\,(a+\varepsilon)\,z-c\,z^2\big)\frac{\partial}{\partial z}
\end{aligned}
$$

$$+\left[\frac{1}{2}\big(y+\hat{h}(t,v)\big)^2 z-\varepsilon\rho\big(y+\hat{h}(t,v)\big)\sqrt{z}\frac{\partial\hat{h}}{\partial v}(t,v)+\frac{\varepsilon^2}{2}\left(\frac{\partial\hat{h}}{\partial v}(t,v)\right)^2\right]\frac{\partial^2}{\partial y^2}$$

$$+\left[2\varepsilon\rho\big(y+\hat{h}(t,v)\big)z^{3/2}-2\varepsilon^2 z\frac{\partial\hat{h}}{\partial v}(t,v)\right]\frac{\partial}{\partial y\partial z}+2\varepsilon^2 z^2\frac{\partial^2}{\partial z^2}.$$

Correspondingly, we write

$$\begin{aligned}\widetilde{f}_2^{*,\varepsilon}(t,y,z) &:= \widetilde{f}_2^{\varepsilon}\big(t,y+\hat{h}(t,\tfrac{1}{2}\log z),\tfrac{1}{2}\log z\big)=\\ &=\mathbb{E}\left[\int_t^{T\wedge\tau_{\hat{h}}^{t,\varepsilon}} e^{-r(u-t)}g_2^{\varepsilon}(u,S_u^{\varepsilon},V_u^{\varepsilon})\,du\middle|S_t^{\varepsilon}=y+\hat{h}(t,\tfrac{1}{2}\log z),V_t^{\varepsilon}=\tfrac{1}{2}\log z\right]\\ &=\mathbb{E}\left[\int_t^{T\wedge\tau_{\hat{h}}^{t,\varepsilon}} e^{-r(u-t)}g_2^{\varepsilon}\big(u,Y_u^{\varepsilon}+\hat{h}(t,\tfrac{1}{2}\log Z_u^{\varepsilon}),\tfrac{1}{2}\log Z_u^{\varepsilon}\big)\,du\middle|Y_t^{\varepsilon}=y,Z_t^{\varepsilon}=z\right].\end{aligned}$$

where the process $(Y^{\varepsilon},Z^{\varepsilon})$ is defined as $Y_t^{\varepsilon}:=S_t^{\varepsilon}-\hat{h}(t,V_t^{\varepsilon})$, $Z_t^{\varepsilon}:=e^{2V_t^{\varepsilon}}$, and the stopping time becomes $\tau_{\hat{h}}^{t,\varepsilon}=\inf\{u\geq t: Y_u^{\varepsilon}\leq 0\}$. We note that, by Itô's formula, the original dynamics (3.12) are transformed into

$$\begin{aligned}dY_t^{\varepsilon}&=\left[rS_t^{\varepsilon}-\frac{\partial\hat{h}(t,V_t^{\varepsilon})}{\partial t}-\left(a-\frac{c}{2}Z_t^{\varepsilon}\right)\frac{\partial\hat{h}(t,V_t^{\varepsilon})}{\partial v}-\frac{\varepsilon^2}{2}\frac{\partial^2\hat{h}(t,V_t^{\varepsilon})}{\partial v^2}\right]dt+S_t^{\varepsilon}\sqrt{Z_t^{\varepsilon}}dW_t^1-\varepsilon\frac{\partial\hat{h}(t,V_t^{\varepsilon})}{\partial v}dW_t^*\\ dZ_t^{\varepsilon}&=\big(2(a+\varepsilon^2)Z_t^{\varepsilon}-c\,(Z_t^{\varepsilon})^2\big)\,dt+2\varepsilon\,Z_t^{\varepsilon}\,dW_t^*.\end{aligned}\tag{3.52}$$

(It is easy to check, by performing the usual computations, that the infinitesimal generator of the diffusion process $(Y^{\varepsilon},Z^{\varepsilon})$ is indeed given by the operator $\mathcal{L}^{*,\varepsilon}$.)

So let us turn to the proof of the equivalent statement that the stochastic representation formula $\widetilde{f}_2^{*,\varepsilon}$ is the unique (classical) solution of the boundary value problem (3.51). We now verify the assumptions of Theorem 2.8:

- *Assumption on the domain.* The domain for the space variables is $D=(0,\infty)^2$, which has all the required properties.
- *Assumptions (i), (v), (vii) and (viii).* It can be immediately checked that these assumptions hold.
- *Assumption (iii).* The matrix of the second-order coefficients is

$$a^*(t,y,z,\varepsilon)=\begin{pmatrix} x^2z-2\varepsilon\rho x\sqrt{z}\frac{\partial\hat{h}(t,v)}{\partial v}+\varepsilon^2\left(\frac{\partial\hat{h}(t,v)}{\partial v}\right)^2 & 2\varepsilon\rho x z^{3/2}-2\varepsilon^2 z\frac{\partial\hat{h}(t,v)}{\partial v}\\ 2\varepsilon\rho x z^{3/2}-2\varepsilon^2 z\frac{\partial\hat{h}(t,v)}{\partial v} & 4\varepsilon^2 z^2\end{pmatrix}$$

  (where $v=\frac{1}{2}\log z$ and $x=y+\hat{h}(t,v)$), and its eigenvalues are given by

$$\begin{aligned}\frac{1}{2}\Bigg[&\Big(\frac{\partial\hat{h}(t,v)}{\partial v}\Big)^2\varepsilon^2-2\rho\varepsilon\frac{\partial\hat{h}(t,v)}{\partial v}x\sqrt{z}+x^2z+4\varepsilon^2z^2\\ &\pm\sqrt{\left(\Big(\frac{\partial\hat{h}(t,v)}{\partial v}\Big)^2\varepsilon^2-2\rho\varepsilon\frac{\partial\hat{h}(t,v)}{\partial v}x\sqrt{z}+x^2z+4\varepsilon^2z^2\right)^2-16\varepsilon^2(1-\rho^2)x^2z^3}\,\Bigg].\end{aligned}$$

  These are bounded away from zero in any bounded open set $B\subset\overline{D}$ because z is strictly positive (since the open set B does not include the boundary $z=0$), as well as x (since $y=x-\hat{h}(t,v)>0$ on B). From the well-known alternative formulation of the ellipticity condition, it follows that assumption (iii) holds in any open set $B\subset\overline{D}$.

- *Assumption (iv).* As shown in the computations in Sections B.3, B.4 and B.5 of Appendix B, the functions $\mathcal{L}_2\hat{f}_0$, $\mathcal{L}_1\hat{f}_1$ and $\mathcal{L}_2\hat{f}_1$ have no singularities in the set $\{(t,x,v) : t \in [0,T], x \geq \hat{h}(t,v)\}$. Therefore, the nonhomogeneity term $h(t,y,z) \equiv -g_2^\varepsilon\big(t, y + \hat{h}(t, \frac{1}{2}\log z), \frac{1}{2}\log z\big)$ satisfies the required continuity property.

- *Assumption (vi).* By Lemma 3.9,

$$|h(t,y,z)| \leq C_1\Big(1 + \big|y + \hat{h}(t, \tfrac{1}{2}\log z)\big|^{2k_1} + |z|^{k_1}\Big) \leq C\big(1 + |y|^k + |z|^k\big), \qquad t \in [0,T], y \geq 0, z \in \mathbb{R}$$

  where we have used the estimate (B.14) from Appendix B in the second inequality. Thus assumption (vi) holds true.

The only hypothesis of Theorem 2.8 which our problem does not satisfy is in fact the assumption (ii) on the linear growth of the coefficients of the diffusion process. But we can show that the theorem nevertheless holds through an adaptation of the proof of Theorem 2.8 given in Rubio [9] — and the adaptation here is simpler than in the proof of Lemma 3.5 because the ellipticity condition does not fail. The changes which should now be considered in the proofs presented in [9] are the following:

- *Proposition 2.5 of [9].* As shown in Subsections 2.1.1 and 2.2.1 of Da Fonseca and Martini [1], the unique solution of the diffusion equations (3.47) with initial conditions $S_t^\varepsilon = x$ and $Z_t^\varepsilon = z$ is

$$Z_u^\varepsilon = \frac{z\exp\{2a(u-t) + 2\varepsilon(W_u^* - W_t^*)\}}{1 + cz\int_t^u \exp\{2as + \varepsilon W_s^*\}\,ds}, \qquad S_u^\varepsilon = x\exp\bigg(r(u-t) - \frac{1}{2}\int_t^u Z_s ds + \int_t^u \sqrt{Z_s} dW_s^1\bigg).$$

  Properties (i), (ii) and (iii) are an immediate consequence of these closed-form expressions and the definition $Y_t^\varepsilon = S_t^\varepsilon - \hat{h}(t, V_t^\varepsilon)$. As for (iv), by Lemma 3.10 and the estimate (B.14) from Appendix B we have

$$\begin{aligned}
\mathbb{E}\bigg[\sup_{t\leq u\leq T}\Big(\big|Y_u^\varepsilon\big|^{2r} + \big|Z_u^\varepsilon\big|^{2r}\Big)\,\bigg|\, Y_t^\varepsilon = y, Z_t^\varepsilon = z\bigg]& \\
\leq \mathbb{E}\bigg[\sup_{t\leq u\leq T}\Big(\big|S_u^\varepsilon\big|^{2r} + \big|Z_u^\varepsilon\big|^{2r}\Big)\,\bigg|\, S_t^\varepsilon = y + \hat{h}\big(t, \tfrac{1}{2}\log z\big), Z_t^\varepsilon = z\bigg]& \\
\leq C_1\Big(1 + \big|y + \hat{h}(t, \tfrac{1}{2}\log z)\big|^{2k} + |z|^{2k}\Big) \leq C\big(1 + |y|^{2k} + |z|^{2k}\big)&
\end{aligned}$$

  for all $t \in [0,T]$, $y > 0$ and $z > 0$, where the constants C and k depend on r. This weaker form of property (iv) will be enough for our purposes.

- *Lemma 3.3, Lemma 3.4 and Theorem 4.1 of [9].* The proofs of these results only use the linear growth assumption via the properties (i)–(iv) in Proposition 2.5 of [9]. As all the arguments are also valid if property (iv) is replaced by our weaker version, the proofs can be carried out in the same manner.

- *Theorem 5.4 of [9].* The result which is invoked in the proof given in [9] depends on the linear growth condition, but it is possible to reach the same conclusion by invoking a different result which does not depend on that assumption. Theorem 10.6.4 and Remark 10.6.5 of Kuo [28], which are also valid in a multidimensional setting (see the comment in page 208 of [28]), ensure that if $(t_n, y_n, z_n) \to (t,y,z)$ and $(Y^{n,\varepsilon}, Z^{n,\varepsilon})$, $(Y^\varepsilon, Z^\varepsilon)$ are the solutions of the SDEs (3.52) with initial conditions $(Y^{n,\varepsilon}_{t_n}, Z^{n,\varepsilon}_{t_n}) = (y_n, z_n)$, $(Y^{\varepsilon}_t, Z^{\varepsilon}_t) = (y, z)$ respectively, then there is mean-square convergence in the sense that

$$\mathbb{E}\left[\left(\sup_{0\le s\le T}\left|\left(Y^{n,\varepsilon}_{t_n+s}, Z^{n,\varepsilon}_{t_n+s}\right) - \left(Y^{\varepsilon}_{t+s}, Z^{\varepsilon}_{t+s}\right)\right|\right)^2\right] \xrightarrow[n\to\infty]{} 0.$$

Since mean-squared convergence implies convergence in probability, the required property follows.

We note that even though the invoked results from Kuo [28] are formulated for the case where t_n is the same for all equations, these results are easily extended to the case where the initial time also differs by adding an extra dimension to the diffusion process, as in Equation (2.3) of Rubio [9].

The other auxiliary results in [9] do not require the linear growth assumption (ii), so they are valid in our setting. It follows that the conclusion of Theorem 2.8 holds for our problem, and the lemma is proved. □

We have completed the proof of the following convergence theorem:

Theorem 3.12. *Let $\hat{f}_0(t,x,v)$ and $\hat{f}_1(t,x,v)$ be, respectively, the zero and first-order term in the first-order expansion* (3.17) *for the price $\hat{f}^{\varepsilon}(t,x,v)$ of a DOC option with barrier function $\hat{h}(t,v)$ under the model* (3.12). *Assume that $K \ge H_1$ and $\psi(t,v) \equiv 1$. Then, there exist positive constants C and m which are independent of $\varepsilon \in [0,1]$ such that*

$$\left|\hat{f}^{\varepsilon}(t,x,v) - \left(\hat{f}_0(t,x,v) + \varepsilon \hat{f}_1(t,x,v)\right)\right| \le C\left(1 + |x|^{2m} + e^{2mv}\right)\varepsilon^2, \qquad t\in[0,T],\ v\in\mathbb{R},\ x \ge \hat{h}(t,v).$$

3.2.7 The multi-stage method

Aiming towards a better approximation between the time and volatility-dependent barrier and a constant barrier H, we now want to generalize our asymptotic pricing technique to the case of a piecewise-smooth barrier function which is of the form (3.13) in each subinterval of time.

Specifically, in analogy with Section 3 of Dorfleitner et al. [20], we now subdivide the interval $[t', T]$ into n subintervals defined by $t' = T_0 < T_1 < \ldots < T_n = T$ and consider the continuous barrier function defined by

$$\hat{h}^{(n)}(t,v) := H_1 \exp\left\{-r(T-t) + \sum_{i=1}^{n} \frac{1+2\beta_i}{2} \mathbb{1}_{\{t<T_i\}} \gamma^2(t \vee T_{i-1}, T_i, V^{t,v}_{t\vee T_{i-1}})\right\}, \qquad (t,v) \in [t', T]\times\mathbb{R} \tag{3.53}$$

which is piecewise of the form (3.13) in the sense that

$$\hat{h}^{(n)}(t,v) := \hat{h}^{(n)}(T_i, V^{t,v}_{T_i}) \exp\left\{-r(T_i - t) + \frac{1+2\beta_i}{2}\gamma^2(t, T_i, v)\right\}, \qquad (t,v)\in[T_{i-1}, T_i]\times\mathbb{R}.$$

In particular, if we set $\beta_i = \beta$ for all $i = 1,\ldots,n$ we obtain (3.13). But, of course, the idea here is to pick $\beta_1,\ldots,\beta_n$ so that $\hat{h}^{(n)}(t,v)$ is closer to the constant barrier H than the single-stage barrier function $\hat{h}(t,v)$: our choice of β_i should ensure that the barrier function is as constant as possible in the interval $[T_{i-1}, T_i]$. Analogously to Subsection 3.2.3, the simplest choice is $H_1 = H$ and $\beta_i = \frac{r(T_i - T_{i-1})}{\gamma^2(T_{i-1}, T_i, V^{t',v'}_{T_{i-1}})} - \frac{1}{2}$. Concerning the choice of n (and in particular the choice between the single and the multi-stage methods), one should try to find a good compromise between computational speed and numerical accuracy, depending on the practical problem at hand. As stated in Subsection 3.2.3, for small maturities it suffices to take $n = 1$, whereas for higher maturities a greater number of stages may be necessary.

As we show below, a stepwise PDE approach in the spirit of Lo et al. [21] and Dorfleitner et al. [20] enables us to derive an explicit formula for the zero and first-order terms of the asymptotic expansion of the price of the option with barrier function (3.53).

The exact price $\hat{f}^{(n)}(t,x,v)$ of the option with piecewise-smooth barrier function (3.53), which is the solution of (3.14) with $\hat{h}(t,v)$ replaced by $\hat{h}^{(n)}(t,v)$, can (at least formally) be expanded as

$$\hat{f}^{(n)}(t,x,v) = \hat{f}_0^{(n)}(t,x,v) + \varepsilon \hat{f}_1^{(n)}(t,x,v) + o(\varepsilon)$$

where the functions $\hat{f}_0^{(n)}$ and $\hat{f}_1^{(n)}$ satisfy (3.16). (Naturally, the nonconstant boundary conditions are now $\hat{f}_j^{(n)}(t,\hat{h}^{(n)}(t,v),v) = 0$ for $j = 0,1$.)

The same argument from the single-stage framework shows that for our fixed initial time t' and initial log-volatility v', the zero-order term is again the price, under the same Black and Scholes model, of a DOC option whose time-dependent barrier function is $\hat{h}^{(n)}(t, V_t^{t',v'})$; that is, $\hat{f}_0^{(n)}(t,x,V_t^{t',v'})$ is the solution of

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\mathrm{bs}}^{t',v'} u(t,x) &= 0, & t \in [t',T],\ x > \hat{h}^{(n)}(t, V_t^{t',v'}) \\ u(T,x) &= (x-K)^+, & x > \hat{h}^{(n)}(T, V_T^{t',v'}) \\ u(t, \hat{h}^{(n)}(t, V_t^{t',v'})) &= 0, & t \in [t',T]. \end{aligned}$$

Since the time-dependent barrier function $\hat{h}^{(n)}(t, V_t^{t',v'})$ is of the form (3.24) in each subinterval $[T_{i-1}, T_i]$, we can use the following multi-stage procedure to obtain the closed-form expression for the zero-order term $\hat{f}_0^{(n)}(t',x,v')$:

1. We compute $\hat{f}_0^{(n)}(T_{n-1}, x, V_{T_{n-1}}^{t',v'})$ via the closed-form expression (3.25), where $\hat{h}(t,v)$ becomes $\hat{h}^{(n)}(T_{n-1}, V_{T_{n-1}}^{t',v'})$ and β is replaced by β_n.
2. For $i = n-2, \ldots, 0$, we use $\hat{f}_0^{(n)}(T_{i+1}, x, V_{T_{i+1}}^{t',v'})$ as the terminal condition for the PDE problem in the interval $[T_i, T_{i+1}]$, and we compute the explicit closed-form expression for $\hat{f}_0^{(n)}(T_i, x, V_{T_i}^{t',v'})$ using the integral representation formula (A11) in Theorem 1 of Dorfleitner et al. [20].

In order to illustrate that the integrals from the representation formula of Theorem 1 of Dorfleitner et al. [20] can be computed in closed form, let us focus on the case $n = 2$, where the integral representation for the zero-order term is

$$\begin{aligned} \hat{f}_0^{(2)}(t',x,v') = e^{-r(T_1-t')} & \int_{\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})}^{\infty} \frac{1}{\sqrt{2\pi}\gamma(t',T_1,v')z} \times \\ & \times \exp\left\{ -\frac{1}{2\gamma^2(t',T_1,v')} \left(\log\left(\frac{z}{x}\right) - r(T_1-t') + \frac{1}{2}\gamma^2(t',T_1,v') \right)^2 \right\} \\ & \times \left[1 - \exp\left\{ -\frac{2}{\gamma^2(t',T_1,v')} \log\left(\frac{\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})}{z} \right) \log\left(\frac{\hat{h}^{(2)}(t',v')}{x} \right) \right\} \right] \end{aligned}$$

$$
\times \left[z\mathcal{N}\big(d_1(T_1, z, V_{T_1}^{t',v'})\big) - Ke^{-r(T-T_1)}\mathcal{N}\big(d_2(T_1, z, V_{T_1}^{t',v'})\big) + \right.
$$

$$
\left. + \left(\frac{\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})}{z} \right)^{2+2\beta_2} \left(-z\mathcal{N}\big(d_3(T_1, z, V_{T_1}^{t',v'})\big) + \frac{Ke^{-r(T-T_1)}z^2}{\big(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\big)^2} \mathcal{N}\big(d_4(T_1, z, V_{T_1}^{t',v'})\big) \right) \right] dz
$$

$$
= e^{-r(T_1-t')} \int_{\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)}^{\infty} \frac{1}{\sqrt{2\pi}\gamma(t', T_1, v')} \times
$$

$$
\times \exp\left\{ -\frac{1}{2\gamma^2(t', T_1, v')} \left(w - \log x - r(T_1 - t') + \frac{1}{2}\gamma^2(t', T_1, v') \right)^2 \right\}
$$

$$
\times \left[1 - \exp\left\{ -\frac{2}{\gamma^2(t', T_1, v')} \left(w - \log\big(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\big) \right) \log\left(\frac{x}{\hat{h}^{(2)}(t', v')} \right) \right\} \right] \times
$$

$$
\times \left[e^w\mathcal{N}\big(d_1(T_1, e^w, V_{T_1}^{t',v'})\big) - Ke^{-r(T-T_1)}\mathcal{N}\big(d_2(T_1, e^w, V_{T_1}^{t',v'})\big) + \right.
$$

$$
\left. + \left(\frac{\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})}{e^w} \right)^{2+2\beta_2} \left(-e^w\mathcal{N}\big(d_3(T_1, e^w, V_{T_1}^{t',v'})\big) + \frac{Ke^{-r(T-T_1)}e^{2w}}{\big(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\big)^2} \mathcal{N}\big(d_4(T_1, e^w, V_{T_1}^{t',v'})\big) \right) \right] dw.
$$

The above integral can be decomposed into eight summands of the form $\int_L^\infty \exp\big(-(Aw^2 + Bw + C)\big)$ $\mathcal{N}(Dw + E)dw$, where $A > 0, B, C, D, E$ and L are parameters which do not depend on w; as shown in Appendix A.3, these can be written in closed form in terms of the bivariate normal cumulative distribution function. So, by carrying out the decomposition we obtain the following closed-form expression for the zero-order term:

$$
\hat{f}_0^{(2)}(t', x, v') = \frac{e^{-r(T_1-t')}}{\sqrt{2\pi}\gamma(t', T_1, v')} \sum_{i=1}^{8} A_{i,1}\, \Upsilon\big(A_{i,2}, A_{i,3}, A_{i,4}, A_{i,5}, A_{i,6}, A_{i,7}\big) \tag{3.54}
$$

where $\Upsilon(A, B, C, D, E, L)$ is defined in Equation (A.14) of Appendix A, and the parameters $A_{i,j}$ are defined in Table 3.1. (It can be verified numerically that this formula coincides with (3.25) in the particular case $\beta_1 = \beta_2 = \beta$, and also that it satisfies the standard monotonicity properties with respect to the barrier function.)

For higher n, a straightforward inductive argument shows that the integral representation formula for $\hat{f}_0^{(n)}(t', x, v')$ can be written in terms of the cumulative distribution function of the n-dimensional normal distribution.

Regarding the first-order term, which is defined as the solution of

$$
\begin{aligned}
\frac{\partial \hat{f}_1^{(n)}}{\partial t}(t, x, v) + \mathcal{L}_0 \hat{f}_1^{(n)}(t, x, v) &= -\mathcal{L}_1 \hat{f}_0^{(n)}(t, x, v), && t \in [t', T],\ x > \hat{h}^{(n)}(t, v) \\
\hat{f}_1^{(n)}(T, x, v) &= 0, && x > \hat{h}^{(n)}(T, v) \\
\hat{f}_1^{(n)}(t, \hat{h}^{(n)}(t, v), v) &= 0, && t \in [t', T],
\end{aligned}
$$

the strategy to obtain an explicit expression for $\hat{f}_1^{(n)}(t', x, v')$ is similar to that from the single-stage framework: first we use the Feynman-Kac theorem to write out the stochastic representation formula for $\hat{f}_1^{(n)}(t, x, v)$; then we fix $v = v'$ and appeal again to the Feynman-Kac theorem to assert that $\hat{f}_1^{(n)}(t, x, V_t^{t',v'})$ is the solution of the terminal and boundary value problem (3.29) (with $\hat{h}(t, V_t^{t',v'})$ replaced by $\hat{h}^{(n)}(t, V_t^{t',v'})$); finally, we decompose $\hat{f}_1^{(n)}(t, x, V_t^{t',v'}) = \hat{f}_1^{(A,n)}(t, x, V_t^{t',v'}) - \hat{f}_1^{(B,n)}(t, x, V_t^{t',v'})$ as in (3.30)–(3.31). We can obtain an explicit expression for $\hat{f}_1^{(A,n)}(t', x, v')$ through a multi-stage procedure:

Table 3.1: Parameters in the closed-form expression (3.54) for $\hat{f}_0^{(2)}(t', x, v')$

i	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$
1	1	$\frac{1}{2\gamma^2(t',T_1,v')}$	$-\frac{1}{\gamma^2(t',T_1,v')}\left[\log x + r(T_1 - t') + \frac{1}{2}\gamma^2(t', T_1, v')\right]$
2	-1	$\frac{1}{2\gamma^2(t',T_1,v')}$	$-\frac{1}{\gamma^2(t',T_1,v')}\left[\log\left(\frac{(\hat{h}^{(2)}(t',v'))^2}{x}\right) + r(T_1 - t') + \frac{1}{2}\gamma^2(t', T_1, v')\right]$
3	$-Ke^{-r(T-T_1)}$	$\frac{1}{2\gamma^2(t',T_1,v')}$	$-\frac{1}{\gamma^2(t',T_1,v')}\left[\log x + r(T_1 - t') - \frac{1}{2}\gamma^2(t', T_1, v')\right]$
4	$Ke^{-r(T-T_1)}$	$\frac{1}{2\gamma^2(t',T_1,v')}$	$-\frac{1}{\gamma^2(t',T_1,v')}\left[\log\left(\frac{(\hat{h}^{(2)}(t',v'))^2}{x}\right) + r(T_1 - t') - \frac{1}{2}\gamma^2(t', T_1, v')\right]$
5	$-\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^{2+2\beta_2}$	$\frac{1}{2\gamma^2(t',T_1,v')}$	$A_{3,3} + 1 + 2\beta_2$
6	$\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^{2+2\beta_2}$	$\frac{1}{2\gamma^2(t',T_1,v')}$	$A_{4,3} + 1 + 2\beta_2$
7	$\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^{2\beta_2} A_{4,1}$	$\frac{1}{2\gamma^2(t',T_1,v')}$	$A_{1,3} + 1 + 2\beta_2$
8	$-A_{7,1}$	$\frac{1}{2\gamma^2(t',T_1,v')}$	$A_{2,3} + 1 + 2\beta_2$

i	$A_{i,4}$	$A_{i,5}$
1	$\frac{1}{2\gamma^2(t',T_1,v')}\left[\log x + r(T_1 - t') - \frac{1}{2}\gamma^2(t', T_1, v')\right]^2$	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
2	$A_{1,4} + \frac{1}{2\gamma^2(t',T_1,v')}\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^2 \cdot \log\left(\frac{(\hat{h}^{(2)}(t',v'))^2}{x^2}\right)$	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
3	$\frac{1}{2\gamma^2(t',T_1,v')}\left[\log x + r(T_1 - t') - \frac{1}{2}\gamma^2(t', T_1, v')\right]^2$	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
4	$A_{3,4} + \frac{1}{2\gamma^2(t',T_1,v')}\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^2 \cdot \log\left(\frac{(\hat{h}^{(2)}(t',v'))^2}{x^2}\right)$	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
5	$\frac{1}{2\gamma^2(t',T_1,v')}\left[\log x + r(T_1 - t') - \frac{1}{2}\gamma^2(t', T_1, v')\right]^2$	$-\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
6	$A_{5,4} + \frac{1}{2\gamma^2(t',T_1,v')}\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^2 \cdot \log\left(\frac{(\hat{h}^{(2)}(t',v'))^2}{x^2}\right)$	$-\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
7	$\frac{1}{2\gamma^2(t',T_1,v')}\left[\log x + r(T_1 - t') - \frac{1}{2}\gamma^2(t', T_1, v')\right]^2$	$-\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$
8	$A_{7,4} + \frac{1}{2\gamma^2(t',T_1,v')}\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)^2 \cdot \log\left(\frac{(\hat{h}^{(2)}(t',v'))^2}{x^2}\right)$	$-\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}$

i	$A_{i,6}$	$A_{i,7}$
1	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[-\log(K \vee H_1) + r(T - T_1) + \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
2	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[-\log(K \vee H_1) + r(T - T_1) + \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
3	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[-\log(K \vee H_1) + r(T - T_1) - \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
4	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[-\log(K \vee H_1) + r(T - T_1) - \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
5	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[\log\left(\frac{(\hat{h}^{(2)}(T_1,V_{T_1}^{t',v'}))^2}{K\vee H_1}\right) + r(T - T_1) + \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
6	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[\log\left(\frac{(\hat{h}^{(2)}(T_1,V_{T_1}^{t',v'}))^2}{K\vee H_1}\right) + r(T - T_1) + \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
7	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[\log\left(\frac{(\hat{h}^{(2)}(T_1,V_{T_1}^{t',v'}))^2}{K\vee H_1}\right) + r(T - T_1) - \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$
8	$\frac{1}{\gamma(T_1,T,V_{T_1}^{t',v'})}\left[\log\left(\frac{(\hat{h}^{(2)}(T_1,V_{T_1}^{t',v'}))^2}{K\vee H_1}\right) + r(T - T_1) - \frac{1}{2}\gamma^2(T_1, T, V_{T_1}^{t',v'})\right]$	$\log\left(\hat{h}^{(2)}(T_1, V_{T_1}^{t',v'})\right)$

1. In the interval $[T_{n-1}, T]$ the function $\hat{f}_1^{(A,n)}(t, x, V_t^{t',v'})$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\rm bs}^{t',v'} u(t,x) &= -\mathcal{L}_1 \hat{f}_0^{(n)}(t, x, V_t^{t',v'}), \quad t \in [T_{n-1}, T],\ x \in \mathbb{R} \\ u(T,x) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R} \end{aligned} \tag{3.55}$$

so we can compute $\hat{f}_1^{(A,n)}(T_{n-1}, x, V_{T_{n-1}}^{t',v'})$ via the explicit expression (3.36).

2. For $i = n-2, \ldots, 0$ the function $\hat{f}_1^{(A,n)}(t, x, V_t^{t',v'})$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\rm bs}^{t',v'} u(t,x) &= -\mathcal{L}_1 \hat{f}_0^{(n)}(t, x, V_t^{t',v'}), \qquad t \in [T_i, T_{i+1}],\ x \in \mathbb{R} \\ u(T_{i+1},x) &= \hat{f}_1^{(A,n)}(T_{i+1}, x, V_{T_{i+1}}^{t',v'}), \qquad\qquad x \in \mathbb{R} \end{aligned} \tag{3.56}$$

where the terminal condition is the function which has been derived in the previous stage. Hence we can compute the explicit expression for $\hat{f}_1^{(A,n)}(T_i, x, V_{T_i}^{t',v'})$ by combining the Feynman-Kac theorem of Heath and Schweizer [8] with the known expression for the law of the process $S^{t',v'}$.

After this, the explicit expression for $\hat{f}_1^{(B,n)}(t', x, v')$ is deduced in a similar fashion:

1. In the interval $[T_{n-1}, T]$ the function $\hat{f}_1^{(B,n)}(t, x, V_t^{t',v'})$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\rm bs}^{t',v'} u(t,x) &= 0, && t \in [T_{n-1}, T],\ x > \hat{h}^{(n)}(t, V_t^{t',v'}) \\ u(T,x) &= 0, && x > \hat{h}^{(n)}(T, V_T^{t',v'}) \\ u(t, \hat{h}^{(n)}(t, V_t^{t',v'})) &= \hat{f}_1^{(A,n)}(t, \hat{h}^{(n)}(t, V_t^{t',v'}), V_t^{t',v'}), && t \in [T_{n-1}, T] \end{aligned}$$

so we can compute $\hat{f}_1^{(B,n)}(T_{n-1}, x, V_{T_{n-1}}^{t',v'})$ as in the single-stage method.

2. For $i = n-2, \ldots, 0$ the function $\hat{f}_1^{(B,n)}(t, x, V_t^{t',v'})$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\rm bs}^{t',v'} u(t,x) &= 0, && t \in [T_i, T_{i+1}],\ x > \hat{h}^{(n)}(t, V_t^{t',v'}) \\ u(T_{i+1},x) &= \hat{f}_1^{(B,n)}(T_{i+1}, x, V_{T_{i+1}}^{t',v'}), && x > \hat{h}^{(n)}(T_{i+1}, V_{T_{i+1}}^{t',v'}) \\ u(t, \hat{h}^{(n)}(t, V_t^{t',v'})) &= \hat{f}_1^{(A,n)}(t, \hat{h}^{(n)}(t, V_t^{t',v'}), V_t^{t',v'}), && t \in [T_i, T_{i+1}] \end{aligned}$$

where once more the terminal condition is the function which has been derived in the previous stage. Therefore we can obtain an explicit representation for $\hat{f}_1^{(B,n)}(T_i, x, V_{T_i}^{t',v'})$ by resorting to formula (A15) in Theorem 1 of Dorfleitner et al. [20].

It is worth pointing out that the justification of the validity of the Feynman-Kac theorems is somewhat more delicate in this multi-stage setting, not only because the barrier function is no longer globally smooth but also because we need to deal with more lengthy analytical expressions in the verification of the growth conditions. We will not deal with these technicalities here, but we do note that the natural strategy to circumvent these difficulties consists in applying the Feynman-Kac theorems sequentially in each interval $[T_{n-1}, T], \ldots, [t', T_1]$.

Chapter 4

Conclusions

In this thesis we successfully achieved our goal of establishing an asymptotic pricing formula for barrier options under the 2-hypergeometric stochastic volatility model. Moreover, we were able to show that our small volatility of volatility expansion method, which is based on that of Privault and She [2], is not just a formal asymptotic technique, as it is indeed possible to prove that it converges when the perturbation parameter tends to zero.

An important feature of our method is that, taking advantage of the analytical tractability of the 2-hypergeometric model, we obtain an explicit pricing formula which only requires the numerical evaluation of a double definite integral. This is much simpler than the computationally intensive methods which are commonly used for numerically computing option prices under stochastic volatility.

The only drawback of our barrier option pricing technique is the fact that, in general, it requires two approximation steps: the first concerns the approximation of the constant barrier by a time and volatility-dependent barrier, and the second is related to the asymptotic nature of our perturbation expansion method. However, this shortcoming is partly offset by the fact that the multi-stage method can be employed whenever one needs to improve the quality of the approximation.

In closing, we would like to point out some relevant topics which, due to time and space constraints, were not covered in this study. It would be interesting to better examine the accuracy of our single and multi-stage approximations through a numerical comparison with (numerically) exact values obtained e.g. through Monte Carlo simulation or a finite difference scheme. In particular, such an implementation would allow us to check if the numerical results endorse our conjecture regarding the validity of the asymptotic expansion for the case of reverse barrier options (cf. Remark 3.6). Lastly, it would be valuable to investigate whether the price of other exotic options (including American-type options) under the 2-hypergeometric stochastic volatility model can also be computed via the small vol of vol expansion method addressed in this work.

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Appendix A

Auxiliary computations

A.1 The expectation of a function of a normal random variable

If $Y \sim \mathrm{Normal}(\mu, \sigma^2)$, then

$$\begin{aligned}
&\mathbb{E}\big[(a_1 Y^2 + a_2 Y + a_3)\exp\left\{-(a_4 Y^2 + a_5 Y + a_6)\right\} \mathbb{1}_{\{Y \le L\}}\big] = \\
&\quad = \int_{-\infty}^{L} (a_1 y^2 + a_2 y + a_3) \frac{1}{\sqrt{2\pi}\sigma} \exp\Bigg\{-\left(a_4 + \frac{1}{2\sigma^2}\right) y^2 - \left(a_5 - \frac{\mu}{\sigma^2}\right) y - \left(a_6 + \frac{\mu^2}{2\sigma^2}\right)\Bigg\} dy \\
&\quad = \frac{1}{\sigma} \int_{-\infty}^{L} (b_0 y^2 + b_1 y + b_2) \frac{1}{\sqrt{2\pi} s} \exp\Bigg\{-\frac{1}{2s^2} y^2 + \frac{m}{s^2} y - \frac{m^2}{2s^2}\Bigg\} dy \\
&\quad = \frac{1}{\sigma} \int_{-\infty}^{L} \Big(b_0 (y-m)^2 + (b_1 + 2b_0 m)(y-m) + (b_2 + b_1 m + b_0 m^2)\Big) \\
&\qquad\qquad \times \frac{1}{\sqrt{2\pi}} \exp\Bigg\{-\frac{1}{2s^2} y^2 + \frac{m}{s^2} y - \frac{m^2}{2s^2}\Bigg\} dy \\
&\quad = \frac{s^2}{\sigma} b_0 \left[s\,\mathcal{N}\left(\frac{L-m}{s}\right) - (L-m)\, n\left(\frac{L-m}{s}\right)\right] \\
&\qquad - \frac{s^2}{\sigma}(b_1 + 2b_0 m)\, n\left(\frac{L-m}{s}\right) + \frac{s}{\sigma}(b_2 + b_1 m + b_0 m^2)\,\mathcal{N}\left(\frac{L-m}{s}\right)
\end{aligned}$$

where

$$\begin{cases}
2a_4 + \frac{1}{\sigma^2} = \frac{1}{s^2} \\
-a_5 + \frac{\mu}{\sigma^2} = \frac{m}{s^2} \\
a_1 \exp\Big\{-\big(a_6 + \frac{\mu^2}{2\sigma^2}\big)\Big\} = b_0 \exp\Big\{-\frac{m^2}{2s^2}\Big\} \\
a_2 \exp\Big\{-\big(a_6 + \frac{\mu^2}{2\sigma^2}\big)\Big\} = b_1 \exp\Big\{-\frac{m^2}{2s^2}\Big\} \\
a_3 \exp\Big\{-\big(a_6 + \frac{\mu^2}{2\sigma^2}\big)\Big\} = b_2 \exp\Big\{-\frac{m^2}{2s^2}\Big\}
\end{cases}
\iff
\begin{cases}
m = \frac{\mu - \sigma^2 a_5}{1 + 2\sigma^2 a_4} \\
s^2 = \frac{\sigma^2}{1 + 2\sigma^2 a_4} \\
b_0 = a_1 \exp\Big\{-\frac{\mu^2}{2\sigma^2} + \frac{(\mu - \sigma^2 a_5)^2}{2\sigma^2(1 + 2\sigma^2 a_4)} - a_6\Big\} \\
b_1 = a_2 \exp\Big\{-\frac{\mu^2}{2\sigma^2} + \frac{(\mu - \sigma^2 a_5)^2}{2\sigma^2(1 + 2\sigma^2 a_4)} - a_6\Big\} \\
b_2 = a_3 \exp\Big\{-\frac{\mu^2}{2\sigma^2} + \frac{(\mu - \sigma^2 a_5)^2}{2\sigma^2(1 + 2\sigma^2 a_4)} - a_6\Big\}
\end{cases}$$

so that

$$\begin{aligned}
&\mathbb{E}\big[(a_1 Y^2 + a_2 Y + a_3)\exp\Big\{-(a_4 Y^2 + a_5 Y + a_6)\Big\}\mathbb{1}_{\{Y \le L\}}\big] = \exp\left\{-\frac{\mu^2}{2\sigma^2} + \frac{(\mu - \sigma^2 a_5)^2}{2\sigma^2(1 + 2\sigma^2 a_4)} - a_6\right\} \times \\
&\quad \times \Bigg[\frac{\sigma^2\, a_1}{(1 + 2\sigma^2 a_4)^{3/2}} \mathcal{N}\Bigg(\frac{1}{\sigma}\sqrt{1 + 2\sigma^2 a_4}\Big(L - \frac{\mu - \sigma^2 a_5}{1 + 2\sigma^2 a_4}\Big)\Bigg) \\
&\qquad - \frac{\sigma\, a_1}{1 + 2\sigma^2 a_4}\left(L - \frac{\mu - \sigma^2 a_5}{1 + 2\sigma^2 a_4}\right) n\Bigg(\frac{1}{\sigma}\sqrt{1 + 2\sigma^2 a_4}\Big(L - \frac{\mu - \sigma^2 a_5}{1 + 2\sigma^2 a_4}\Big)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\sigma}{1+2\sigma^2 a_4}\left(2a_1 \frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4} + a_2\right) n\left(\frac{1}{\sigma}\sqrt{1+2\sigma^2 a_4}\Big(L - \frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4}\Big)\right)\\
&+ \frac{1}{\sqrt{1+2\sigma^2 a_4}}\left(a_1\Big(\frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4}\Big)^2 + a_2 \frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4} + a_3\right)\mathcal{N}\left(\frac{1}{\sigma}\sqrt{1+2\sigma^2 a_4}\Big(L - \frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4}\Big)\right)\Bigg].
\end{aligned}
\tag{A.1}
$$

Taking the limit $L \to +\infty$ we obtain the particular case

$$
\begin{aligned}
&\mathbb{E}\big[(a_1 Y^2 + a_2 Y + a_3)\exp\{-(a_4 Y^2 + a_5 Y + a_6)\}\big] = \frac{1}{\sqrt{1+2\sigma^2 a_4}}\times\\
&\times \exp\left\{-\frac{\mu^2}{2\sigma^2} + \frac{(\mu-\sigma^2 a_5)^2}{2\sigma^2(1+2\sigma^2 a_4)} - a_6\right\}\left[a_1\left(\Big(\frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4}\Big)^2 + \frac{\sigma^2}{1+2\sigma^2 a_4}\right) + a_2 \frac{\mu-\sigma^2 a_5}{1+2\sigma^2 a_4} + a_3\right].
\end{aligned}
\tag{A.2}
$$

Moreover, a judicious choice of the parameters shows that

$$
\int_{-\infty}^{L} (c_1 w + c_2)\exp\left\{-(c_3 w^2 + c_4 w + c_5)\right\} dw = \sqrt{\frac{\pi}{c_3}}\exp\left\{\frac{c_4^2}{4c_3} - c_5\right\}\mathbb{E}\left[(c_1 W + c_2)\mathbb{1}_{\{W\le L\}}\right]
$$

where $W \sim \mathrm{Normal}\big(-\frac{c_4}{2c_3}, \frac{1}{2c_3}\big)$, so (A.1) yields

$$
\begin{aligned}
&\int_{-\infty}^{L} (c_1 w + c_2)\exp\left\{-(c_3 w^2 + c_4 w + c_5)\right\} dw =\\
&= \sqrt{\frac{\pi}{c_3}}\exp\left\{\frac{c_4^2}{4c_3} - c_5\right\}\left[-\frac{c_1}{\sqrt{2c_3}}\, n\left(\sqrt{2c_3}\Big(L + \frac{c_4}{2c_3}\Big)\right) + \left(-\frac{c_1 c_4}{2c_3} + c_2\right)\mathcal{N}\left(\sqrt{2c_3}\Big(L + \frac{c_4}{2c_3}\Big)\right)\right].
\end{aligned}
\tag{A.3}
$$

A.2 Computation of the expectation in Equation (3.35)

Here we will compute each of the terms of the expectation $\mathbb{E}\Big[S_u^{t',v'}\frac{\partial \hat{f}_0}{\partial x \partial v}(u, S_u^{t',v'}, V_u^{t',v'})\Big| S_{t'}^{t',v'} = x\Big]$. (All the expectations below are conditional to $S_{t'}^{t',v'} = x$, but for brevity we will omit the indication.)

The function $E_{1,0}(t',u,x,v')$**:**

$$
\begin{aligned}
E_{1,0} \equiv E_{1,0}(t',u,x,v') &= \mathbb{E}\left[S_u^{t',v'}\, n\big(d_1(u, S_u^{t',v'}, V_u^{t',v'})\big)\right]\\
&= \mathbb{E}\left[S_u^{t',v'}\, n\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\Big(\frac{S_u^{t',v'}}{K\vee H_1}\Big) + r(T-u) + \frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)\right)\right]\\
&= \frac{K\vee H_1}{\sqrt{2\pi}}\mathbb{E}\left[(a_2 Y + a_3)\exp\left\{-(a_4 Y^2 + a_5 Y + a_6)\right\}\right]
\end{aligned}
$$

where $Y = \log\left(\frac{S_u^{t',v'}}{K\vee H_1}\right) \sim \mathrm{Normal}\Big(\log\left(\frac{x}{K\vee H_1}\right) + r(u-t') - \frac{1}{2}\gamma^2(t',u,v'), \gamma^2(t',u,v')\Big)$ and

$$
a_2 = 0,\quad a_3 = 1,\quad a_4 = \frac{1}{2\gamma^2(u,T,V_u^{t',v'})},\quad a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})} - \frac{1}{2},\quad a_6 = \frac{\big(r(T-u) + \frac{\gamma^2(u,T,V_u^{t',v'})}{2}\big)^2}{2\gamma^2(u,T,V_u^{t',v'})}.
$$

Using (A.2), we get

$$
\begin{aligned}
E_{1,0} &= \frac{(K\vee H_1)\gamma(u,T,V_u^{t',v'})}{\sqrt{2\pi}\gamma(t',T,v')}\exp\Bigg\{-\frac{1}{2\gamma^2(t',T,v')}\left(\log\Big(\frac{x}{K\vee H_1}\Big) + r(T-t')\right)^2\\
&\qquad + \frac{1}{2}\left[r\big((T-u)-(u-t')\big) - \log\Big(\frac{x}{K\vee H_1}\Big) + \frac{1}{4}\gamma^2(t',T,v')\right]\Bigg\}\\
&= e^{-r(T-u)}(K\vee H_1)\frac{\gamma(u,T,V_u^{t',v'})}{\gamma(t',T,v')} n\big(d_2(t',x,v')\big).
\end{aligned}
\tag{A.4}
$$

The function $E_{1,1}(t',u,x,v')$**:**

$$
\begin{aligned}
E_{1,1} &= \mathbb{E}\left[S_u^{t',v'} d_1(u,S_u^{t',v'},V_u^{t',v'})\, n\big(d_1(u,S_u^{t',v'},V_u^{t',v'})\big)\right] \\
&= \mathbb{E}\Bigg[\frac{S_u^{t',v'}}{\gamma(u,T,V_u^{t',v'})}\bigg(\log\Big(\frac{S_u^{t',v'}}{K\vee H_1}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\bigg) \\
&\qquad \times n\bigg(\frac{1}{\gamma(u,T,V_u^{t',v'})}\bigg(\log\Big(\frac{S_u^{t',v'}}{K\vee H_1}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\bigg)\bigg)\Bigg] \\
&= \frac{1}{\sqrt{2\pi}}(K\vee H_1)\,\gamma(u,T,V_u^{t',v'})\,\mathbb{E}\left[(a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]
\end{aligned}
$$

where we now have

$$
\begin{gathered}
a_2 = 2a_4 = \frac{1}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_3 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}+\frac{1}{2}, \\
a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}-\frac{1}{2}, \qquad a_6 = \frac{\big(r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\big)^2}{2\gamma^2(u,T,V_u^{t',v'})},
\end{gathered}
$$

so that (A.2) gives

$$
\begin{aligned}
E_{1,1} &= \frac{(K\vee H_1)\gamma(u,T,V_u^{t',v'})}{\sqrt{2\pi\big(1+\frac{\gamma^2(t',u,v')}{\gamma^2(u,T,V_u^{t',v'})}\big)}} \\
&\quad\times\Bigg(\frac{1}{\gamma^2(t',T,v')}\bigg(\log\Big(\frac{x}{K\vee H_1}\Big)+r\Big((u-t')-\frac{\gamma^2(t',u,v')}{\gamma^2(u,T,V_u^{t',v'})}(T-u)\Big)\bigg)+\frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}+\frac{1}{2}\Bigg) \\
&\quad\times\exp\Bigg\{-\frac{1}{2\gamma^2(t',u,v')}\bigg(\log\Big(\frac{x}{K\vee H_1}\Big)+r(u-t')-\frac{1}{2}\gamma^2(t',u,v')\bigg)^2 \\
&\qquad+\frac{\Big(\log\big(\frac{x}{K\vee H_1}\big)+r\Big((u-t')-\frac{\gamma^2(t',u,v')}{\gamma^2(u,T,V_u^{t',v'})}(T-u)\Big)\Big)^2}{2\gamma^2(t',u,v')\big(1+\frac{\gamma^2(t',u,v')}{\gamma^2(u,T,V_u^{t',v'})}\big)}-\frac{(r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'}))^2}{2\gamma^2(u,T,V_u^{t',v'})}\Bigg\} \\
&= e^{-r(T-u)}(K\vee H_1)\frac{\gamma^2(u,T,V_u^{t',v'})}{\gamma^2(t',T,v')}\,d_1(t',x,v')\,n\big(d_2(t',x,v')\big).
\end{aligned}
\tag{A.5}
$$

The function $E_{1,2}(t',u,x,v')$**:**

$$
\begin{aligned}
E_{1,2} &= \mathbb{E}\left[S_u^{t',v'}\big(d_1(u,S_u^{t',v'},V_u^{t',v'})\big)^2\, n\big(d_1(u,S_u^{t',v'},V_u^{t',v'})\big)\right] \\
&= \mathbb{E}\Bigg[\frac{S_u^{t',v'}}{\gamma^2(u,T,V_u^{t',v'})}\bigg(\log\Big(\frac{S_u^{t',v'}}{K\vee H_1}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\bigg)^2 \\
&\qquad \times n\bigg(\frac{1}{\gamma(u,T,V_u^{t',v'})}\bigg(\log\Big(\frac{S_u^{t',v'}}{K\vee H_1}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\bigg)\bigg)\Bigg] \\
&= \frac{K\vee H_1}{\sqrt{2\pi}}\mathbb{E}\left[(a_1Y^2+a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]
\end{aligned}
$$

with

$$
\begin{gathered}
a_1 = 2a_4 = \frac{1}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_2 = \frac{2r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}+1, \\
a_3 = 2a_6 = \frac{\big(r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\big)^2}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}-\frac{1}{2},
\end{gathered}
$$

and we use (A.2) to obtain

$$
E_{1,2} = e^{-r(T-u)}(K\vee H_1)\frac{\gamma(u,T,V_u^{t',v'})}{\gamma^3(t',T,v')}\,\Big(d_1^2(t',x,v')\gamma^2(u,T,V_u^{t',v'})+\gamma^2(t',u,v')\Big)\,n\big(d_2(t',x,v')\big). \tag{A.6}
$$

The function $E_2(t',u,x,v')$**:**

$$
\begin{aligned}
E_2 &= \mathbb{E}\Bigg[\Big(S_u^{t',v'}\Big)^{-(1+2\beta)}\mathcal{N}\Bigg(\frac{1}{\gamma(u,T,V_u^{t',v'})}\Bigg(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\Bigg)\Bigg)\Bigg]\\
&= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}\gamma(u,T,V_u^{t',v'})}\mathbb{E}\Bigg[\Big(S_u^{t',v'}\Big)^{-(1+2\beta)}\\
&\qquad\times\exp\Bigg\{\frac{1}{2\gamma^2(u,T,V_u^{t',v'})}\Bigg(w+\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\Bigg)^2\Bigg\}\Bigg]dw\\
&= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}\gamma(u,T,V_u^{t',v'})}\Bigg(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Bigg)^{1+2\beta}\mathbb{E}\left[(a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]dw
\end{aligned}
$$

where $Y=\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)\sim\mathrm{Normal}\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)-r(u-t')+\frac{1}{2}\gamma^2(t',u,v'),\gamma^2(t',u,v')\Big)$ and

$$
\begin{aligned}
&a_2=0,\qquad a_3=1,\qquad a_4=\frac{1}{2\gamma^2(u,T,V_u^{t',v'})},\\
&a_5=\frac{w+r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}-\frac{1}{2}-2\beta,\qquad a_6=\frac{\big(w+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\big)^2}{2\gamma^2(u,T,V_u^{t',v'})}.
\end{aligned}
$$

Using (A.2), we get

$$
\begin{aligned}
E_2 &= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}\gamma(u,T,V_u^{t',v'})\sqrt{1+\frac{\gamma^2(t',u,v')}{\gamma^2(u,T,V_u^{t',v'})}}}\Bigg(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Bigg)^{1+2\beta}\\
&\quad\times\exp\Bigg\{-\frac{1}{2\gamma^2(t',u,v')}\Bigg(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)-r(u-t')+\frac{1}{2}\gamma^2(t',u,v')\Bigg)^2+\\
&\qquad+\frac{1}{2\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})\gamma^2(t',T,v')}\Bigg(\gamma^2(u,T,V_u^{t',v'})\Bigg(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)-r(u-t')\Bigg)\\
&\qquad\qquad-\gamma^2(t',u,v')\big(w+r(T-u)\big)+\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})(1+2\beta)\Bigg)^2\\
&\qquad-\frac{1}{2\gamma^2(u,T,V_u^{t',v'})}\Bigg(w+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\Bigg)^2\Bigg\}dw\\
&= \frac{1}{\sqrt{2\pi}\gamma(t',T,v')}\Bigg(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Bigg)^{1+2\beta}\int_{-\infty}^{0}(c_1w+c_2)\exp\big\{-(c_3w^2+c_4w+c_5)\big\}dw
\end{aligned}
$$

where

$$
\begin{aligned}
&c_1=0,\qquad c_2=1,\qquad c_3=\frac{1}{2\gamma^2(t',T,v')},\\
&c_4=\frac{1}{\gamma^2(t',T,v')}\Bigg(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)+r\big((T-u)-(u-t')\big)+\gamma^2(t',u,v')(1+2\beta)\Bigg)+\frac{1}{2},\\
&c_5=\frac{\gamma^2(t',T,v')}{8}+\frac{1}{2}\Bigg(\log\Big(\frac{H_1^2}{x(K\vee H_1)}\Big)+r\big((T-u)-(u-t')\big)\Bigg)+\\
&\quad+\frac{1}{2\gamma^2(t',T,v')}\Bigg\{\Bigg(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)+r\big((T-u)-(u-t')\big)\Bigg)^2\\
&\qquad-(1+2\beta)\Bigg[\gamma^2(u,T,V_u^{t',v'})\Bigg(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)-r(u-t')\Bigg)-\gamma^2(t',u,v')r(T-u)\Bigg]\\
&\qquad-(1+2\beta)^2\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})\Bigg\}.
\end{aligned}
$$

By (A.3),

$$
\begin{aligned}
E_2 &= \left(\frac{K\vee H_1}{\hat h^2(u,V_u^{t',v'})}\right)^{1+2\beta}\exp\left\{(1+2\beta)\left(\log\left(\frac{\hat h^2(u,V_u^{t',v'})}{x(K\vee H_1)}\right)-r(u-t')+\gamma^2(t',u,v')(1+\beta)\right)\right\}\\
&\quad\times\mathcal N\left(\frac{1}{\gamma(t',T,v')}\left(\log\left(\frac{\hat h^2(u,V_u^{t',v'})}{x(K\vee H_1)}\right)+r\big((T-u)-(u-t')\big)+\gamma^2(t',u,v')(1+2\beta)+\frac12\gamma^2(t',T,v')\right)\right)\\
&= x^{-(1+2\beta)}\exp\big\{(1+2\beta)[-r(u-t')+\gamma^2(t',u,v')(1+\beta)]\big\}\,\mathcal N\left(q_2(t',u,x,v')\right) \qquad \text{(A.7)}
\end{aligned}
$$

where $q_2(t',u,x,v') := \frac{1}{\gamma(t',T,v')}\Big(\log\Big(\frac{\hat h^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)+r\big((T-u)-(u-t')\big)+\gamma^2(t',u,v')(1+2\beta)+\frac12\gamma^2(t',T,v')\Big)$.

The function $E_{3,0}(t',u,x,v')$**:**

$$
\begin{aligned}
E_{3,0} &= \mathbb E\left[\left(S_u^{t',v'}\right)^{-(1+2\beta)}\; n\big(d_3(u,S_u^{t',v'},V_u^{t',v'})\big)\right]\\
&= \mathbb E\left[\left(S_u^{t',v'}\right)^{-(1+2\beta)} n\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\left(\frac{\hat h^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\right)+r(T-u)+\frac12\gamma^2(u,T,V_u^{t',v'})\right)\right)\right]\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{K\vee H_1}{\hat h^2(u,V_u^{t',v'})}\right)^{1+2\beta}\mathbb E\left[(a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]
\end{aligned}
$$

where $Y=\log\Big(\frac{\hat h^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)\sim \mathrm{Normal}\Big(\log\big(\frac{\hat h^2(u,V_u^{t',v'})}{x(K\vee H_1)}\big)-r(u-t')+\frac12\gamma^2(t',u,v'),\gamma^2(t',u,v')\Big)$ and

$$
a_2=0,\qquad a_3=1,\qquad a_4=\frac{1}{2\gamma^2(u,T,V_u^{t',v'})},
$$
$$
a_5=\frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}-\frac12-2\beta,\qquad a_6=\frac{(r(T-u)+\frac12\gamma^2(u,T,V_u^{t',v'}))^2}{2\gamma^2(u,T,V_u^{t',v'})}.
$$

Using (A.2), we get

$$
\begin{aligned}
E_{3,0} &= \frac{\gamma(u,T,V_u^{t',v'})}{\sqrt{2\pi}\gamma(t',T,v')}\Big(\frac{K\vee H_1}{\hat h^2(u,V_u^{t',v'})}\Big)^{1+2\beta}\times\\
&\quad\times\exp\left\{-\frac{1}{2\gamma^2(t',u,v')}\left(\log\Big(\frac{\hat h^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)-r(u-t')+\frac12\gamma^2(t',u,v')\right)^2+\right.\\
&\qquad+\frac{1}{2\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})\gamma^2(t',T,v')}\left(\gamma^2(u,T,V_u^{t',v'})\left(\log\Big(\frac{\hat h^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big)-r(u-t')\right)\right.\\
&\qquad\qquad\left.-\gamma^2(t',u,v')r(T-u)+\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})(1+2\beta)\right)^2\\
&\qquad\left.-\frac{1}{2\gamma^2(u,T,V_u^{t',v'})}\left(r(T-u)+\frac12\gamma^2(u,T,V_u^{t',v'})\right)^2\right\}\\
&= \frac{\gamma(u,T,V_u^{t',v'})}{x^{1+2\beta}\,\gamma(t',T,v')}\exp\left\{(1+2\beta)\left[-r(u-t')+\gamma^2(t',u,v')(1+\beta)\right]\right\}\,n\big(q_2(t',u,x,v')\big). \qquad \text{(A.8)}
\end{aligned}
$$

The function $E_{3,1}(t',u,x,v')$**:**

$$
\begin{aligned}
E_{3,1} &= \mathbb E\left[\left(S_u^{t',v'}\right)^{-(1+2\beta)}\; d_3(u,S_u^{t',v'},V_u^{t',v'})\,n\big(d_3(u,S_u^{t',v'},V_u^{t',v'})\big)\right]\\
&= \mathbb E\left[\frac{(S_u^{t',v'})^{-(1+2\beta)}}{\gamma(u,T,V_u^{t',v'})}\left(\log\left(\frac{\hat h^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\right)+r(T-u)+\frac12\gamma^2(u,T,V_u^{t',v'})\right)\right.\\
&\qquad\left.\times\, n\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\left(\frac{\hat h^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\right)+r(T-u)+\frac12\gamma^2(u,T,V_u^{t',v'})\right)\right)\right]
\end{aligned}
$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\right)^{1+2\beta}\gamma(u,T,V_u^{t',v'})\,\mathbb{E}\left[(a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]$$

where we now have

$$a_2 = 2a_4 = \frac{1}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_3 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}+\frac{1}{2},$$
$$a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}-\frac{1}{2}-2\beta, \qquad a_6 = \frac{(r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'}))^2}{2\gamma^2(u,T,V_u^{t',v'})}.$$

so, invoking (A.2),

$$E_{3,1} = \frac{\gamma^2(u,T,V_u^{t',v'})}{x^{1+2\beta}\,\gamma^2(t',T,v')}\exp\left\{(1+2\beta)\left[-r(u-t')+\gamma^2(t',u,v')(1+\beta)\right]\right\}q_2(t',u,x,v')\,n\big(q_2(t',u,x,v')\big). \tag{A.9}$$

The function $E_{3,2}(t',u,x,v')$**:**

$$\begin{aligned}
E_{3,2} &= \mathbb{E}\left[\left(S_u^{t',v'}\right)^{-(1+2\beta)}\ \left(d_3(u,S_u^{t',v'},V_u^{t',v'})\right)^2 n\big(d_3(u,S_u^{t',v'},V_u^{t',v'})\big)\right]\\
&= \mathbb{E}\Bigg[\frac{(S_u^{t',v'})^{-(1+2\beta)}}{\gamma^2(u,T,V_u^{t',v'})}\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)^2\\
&\qquad\times n\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)+r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)\right)\Bigg]\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\right)^{1+2\beta}\mathbb{E}\left[(a_1Y^2+a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]
\end{aligned}$$

with

$$a_1 = 2a_4 = \frac{1}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_2 = \frac{2r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}+1,$$
$$a_3 = 2a_6 = \frac{(r(T-u)+\frac{1}{2}\gamma^2(u,T,V_u^{t',v'}))^2}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})}-\frac{1}{2}-2\beta.$$

Employing again (A.2),

$$\begin{aligned}
E_{3,2} = \frac{\gamma(u,T,V_u^{t',v'})}{x^{1+2\beta}\,\gamma^3(t',T,v')}&\exp\left\{(1+2\beta)\left[-r(u-t')+\gamma^2(t',u,v')(1+\beta)\right]\right\}\times\\
&\times\left[\gamma^2(u,T,V_u^{t',v'})q_2^2(t',u,x,v')+\gamma^2(t',u,v')\right]\,n\big(q_2(t',u,x,v')\big).
\end{aligned} \tag{A.10}$$

The function $E_4(t',u,x,v')$**:**

$$\begin{aligned}
E_4 &= \mathbb{E}\left[\left(S_u^{t',v'}\right)^{-2\beta}\mathcal{N}\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)+r(T-u)-\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)\right)\right]\\
&= \int_{-\infty}^0\frac{1}{\sqrt{2\pi}\gamma(u,T,V_u^{t',v'})}\mathbb{E}\Bigg[\left(S_u^{t',v'}\right)^{-2\beta}\\
&\qquad\times\exp\left\{\frac{1}{2\gamma^2(u,T,V_u^{t',v'})}\left(w+\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big)+r(T-u)-\frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)^2\right\}\Bigg]dw\\
&= \int_{-\infty}^0\frac{1}{\sqrt{2\pi}\gamma(u,T,V_u^{t',v'})}\left(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\right)^{2\beta}\mathbb{E}\left[(a_2Y+a_3)\exp\left\{-(a_4Y^2+a_5Y+a_6)\right\}\right]dw
\end{aligned}$$

where $Y = \log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big) \sim \mathrm{Normal}\Big(\log\big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\big) - r(u-t') + \frac{1}{2}\gamma^2(t',u,v'), \gamma^2(t',u,v')\Big)$ and

$$a_2 = 0, \qquad a_3 = 1, \qquad a_4 = \frac{1}{2\gamma^2(u,T,V_u^{t',v'})},$$

$$a_5 = \frac{w + r(T-u)}{\gamma^2(u,T,V_u^{t',v'})} - \frac{1}{2} - 2\beta, \qquad a_6 = \frac{(w + r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'}))^2}{2\gamma^2(u,T,V_u^{t',v'})}.$$

Using (A.2), we get

$$\begin{aligned}
E_4 = &\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\gamma(u,T,V_u^{t',v'})\sqrt{1+\frac{\gamma^2(t',u,v')}{\gamma^2(u,T,V_u^{t',v'})}}}\Big(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Big)^{2\beta} \\
&\times \exp\bigg\{-\frac{1}{2\gamma^2(t',u,v')}\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) - r(u-t') + \frac{1}{2}\gamma^2(t',u,v')\Big)^2 + \\
&\quad + \frac{1}{2\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})\gamma^2(t',T,v')}\Big(\gamma^2(u,T,V_u^{t',v'})\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) - r(u-t')\Big) \\
&\qquad - \gamma^2(t',u,v')\big(w + r(T-u)\big) + \gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})(1+2\beta)\Big)^2 \\
&\quad - \frac{1}{2\gamma^2(u,T,V_u^{t',v'})}\Big(w + r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\Big)^2\bigg\}dw \\
= &\frac{1}{\sqrt{2\pi}\gamma(t',T,v')}\Big(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Big)^{2\beta}\int_{-\infty}^{0}(c_1 w + c_2)\exp\big\{-(c_3 w^2 + c_4 w + c_5)\big\}dw
\end{aligned}$$

where

$$c_1 = 0, \qquad c_2 = 1, \qquad c_3 = \frac{1}{2\gamma^2(t',T,v')},$$

$$c_4 = \frac{1}{\gamma^2(t',T,v')}\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) + r\big((T-u)-(u-t')\big) + \gamma^2(t',u,v')(1+2\beta)\Big) - \frac{1}{2},$$

$$\begin{aligned}
c_5 = &\frac{\gamma^2(t',T,v')}{8} + \frac{1}{2}\Big(\log\Big(\frac{H_1^2}{x(K\vee H_1)}\Big) - r(T-t')\Big) + \\
&+ \frac{1}{2\gamma^2(t',T,v')}\bigg\{\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) + r\big((T-u)-(u-t')\big)\Big)^2 \\
&\qquad - (1+2\beta)\Big[\gamma^2(u,T,V_u^{t',v'})\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) - r(u-t')\Big) - \gamma^2(t',u,v')r(T-u)\Big] \\
&\qquad - (1+2\beta)^2\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})\bigg\}.
\end{aligned}$$

By (A.3),

$$\begin{aligned}
E_4 = &\Big(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Big)^{2\beta}\exp\bigg\{2\beta\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) - r(u-t') + \gamma^2(t',u,v')\Big(\frac{1}{2}+\beta\Big)\Big)\bigg\} \\
&\times \mathcal{N}\bigg(\frac{1}{\gamma(t',T,v')}\Big(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) + r\big((T-u)-(u-t')\big) + \gamma^2(t',u,v')(1+2\beta) - \frac{1}{2}\gamma^2(t',T,v')\Big)\bigg) \\
= &\, x^{-2\beta}\exp\big\{-2\beta\, r(u-t') + \gamma^2(t',u,v')(\beta+2\beta^2)\big\}\,\mathcal{N}\left(q_4(t',u,x,v')\right) \qquad \text{(A.11)}
\end{aligned}$$

where $q_4(t',u,x,v') = q_2(t',u,x,v') - \gamma(t',T,v')$.

The function $E_{5,0}(t',u,x,v')$:

$$E_{5,0} = \mathbb{E}\left[\Big(S_u^{t',v'}\Big)^{-2\beta} n\big(d_4(u,S_u^{t',v'},V_u^{t',v'})\big)\right]$$

$$= \mathbb{E}\left[\left(S_u^{t',v'}\right)^{-2\beta} n\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\left(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\right) + r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\right)^{2\beta} \mathbb{E}\left[(a_2Y + a_3)\exp\left\{-(a_4Y^2 + a_5Y + a_6)\right\}\right]$$

where $Y = \log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big) \sim \mathrm{Normal}\Big(\log\big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\big) - r(u-t') + \frac{1}{2}\gamma^2(t',u,v'), \gamma^2(t',u,v')\Big)$ and

$$a_2 = 0, \qquad a_3 = 1, \qquad a_4 = \frac{1}{2\gamma^2(u,T,V_u^{t',v'})},$$

$$a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})} - \frac{1}{2} - 2\beta, \qquad a_6 = \frac{(r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'}))^2}{2\gamma^2(u,T,V_u^{t',v'})}.$$

Using (A.2), we get

$$\begin{aligned}
E_{5,0} &= \frac{\gamma(u,T,V_u^{t',v'})}{\sqrt{2\pi}\gamma(t',T,v')}\Big(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\Big)^{2\beta} \times \\
&\times \exp\Bigg\{-\frac{1}{2\gamma^2(t',u,v')}\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) - r(u-t') + \frac{1}{2}\gamma^2(t',u,v')\right)^2 + \\
&\quad + \frac{1}{2\gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})\gamma^2(t',T,v')}\Bigg(\gamma^2(u,T,V_u^{t',v'})\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{x(K\vee H_1)}\Big) - r(u-t')\right) \\
&\qquad - \gamma^2(t',u,v')r(T-u) + \gamma^2(t',u,v')\gamma^2(u,T,V_u^{t',v'})(1+2\beta)\Bigg)^2 \\
&\quad - \frac{1}{2\gamma^2(u,T,V_u^{t',v'})}\Big(r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\Big)^2\Bigg\} \\
&= \frac{\gamma(u,T,V_u^{t',v'})}{x^{2\beta}\,\gamma(t',T,v')}\exp\left\{-2\beta\, r(u-t') + \gamma^2(t',u,v')(\beta + 2\beta^2)\right\}\, n\big(q_4(t',u,x,v')\big).
\end{aligned} \tag{A.12}$$

The function $E_{5,1}(t',u,x,v')$:

$$\begin{aligned}
E_{5,1} &= \mathbb{E}\left[\left(S_u^{t',v'}\right)^{-2\beta}\, d_4(u,S_u^{t',v'},V_u^{t',v'})\, n\big(d_4(u,S_u^{t',v'},V_u^{t',v'})\big)\right] \\
&= \mathbb{E}\Bigg[\frac{\big(S_u^{t',v'}\big)^{-2\beta}}{\gamma(u,T,V_u^{t',v'})}\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big) + r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right) \\
&\qquad \times n\left(\frac{1}{\gamma(u,T,V_u^{t',v'})}\left(\log\Big(\frac{\hat{h}^2(u,V_u^{t',v'})}{S_u^{t',v'}(K\vee H_1)}\Big) + r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'})\right)\right)\Bigg] \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{K\vee H_1}{\hat{h}^2(u,V_u^{t',v'})}\right)^{2\beta}\gamma(u,T,V_u^{t',v'})\,\mathbb{E}\left[(a_2Y + a_3)\exp\left\{-(a_4Y^2 + a_5Y + a_6)\right\}\right]
\end{aligned}$$

where we now have

$$a_2 = 2a_4 = \frac{1}{\gamma^2(u,T,V_u^{t',v'})}, \qquad a_3 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})} - \frac{1}{2},$$

$$a_5 = \frac{r(T-u)}{\gamma^2(u,T,V_u^{t',v'})} - \frac{1}{2} - 2\beta, \qquad a_6 = \frac{(r(T-u) - \frac{1}{2}\gamma^2(u,T,V_u^{t',v'}))^2}{2\gamma^2(u,T,V_u^{t',v'})}.$$

so, invoking (A.2),

$$E_{5,1} = \frac{\gamma^2(u,T,V_u^{t',v'})}{x^{2\beta}\,\gamma^2(t',T,v')}\exp\left\{-2\beta\, r(u-t') + \gamma^2(t',u,v')(\beta + 2\beta^2)\right\} q_4(t',u,x,v')\, n\big(q_4(t',u,x,v')\big). \tag{A.13}$$

A.3 Writing an integral in terms of the bivariate normal distribution function

We compute

$$
\begin{aligned}
&\Upsilon(A,B,C,D,E,L) := \int_L^\infty \exp\left\{-(Aw^2+Bw+C)\right\}\mathcal{N}(Dw+E)dw \\
&= \int_L^\infty \exp\left\{-(Aw^2+Bw+C)\right\}\int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(z+Dw+E)^2\right\}dz\,dw \\
&= \int_L^\infty\int_{-\infty}^0 \frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\Big(A+\frac{D^2}{2}\Big)w^2+\frac{1}{2}z^2+Dzw+(B+DE)w+Ez+\Big(C+\frac{E^2}{2}\Big)\right]\right\}dz\,dw \\
&= e^{-K}\sqrt{2\pi}\sigma_1\sigma_2\sqrt{1-\rho^2}\int_L^\infty\int_{-\infty}^0 \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\times \\
&\qquad\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}(w-\mu_1)^2+\frac{1}{\sigma_2^2}(z-\mu_2)^2-\frac{2\rho}{\sigma_1\sigma_2}(w-\mu_1)(z-\mu_2)\right]\right\}dz\,dw \\
&= e^{-K}\sqrt{2\pi}\sigma_1\sigma_2\sqrt{1-\rho^2}\left[\mathcal{N}\left(-\frac{\mu_2}{\sigma_2}\right)-\mathcal{N}_2\left(\frac{L-\mu_1}{\sigma_1},-\frac{\mu_2}{\sigma_2};\rho\right)\right]
\end{aligned}
$$

where $\mathcal{N}_2(\,\cdot\,,\cdot\,;\rho)$ denotes the cumulative distribution function of a bivariate normal random variable with zero means, unit variances and correlation ρ, and the parameters μ_1, μ_2, σ_1, σ_2, ρ and K are chosen as follows:

$$
\begin{cases}
A+\frac{D^2}{2}=\frac{1}{2(1-\rho^2)\sigma_1^2}\\
\frac{1}{2}=\frac{1}{2(1-\rho^2)\sigma_2^2}\\
D=\frac{-\rho}{(1-\rho^2)\sigma_1\sigma_2}\\
B+DE=\frac{1}{1-\rho^2}\left(-\frac{\mu_1}{\sigma_1^2}+\frac{\rho\mu_2}{\sigma_1\sigma_2}\right)\\
E=\frac{1}{1-\rho^2}\left(-\frac{\mu_2}{\sigma_2^2}+\frac{\rho\mu_1}{\sigma_1\sigma_2}\right)\\
C+\frac{E^2}{2}=\frac{1}{2(1-\rho^2)}\left(\frac{\mu_1^2}{\sigma_1^2}+\frac{\mu_2^2}{\sigma_2^2}-\frac{2\rho\mu_1\mu_2}{\sigma_1\sigma_2}\right)+K
\end{cases}
\iff
\begin{cases}
\mu_1=-\frac{B}{2A}\\
\mu_2=\frac{BD}{2A}-E\\
\sigma_1=\frac{1}{\sqrt{2A}}\\
\sigma_2=\frac{\sqrt{2A+D^2}}{\sqrt{2A}}\\
\rho=-\frac{D}{\sqrt{2A+D^2}}\\
K=C-\frac{B^2}{4A}
\end{cases}.
$$

Returning to the original variables, we get

$$
\begin{aligned}
\Upsilon(A,B,C,D,E,L) = \sqrt{\frac{\pi}{A}}\exp\Big\{\frac{B^2}{4A}-C\Big\}\Big[\mathcal{N}\Big(\frac{2AE-BD}{\sqrt{2A(2A+D^2)}}\Big) \\
-\mathcal{N}_2\Big(L\sqrt{2A}+\frac{B}{\sqrt{2A}},\frac{2AE-BD}{\sqrt{2A(2A+D^2)}};-\frac{D}{\sqrt{2A+D^2}}\Big)\Big].
\end{aligned}
\tag{A.14}
$$

Appendix B

Estimates for the derivatives of the zero and first-order terms

Here we will carry out the computations which show that the estimates in Equations (3.41) and (3.46) hold true. We will be assuming that $K \geq H_1$ (so that $A = 0$) and that $\psi(t, v) \equiv 1$.

Remark B.1. Throughout this appendix, the same letters C and k will be used to denote positive constants which may vary from line to line.

B.1 Auxiliary estimates

In this section we collect some estimates for the log-volatility function $V_u^{t,v}$, the integrated variance function $\gamma^2(t, T, v)$, the barrier function $\hat{h}(t, v)$ and the auxiliary functions $d_i(t, x, v)$, as well as their derivatives with respect to v.

Our starting observation is that if $e^{2v} > \frac{2a}{c}$ (respectively $e^{2v} \leq \frac{2a}{c}$) then the function $u \mapsto V_u^{t,v}$ is decreasing (resp. increasing) in the interval $[t, T]$ and therefore $e^{2V_u^{T,v}} \leq e^{2V_u^{t,v}} \leq e^{2v}$ (resp. $e^{2v} \leq e^{2V_u^{t,v}} \leq e^{2V_u^{T,v}}$) for $u \in [t, T]$. So for all $0 \leq t \leq u < T$ and all $v \in \mathbb{R}$ we have

$$e^{2V_u^{t,v}} = \frac{e^{2a(u-t)}e^{2v}}{1 + \frac{c}{2a}e^{2v}\big(e^{2a(u-t)} - 1\big)} \leq Ce^{2v} \tag{B.1}$$

$$0 < \frac{\partial e^{2V_u^{t,v}}}{\partial v} = \frac{2e^{2a(u-t)}e^{2v}}{\big(1 + \frac{c}{2a}e^{2v}\big(e^{2a(u-t)} - 1\big)\big)^2} = 2e^{-2a(u-t)}e^{-2v}e^{4V_u^{t,v}} \leq Ce^{2v} \tag{B.2}$$

$$\left|\frac{\partial^2 e^{2V_u^{t,v}}}{\partial v^2}\right| = e^{-2a(u-t)}\left|-4e^{-2v}e^{4V_u^{t,v}} + 4e^{-2v}e^{2V_u^{t,v}}\frac{\partial e^{2V_u^{t,v}}}{\partial v}\right| \leq Ce^{2v} \tag{B.3}$$

$$\left|\frac{\partial^3 e^{2V_u^{t,v}}}{\partial v^3}\right| = e^{-2a(u-t)}e^{-2v}\left|8e^{4V_u^{t,v}} - 16e^{2V_u^{t,v}}\frac{\partial e^{2V_u^{t,v}}}{\partial v} + 4\left(\frac{\partial e^{2V_u^{t,v}}}{\partial v}\right)^2 + 4e^{2V_u^{t,v}}\frac{\partial^2 e^{2V_u^{t,v}}}{\partial v^2}\right| \leq Ce^{2v} \tag{B.4}$$

$$0 < \frac{\partial V_u^{t,v}}{\partial v} = 1 + \frac{\frac{c}{2a}e^{2v}\big(e^{2a(u-t)} - 1\big)}{1 + \frac{c}{2a}e^{2v}\big(e^{2a(u-t)} - 1\big)} \leq 2 \tag{B.5}$$

$$\left|\frac{\partial^2 V_u^{t,v}}{\partial v^2}\right| = \left|\frac{\frac{c}{a}e^{2v}\big(e^{2a(u-t)} - 1\big)}{1 + \frac{c}{2a}e^{2v}\big(e^{2a(u-t)} - 1\big)} - \frac{\frac{c^2}{2a^2}e^{4v}\big(e^{2a(u-t)} - 1\big)^2}{\big(1 + \frac{c}{2a}e^{2v}\big(e^{2a(u-t)} - 1\big)\big)^2}\right| \leq 4 \tag{B.6}$$

$$\gamma^2(t,T,v) = \frac{1}{c}\log\Big(1+\frac{c}{2a}e^{2v}(e^{2a(T-t)}-1)\Big) = \int_t^T e^{2V_u^{t,v}}du$$

$$\geq \begin{cases} (T-t)e^{2V_T^{t,v}} = (T-t)\frac{e^{2v}e^{2a(T-t)}}{1+\frac{c}{2a}e^{2v}(e^{2a(T-t)}-1)}, & \text{if } e^{2v} > \frac{2a}{c} \\ (T-t)e^{2v}, & \text{if } e^{2v} \leq \frac{2a}{c} \end{cases} \tag{B.7}$$

$$e^{\mu\gamma^2(t,u,v)} = \Big(1+\frac{c}{2a}e^{2v}(e^{2a(u-t)}-1)\Big)^{\frac{\mu}{c}} \leq C(1+e^{kv}), \qquad \mu\in\mathbb{R} \tag{B.8}$$

$$0 < \frac{1}{\gamma(t,T,v)} \leq \begin{cases} C\frac{1}{\sqrt{T-t}}e^{-v}\sqrt{1+\frac{c}{2a}e^{2v}(e^{2|a|(T-t)}-1)}, & \text{if } e^{2v} > \frac{2a}{c} \\ C\frac{1}{\sqrt{T-t}}e^{-v}, & \text{if } e^{2v} \leq \frac{2a}{c} \end{cases}$$
$$\leq C\frac{1}{\sqrt{T-t}}e^{-v}(1+e^{2v}), \tag{B.9}$$

$$0 < \frac{\partial\gamma^2}{\partial v}(t,T,v) = \int_t^T \frac{\partial e^{2V_u^{t,v}}}{\partial v}du \leq C(T-t)e^{2v} \tag{B.10}$$

$$\left|\frac{\partial^j\gamma^2}{\partial v^j}(t,T,v)\right| = \left|\int_t^T \frac{\partial^j e^{2V_u^{t,v}}}{\partial v^j}du\right| \leq C(T-t)e^{2v}, \qquad j=2,3 \tag{B.11}$$

$$0 < \frac{\partial\gamma}{\partial v}(t,T,v) = \frac{1}{2\gamma(t,T,v)}\frac{\partial\gamma^2}{\partial v}(t,T,v) \leq C\frac{1}{\sqrt{T-t}}e^{-v}(1+e^{2v})(T-t)e^{2v} \leq C\sqrt{T-t}\,e^v(1+e^{2v}) \tag{B.12}$$

$$\left|\frac{\partial^2\gamma}{\partial v^2}(t,T,v)\right| = \left|\frac{1}{2\gamma(t,T,v)}\frac{\partial^2\gamma^2}{\partial v^2}(t,T,v) - \frac{1}{8\gamma^3(t,T,v)}\Big(\frac{\partial\gamma^2}{\partial v}(t,T,v)\Big)^2\right| \leq C\sqrt{T-t}\,e^v(1+e^{kv}) \tag{B.13}$$

where the letters C and k denote positive constants whose exact value may change within each inequality. (We have used the fact that for every constant $\mu\in\mathbb{R}$ the function $e^{\mu(T-t)}$ is bounded on the domain $t\in[t',T]$.) Furthermore, we also have

$$0 < \hat{h}^\mu(t,v) = H_1^\mu \exp\left\{\mu\left(-r(T-t)+\frac{(1+2\beta)}{2}\gamma^2(t,T,v)\right)\right\} \leq C\exp\left\{\frac{\mu(1+2\beta)}{2}\gamma^2(t,T,v)\right\}$$
$$= C\Big(1+\frac{c}{2a}e^{2v}(e^{2a(T-t)}-1)\Big)^{\frac{\mu(1+2\beta)}{2c}} \leq C\left(1+e^{kv}\right), \qquad \mu\in\mathbb{R}, \tag{B.14}$$

$$\left|\frac{\partial\log\hat{h}}{\partial v}(t,v)\right| = \frac{|1+2\beta|}{2}\frac{\partial\gamma^2}{\partial v}(t,T,v) \leq C(T-t)e^{2v}, \tag{B.15}$$

$$\left(\frac{\hat{h}(t,v)}{x}\right)^\mu \leq \begin{cases} 1, & \text{if } \mu\geq 0 \\ Cx^{-\mu}\left(1+e^{kv}\right), & \text{if } \mu<0 \end{cases} \leq C\left(1+x^k+e^{kv}\right), \qquad \mu\in\mathbb{R}, \tag{B.16}$$

(where this last estimate holds provided that $x\geq\hat{h}(t,v)$) and

$$\left|\frac{\partial\hat{h}^\mu}{\partial v}(t,v)\right| = \frac{1}{2}|\mu(1+2\beta)|\frac{\partial\gamma^2}{\partial v}(t,T,v)\hat{h}^\mu(t,v) \leq C(T-t)(1+e^{kv}) \tag{B.17}$$

$$\left|\frac{\partial^2\hat{h}^\mu}{\partial v^2}(t,v)\right| = \left|\frac{\mu}{2}(1+2\beta)\frac{\partial^2\gamma^2}{\partial v^2}(t,T,v)+\frac{\mu^2}{4}(1+2\beta)^2\Big(\frac{\partial\gamma^2}{\partial v}(t,T,v)\Big)^2\right|\hat{h}^\mu(t,v)$$
$$\leq C(T-t)(1+e^{kv}). \tag{B.18}$$

$$\left|\frac{\partial^3\hat{h}^\mu}{\partial v^3}(t,v)\right| = \left|\frac{\mu}{2}(1+2\beta)\frac{\partial^3\gamma^2}{\partial v^3}(t,T,v)+\mu^2(1+2\beta)^2\frac{\partial\gamma^2}{\partial v}(t,T,v)\frac{\partial^2\gamma^2}{\partial v^2}(t,T,v)\right.$$
$$\left.+\frac{\mu^3}{8}(1+2\beta)^3\Big(\frac{\partial\gamma^2}{\partial v}(t,T,v)\Big)^3\right|\hat{h}^\mu(t,v)$$
$$\leq C(T-t)(1+e^{kv}). \tag{B.19}$$

As for the auxiliary functions $d_i(t,x,v)$ defined in (3.26), we have

$$\left|\frac{\partial d_1}{\partial v}(t,x,v)\right| = \left|\frac{\partial \gamma}{\partial v}(t,T,v)\Big(1-\frac{d_1(t,x,v)}{\gamma(t,T,v)}\Big)\right| \le C(1+e^{kv})\big(1+|d_1(t,x,v)|\big) \tag{B.20}$$

$$\begin{aligned}\left|\frac{\partial^2 d_1}{\partial v^2}(t,x,v)\right| &= \\ &= \left|\frac{\partial^2\gamma}{\partial v^2}(t,T,v)\Big(1-\frac{d_1(t,x,v)}{\gamma(t,T,v)}\Big) + \Big(\frac{\partial\gamma}{\partial v}(t,T,v)\Big)^2\frac{d_1(t,x,v)}{\gamma^2(t,T,v)} - \frac{1}{\gamma(t,T,v)}\frac{\partial\gamma}{\partial v}(t,T,v)\frac{\partial d_1}{\partial v}(t,x,v)\right| \\ &\le C(1+e^{kv})\big(1+|d_1(t,x,v)|\big)\end{aligned} \tag{B.21}$$

$$\left|\frac{\partial d_2}{\partial v}(t,x,v)\right| \le C(1+e^{kv})\big(1+|d_2(t,x,v)|\big) \tag{B.22}$$

$$\left|\frac{\partial^2 d_2}{\partial v^2}(t,x,v)\right| \le C(1+e^{kv})\big(1+|d_2(t,x,v)|\big). \tag{B.23}$$

$$\begin{aligned}\left|\frac{\partial d_3}{\partial v}(t,x,v)\right| &= \left|\frac{\partial\gamma}{\partial v}(t,T,v)\Big(1-\frac{d_3(t,x,v)}{\gamma(t,T,v)}\Big) + \frac{2}{\gamma(t,T,v)}\frac{\partial \log \hat h}{\partial v}(t,v)\right| \\ &\le C(1+e^{kv})\big(1+|d_3(t,x,v)|\big)\end{aligned} \tag{B.24}$$

$$\begin{aligned}\left|\frac{\partial^2 d_3}{\partial v^2}(t,x,v)\right| &= \left|\frac{\partial^2\gamma}{\partial v^2}(t,T,v)\Big(1-\frac{d_3(t,x,v)}{\gamma(t,T,v)}\Big) + \Big(\frac{\partial\gamma}{\partial v}(t,T,v)\Big)^2\frac{d_3(t,x,v)}{\gamma^2(t,T,v)}\right. \\ &\qquad - \frac{1}{\gamma(t,T,v)}\frac{\partial\gamma}{\partial v}(t,T,v)\frac{\partial d_3}{\partial v}(t,x,v) - \frac{2}{\gamma^2(t,T,v)}\frac{\partial\gamma}{\partial v}(t,T,v)\frac{\partial\log\hat h}{\partial v}(t,v) \\ &\qquad \left. + \frac{2}{\gamma(t,T,v)}\frac{\partial^2\log\hat h}{\partial v^2}(t,v)\right| \\ &\le C(1+e^{kv})\big(1+|d_3(t,x,v)|\big)\end{aligned} \tag{B.25}$$

$$\left|\frac{\partial^3 d_3}{\partial v^3}(t,x,v)\right| \le C(1+e^{kv})\big(1+|d_3(t,x,v)|\big) \tag{B.26}$$

$$\left|\frac{\partial d_4}{\partial v}(t,x,v)\right| \le C(1+e^{kv})\big(1+|d_4(t,x,v)|\big) \tag{B.27}$$

$$\left|\frac{\partial^2 d_4}{\partial v^2}(t,x,v)\right| \le C(1+e^{kv})\big(1+|d_4(t,x,v)|\big). \tag{B.28}$$

(The derivation of the estimates for $d_2(t,x,v)$ and $d_4(t,x,v)$ is similar to that of the estimates for $d_1(t,x,v)$ and $d_3(t,x,v)$, respectively, and the upper bound for $\frac{\partial^3 d_3}{\partial v^3}(t,x,v)$ is obtained effortlessly after computing the derivative with respect to v of the expression in (B.25).)

B.2 The function $\mathcal{L}_1\hat f_0$

We are now ready to deduce an upper bound for

$$\begin{aligned}\mathcal{L}_1\hat f_0(t,x,v) = \rho x e^v \Bigg[&\frac{\partial d_1}{\partial v}(t,x,v)\, n(d_1(t,x,v)) && \text{(B.29)} \\ &+ \frac{1+2\beta}{x^{2+2\beta}}\frac{\partial \hat h^{2+2\beta}}{\partial v}(t,v)\,\mathcal{N}(d_3(t,x,v)) && \text{(B.30)} \\ &+ (1+2\beta)\bigg(\frac{\hat h(t,v)}{x}\bigg)^{2+2\beta}\frac{\partial d_3}{\partial v}(t,x,v)\, n(d_3(t,x,v)) && \text{(B.31)} \\ &- Ke^{-r(T-t)}\frac{2\beta}{x^{1+2\beta}}\frac{\partial\hat h^{2\beta}}{\partial v}(t,v)\,\mathcal{N}(d_4(t,x,v)) && \text{(B.32)} \\ &+ Ke^{-r(T-t)}\frac{2\beta}{x^{1+2\beta}}\,\hat h^{2\beta}(t,v)\,\frac{\partial d_4}{\partial v}(t,x,v)\, n(d_4(t,x,v))\Bigg]. && \text{(B.33)}\end{aligned}$$

(cf. Equation (3.34) with $A = 0$ and $\psi(t, v) \equiv 1$).

Using the estimates (B.1)–(B.28) and the fact that the functions $\mathcal{N}(y)$ and $|y|n(y)$ are bounded, we obtain the following estimates for each of the individual terms (B.29)–(B.33) in the expression for $\mathcal{L}_1\hat{f}_0$:

$$\left|\rho x e^v \frac{\partial d_1}{\partial v}(t, T, v)\, n(d_1(t, x, v))\right| \le C x e^v (1 + e^{kv})\big(1 + |d_1(t, x, v)|\big)\, n(d_1(t, x, v)) \le C(1 + x^k + e^{kv}),$$

$$\left|\rho e^v \frac{(1 + 2\beta)}{x^{1+2\beta}} \frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t, v)\, \mathcal{N}(d_3(t, x, v))\right|$$

$$\le C \frac{e^v}{x^{1+2\beta}} \frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t, v) \le C \frac{e^v}{x^{1+2\beta}} \frac{\partial \gamma^2}{\partial v}(t, T, v)\, \hat{h}^{2+2\beta}(t, v) \le C(T - t)(1 + x^k + e^{kv}),$$

$$\left|\rho e^v \frac{(1 + 2\beta)}{x^{1+2\beta}}\, \hat{h}^{2+2\beta}(t, v)\, \frac{\partial d_3}{\partial v}(t, T, v)\, n(d_3(t, x, v))\right|$$

$$\le C e^v \frac{\hat{h}^{1+2\beta}(t, v)}{x^{1+2\beta}} (1 + e^{kv})\big(1 + |d_3(t, x, v)|\big)\, n(d_3(t, x, v)) \le C\left(1 + x^k + e^{kv}\right),$$

$$\left|\rho e^v K e^{-r(T-t)} \frac{2\beta}{x^{2\beta}} \frac{\partial \hat{h}^{2\beta}}{\partial v}(t, v)\, \mathcal{N}(d_4(t, x, v))\right| \le C(T - t)(1 + x^k + e^{kv}),$$

$$\left|\rho e^v K e^{-r(T-t)} \frac{2\beta}{x^{2\beta}} \hat{h}^{2\beta}(t, v)\, \frac{\partial d_4}{\partial v}(t, T, v)\, n(d_4(t, x, v))\right| \le C\left(1 + x^k + e^{kv}\right).$$

Finally, from the above estimates we conclude that there exists $k \in \mathbb{N}$ such that

$$\mathcal{L}_1\hat{f}_0(t, x, v) \le C(1 + x^k + e^{kv}). \tag{B.34}$$

B.3 The function $\mathcal{L}_2\hat{f}_0$

Let us obtain an upper bound for

$$\begin{aligned}
\mathcal{L}_2\hat{f}_0(t, x, v) &= \frac{1}{2}\frac{\partial^2}{\partial v^2}\Bigg(x\,\mathcal{N}(d_1(t, x, v)) - K e^{-r(T-t)}\mathcal{N}(d_2(t, x, v)) \\
&\qquad + \left(\frac{\hat{h}(t, v)}{x}\right)^{1+2\beta}\left[-\hat{h}(t, v)\,\mathcal{N}(d_3(t, x, v)) + \frac{Kx}{\hat{h}(t, v)} e^{-r(T-t)}\mathcal{N}(d_4(t, x, v))\right]\Bigg) \\
&= \frac{1}{2}\frac{\partial}{\partial v}\Bigg(x\frac{\partial d_1}{\partial v}(t, x, v)\, n(d_1(t, x, v)) - K e^{-r(T-t)}\frac{\partial d_2}{\partial v}(t, x, v)\, n(d_2(t, x, v)) \\
&\qquad + \frac{1}{x^{1+2\beta}}\Bigg[-\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t, v)\,\mathcal{N}(d_3(t, x, v)) - \hat{h}^{2+2\beta}(t, v)\,\frac{\partial d_3}{\partial v}(t, x, v)\, n(d_3(t, x, v)) \\
&\qquad + x\frac{\partial \hat{h}^{2\beta}}{\partial v}(t, v)\,\mathcal{N}(d_4(t, x, v)) + x\hat{h}^{2\beta}(t, v)\,\frac{\partial d_4}{\partial v}(t, x, v)\, n(d_4(t, x, v))\Bigg]\Bigg)
\end{aligned}$$

$$= \frac{1}{2}\Bigg\{x\left[\frac{\partial^2 d_1}{\partial v^2}(t, x, v) - \left(\frac{\partial d_1}{\partial v}(t, x, v)\right)^2 d_1(t, x, v)\right] n(d_1(t, x, v)) \tag{B.35}$$

$$- K e^{-r(T-t)}\left[\frac{\partial^2 d_2}{\partial v^2}(t, x, v) - \left(\frac{\partial d_2}{\partial v}(t, x, v)\right)^2 d_2(t, x, v)\right] n(d_2(t, x, v)) \tag{B.36}$$

$$- \frac{1}{x^{1+2\beta}}\frac{\partial^2 \hat{h}^{2+2\beta}}{\partial v^2}(t, v)\,\mathcal{N}(d_3(t, x, v)) \tag{B.37}$$

$$- \frac{2}{x^{1+2\beta}}\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t, v)\,\frac{\partial d_3}{\partial v}(t, x, v)\, n(d_3(t, x, v)) \tag{B.38}$$

$$- \left(\frac{\hat{h}(t, v)}{x}\right)^{1+2\beta}\hat{h}(t, v)\left(\frac{\partial^2 d_3}{\partial v^2}(t, x, v) - \left(\frac{\partial d_3}{\partial v}(t, x, v)\right)^2 d_3(t, x, v)\right) n(d_3(t, x, v)) \tag{B.39}$$

$$+ Ke^{-r(T-t)}\frac{1}{x^{2\beta}}\frac{\partial^2\hat{h}^{2+2\beta}}{\partial v^2}(t,v)\,\mathcal{N}(d_4(t,x,v)) \tag{B.40}$$

$$+ Ke^{-r(T-t)}\frac{2}{x^{2\beta}}\frac{\partial\hat{h}^{2\beta}}{\partial v}(t,v)\,\frac{\partial d_4}{\partial v}(t,x,v)\,n(d_4(t,x,v)) \tag{B.41}$$

$$+ Ke^{-r(T-t)}\left(\frac{\hat{h}(t,v)}{x}\right)^{2\beta}\left(\frac{\partial^2 d_4}{\partial v^2}(t,x,v) - \left(\frac{\partial d_4}{\partial v}(t,x,v)\right)^2 d_4(t,x,v)\right)n(d_4(t,x,v))\Bigg\}. \tag{B.42}$$

The upper bounds for (B.35) and (B.36) are derived as follows: if we recall the estimates (B.20)–(B.23) and the fact that the function $|y|^{\mu}n(y)$ is bounded for any $\mu \geq 0$ we conclude that

$$\left|x\frac{\partial^2 d_1}{\partial v^2}(t,x,v)\,n(d_1(t,x,v))\right| \leq Cx(1+e^{kv}) \leq C(1+x^k+e^{kv})$$
$$\left|x\left(\frac{\partial d_1}{\partial v}(t,x,v)\right)^2 d_1(t,x,v)\,n(d_1(t,x,v))\right| \leq C(1+x^k+e^{kv}).$$
$$\left|Ke^{-r(T-t)}\frac{\partial^2 d_2}{\partial v^2}(t,x,v)\,n(d_2(t,x,v))\right| \leq C(1+e^{kv})$$
$$\left|Ke^{-r(T-t)}\left(\frac{\partial d_2}{\partial v}(t,x,v)\right)^2 d_2(t,x,v)\,n(d_2(t,x,v))\right| \leq C(1+e^{kv}).$$

and therefore $|(\text{B.35})|, |(\text{B.36})| \leq C(1+x^k+e^{kv})$ on the domain $t \in [0,T]$. Moreover, by virtue of the upper bounds (B.24)–(B.28), (B.14) and (B.16) we can likewise conclude that $|(\text{B.39})|, |(\text{B.42})| \leq C(1+x^k+e^{kv})$.

The estimates for the remaining terms are obtained analogously. For instance

$$|(\text{B.38})| \leq C\left(\frac{\hat{h}(t,v)}{x}\right)^{1+2\beta}\hat{h}(t,v)\,\frac{\partial\gamma^2}{\partial v}(t,T,v)\,(1+e^{kv})\big(1+|d_3(t,x,v)|\big)\,n(d_3(t,x,v)) \leq C(1+x^k+e^{kv}),$$

and the same kind of arguments show that $|(\text{B.37})| \leq C(1+x^k+e^{kv})$, $|(\text{B.40})| \leq C(1+x^k+e^{kv})$ and $|(\text{B.41})| \leq C(1+x^k+e^{kv})$. Hence

$$\mathcal{L}_2\hat{f}_0(t,x,v) \leq C(1+x^k+e^{kv}) \tag{B.43}$$

for some constants $C > 0, k \in \mathbb{N}$, which is the desired conclusion.

B.4 The function $\mathcal{L}_1\hat{f}_1$

We shall now deduce an estimate for $\mathcal{L}_1\hat{f}_1(t,x,v) = \rho x e^v \frac{\partial^2}{\partial x\partial v}\hat{f}_1(t,x,v)$, where $\hat{f}_1(t,x,v)$ is given in Equation (3.38).

Our first step is to claim that $\hat{f}_1(t,x,v)$ can be equivalently written as

$$\hat{f}_1(t,x,v) = \int_t^T e^{-r(s-t)}\int_{\hat{h}(s,V_s^{t,v})}^{\infty} G^+(w,s,x,t;v)\,\mathcal{L}_1\hat{f}_0(s,w,V_s^{t,v})\,dw\,ds \tag{B.44}$$

where $G^+(w,s,x,t;v)$ is the Green function from (3.37). Indeed, the adaptation of formula (4.96) of Ilyin et al. [29] to the case of an unbounded and time-dependent domain yields that the solution of the boundary value problem (3.29) admits the integral representation formula (B.44), so the equivalence of the two formulas follows from our decomposition $\hat{f}_1 = \hat{f}_1^{(A)} - \hat{f}_1^{(B)}$, combined with the fact that each of the boundary value problems (3.29), (3.30) and (3.31), associated respectively to $\hat{f}_1$, $\hat{f}_1^{(A)}$ and $\hat{f}_1^{(B)}$, has a unique solution. As we will see, the representation (B.44) is convenient for the estimation of $\mathcal{L}_1\hat{f}_1(t,x,v)$.

By the Leibniz integral rule,

$$\mathcal{L}_1\hat{f}_1(t,x,v) = \rho x e^v \frac{\partial^2}{\partial x \partial v}\left(\int_t^T e^{-r(s-t)} \int_{\hat{h}(s,V_s^{t,v})}^{\infty} G^+(w,s,x,t;v)\,\mathcal{L}_1\hat{f}_0(s,w,V_s^{t,v})\,dw\,ds\right)$$

$$= \rho x e^v \int_t^T e^{-r(s-t)} \frac{\partial}{\partial v}\left(\int_{\hat{h}(s,V_s^{t,v})}^{\infty} \frac{\partial G^+}{\partial x}(w,s,x,t;v)\,\mathcal{L}_1\hat{f}_0(s,w,V_s^{t,v})\,dw\,ds\right).$$

$$= \rho x e^v \int_t^T e^{-r(s-t)} \left(\int_{\hat{h}(s,V_s^{t,v})}^{\infty} \frac{\partial^2 G^+}{\partial x \partial v}(w,s,x,t;v)\,\mathcal{L}_1\hat{f}_0(s,w,V_s^{t,v})\,dw \right. \tag{B.45}$$

$$\left. + \int_{\hat{h}(s,V_s^{t,v})}^{\infty} \frac{\partial G^+}{\partial x}(w,s,x,t;v)\,\frac{\partial \mathcal{L}_1\hat{f}_0}{\partial v}(s,w,V_s^{t,v})\,\frac{\partial V_s^{t,v}}{\partial v}\,dw\right) ds \tag{B.46}$$

where we took into account the fact that $G^+(\hat{h}(s,V_s^{t,v}),s,x,t;v) = 0 = \frac{\partial G^+}{\partial x}(\hat{h}(s,V_s^{t,v}),s,x,t;v)$. Differentiating (3.37) we get

$$\frac{\partial G^+}{\partial x}(w,s,x,t;v) = \frac{1}{\sqrt{2\pi}\gamma^2(t,s,v)wx}$$

$$\times\left[\frac{1}{\gamma(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)\exp\left\{-\frac{1}{2\gamma^2(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)^2\right\}\right. \tag{B.47}$$

$$-\left(\frac{w}{\hat{h}(s,V_s^{t,v})}\right)^{2\beta}\frac{1}{\gamma(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)$$

$$\left.\times\exp\left\{-\frac{1}{2\gamma^2(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)^2\right\}\right]. \tag{B.48}$$

Let us first examine the term (B.46). Our claim is that $\frac{\partial \mathcal{L}_1\hat{f}_0}{\partial v}(t,x,v) \leq C(1+x^k+e^{kv})$. In order to prove this claim, it is enough to show that the absolute value of the derivative with respect to v of the terms (B.30) and (B.31) is upper bounded by $C(1+x^k+e^{kv})$, because then the result follows by applying the same procedure to the terms (B.29), (B.32) and (B.33).

$$\left|\frac{\partial}{\partial v}\text{(B.30)}\right| = \frac{|1+2\beta|}{x^{2+2\beta}}\left|\frac{\partial^2\hat{h}^{2+2\beta}}{\partial v^2}(t,v)\,\mathcal{N}(d_3(t,x,v)) + \frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t,v)\frac{\partial d_3}{\partial v}(t,x,v)\,n(d_3(t,x,v))\right|.$$

Due to (B.18) and (B.16) we have

$$\frac{1}{x^{2+2\beta}}\left|\frac{\partial^2\hat{h}^{2+2\beta}}{\partial v^2}(t,v)\,\mathcal{N}(d_3(t,x,v))\right| \leq C(T-t)(1+x^k+e^{kv})$$

and using (B.24) it is also easily seen that

$$\frac{1}{x^{2+2\beta}}\left|\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t,v)\,\frac{\partial d_3}{\partial v}(t,x,v)\right| n(d_3(t,x,v)) \leq C(T-t)(1+x^k+e^{kv})$$

so it follows that $\left|\frac{\partial}{\partial v}\text{(B.30)}\right| \leq C(T-t)(1+x^k+e^{kv})$.

$$\left|\frac{\partial}{\partial v}\text{(B.31)}\right| = \left|\frac{1}{\hat{h}^{2+2\beta}(t,v)}\frac{\partial \hat{h}^{2+2\beta}}{\partial v}(t,v)\times\text{(B.31)} + (1+2\beta)\left(\frac{\hat{h}(t,v)}{x}\right)^{2+2\beta}\frac{\partial^2 d_3}{\partial v^2}(t,x,v)\right.$$

$$\left. + \left(\frac{\partial d_3}{\partial v}(t,x,v)\right)^2 d_3(t,x,v)\right| n(d_3(t,x,v))$$

$$\leq C(1+x^k+e^{kv})$$

where the inequality is straightforwardly obtained through (B.24) and (B.25). Therefore we can plug the inequality $\frac{\partial \mathcal{L}_1 \hat{f}_0}{\partial v}(t,x,v) \le C(1+x^k+e^{kv})$ (and also the inequality (B.5)) into (B.46) and conclude that

$$|\text{(B.46)}| \le Cxe^v \int_t^T \int_{\hat{h}(s,V_s^{t,v})}^{\infty} \left| \frac{\partial G^+}{\partial x}(w,s,x,t;v) \right| (1+w^k+e^{kV_s^{t,v}})\, dw\, ds$$

where $\left| \frac{\partial G^+}{\partial x}(w,s,x,t;v) \right| \le |\text{(B.47)}| + |\text{(B.48)}|$ and

$$\begin{aligned}
&\int_{\hat{h}(s,V_s^{t,v})}^{\infty} |\text{(B.47)}| \times (1+w^k+e^{kV_s^{t,v}})\, dw \\
&= \int_{\hat{h}(s,V_s^{t,v})}^{\infty} (1+w^k+e^{kV_s^{t,v}}) \left| \frac{1}{\gamma(t,s,v)} \left(\log\Big(\frac{w}{x}\Big) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v) \right) \right|^{\nu} \times \\
&\qquad\qquad \times \frac{1}{\sqrt{2\pi}\gamma^2(t,s,v)wx} \exp\left\{ \frac{1}{2\gamma^2(t,s,v)} \left(\log\Big(\frac{w}{x}\Big) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v) \right)^2 \right\} dw \\
&= \frac{1}{x\gamma(t,s,v)} \mathbb{E}\left[(1+W^k+e^{kV_s^{t,v}}) \left| \frac{1}{\sigma_W} (\log W - \mu_W) \right|^{\nu} \mathbb{1}_{\{W > \hat{h}(s,V_s^{t,v})\}} \right]
\end{aligned} \tag{B.49}$$

where $W \sim \mathrm{Lognormal}\big(\mu_W = \log x + r(s-t) - \frac{1}{2}\gamma^2(t,s,v),\ \sigma_W^2 = \gamma^2(t,s,v)\big)$ and $\nu = 1$; by the Cauchy-Schwarz moment inequality,

$$\begin{aligned}
\le \frac{C}{x\gamma(t,s,v)} \Big(\mathbb{E}\left[1 + W^{2k} + e^{2kV_s^{t,v}} \right] \Big)^{1/2} &\le \frac{C}{x\gamma(t,s,v)} \big(1 + e^{2k\mu_W + 2k^2\sigma_W^2} + e^{2kV_s^{t,v}}\big)^{1/2} \\
&\le \frac{C}{x\gamma(t,s,v)} (1+x^k+e^{kv})
\end{aligned}$$

where we have used the well-known formula for the moments of a lognormal random variable, as well as the estimates (B.1) and (B.8). Hence

$$\begin{aligned}
xe^v \int_t^T \int_{\hat{h}(s,V_s^{t,v})}^{\infty} |\text{(B.47)}| \times (1+w^k+e^{kV_s^{t,v}})\, dw\, ds &\le Ce^v(1+x^k+e^{kv}) \int_t^T \frac{1}{\gamma(t,s,v)}\, ds \\
&\le C(1+x^k+e^{kv}) \int_t^T \frac{1}{\sqrt{s-t}}\, ds \le C\sqrt{T-t}(1+x^k+e^{kv}).
\end{aligned}$$

Similar computations show that $xe^v \int_t^T \int_{\hat{h}(s,V_s^{t,v})}^{\infty} |\text{(B.48)}| \times (1+w^k+e^{kV_s^{t,v}})\, dw\, ds \le C\sqrt{T-t}(1+x^k+e^{kv})$, and we conclude that

$$|\text{(B.46)}| \le C\sqrt{T-t}(1+x^k+e^{kv}).$$

We still need to obtain an estimate for the term (B.45):

$$\frac{\partial^2 G^+}{\partial x \partial v}(w,s,x,t;v) = \frac{-2}{\gamma(t,s,v)} \frac{\partial \gamma}{\partial v}(t,s,v) \frac{\partial G^+}{\partial x}(w,s,x,t;v) \tag{B.50}$$

$$\begin{aligned}
&+ \frac{1}{\sqrt{2\pi}\gamma^2(t,s,v)wx} \left[\frac{\partial \gamma}{\partial v}(t,s,v) \left(1 - \frac{1}{\gamma^2(t,s,v)} \left(\log\Big(\frac{w}{x}\Big) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v) \right) \right) \right. \\
&\qquad \times \left(1 - \frac{1}{\gamma^2(t,s,v)} \left(\log\Big(\frac{w}{x}\Big) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v) \right)^2 \right) \\
&\qquad \times \exp\left\{ -\frac{1}{2\gamma^2(t,s,v)} \left(\log\Big(\frac{w}{x}\Big) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v) \right)^2 \right\}
\end{aligned} \tag{B.51}$$

$$
\begin{aligned}
&+\Big(\frac{w}{\hat h(s,V_s^{t,v})}\Big)^{2\beta}\Bigg\{\beta(1+2\beta)\frac{\partial\gamma^2}{\partial v}(s,T,V_s^{t,v})\frac{\partial V_s^{t,v}}{\partial v}\\
&\qquad\times\frac{1}{\gamma(t,s,v)}\bigg(\log\Big(\frac{\hat h^2(s,V_s^{t,v})}{wx}\Big)-r(s-t)+\frac12\gamma^2(t,s,v)\bigg)\\
&\qquad-\bigg(\frac{\partial\gamma}{\partial v}(t,s,v)\bigg[1-\frac{1}{\gamma^2(t,s,v)}\bigg(\log\Big(\frac{\hat h^2(s,V_s^{t,v})}{wx}\Big)-r(s-t)+\frac12\gamma^2(t,s,v)\bigg)\bigg]\\
&\qquad\qquad+\frac{2}{\gamma(t,s,v)}\frac{\partial\log\hat h}{\partial v}(s,V_s^{t,v})\frac{\partial V_s^{t,v}}{\partial v}\bigg)\\
&\qquad\times\bigg(1-\frac{1}{\gamma^2(t,s,v)}\bigg(\log\Big(\frac{\hat h^2(s,V_s^{t,v})}{wx}\Big)-r(s-t)+\frac12\gamma^2(t,s,v)\bigg)^2\bigg)\Bigg\}\\
&\times\exp\bigg\{-\frac{1}{2\gamma^2(t,s,v)}\bigg(\log\Big(\frac{\hat h^2(s,V_s^{t,v})}{wx}\Big)-r(s-t)+\frac12\gamma^2(t,s,v)\bigg)^2\bigg\}\Bigg].
\end{aligned}
\tag{B.52}
$$

We have

$$
|\text{(B.45)}|\le Cxe^v\int_t^T\int_{\hat h(s,V_s^{t,v})}^\infty\big(\big|\text{(B.50)}\big|+\big|\text{(B.51)}\big|+\big|\text{(B.52)}\big|\big)(1+w^k+e^{kV_s^{t,v}})\,dw\,ds.
$$

If we observe that $\frac{1}{\gamma(t,s,v)}\frac{\partial\gamma}{\partial v}(t,s,v)\le C(1+e^{kv})$ and recall our previous estimate for the integral $xe^v\int_t^T\int_{\hat h(s,V_s^{t,v})}^\infty\big|\frac{\partial G^+}{\partial x}(w,s,x,t;v)\big|(1+w^k+e^{kV_s^{t,v}})\,dw\,ds$, we conclude that

$$
xe^v\int_t^T\int_{\hat h(s,V_s^{t,v})}^\infty\big|\text{(B.50)}\big|(1+w^k+e^{kV_s^{t,v}})\,dw\,ds\le C\sqrt{T-t}(1+x^k+e^{kv}).
$$

In turn,

$$
\begin{aligned}
&\int_{\hat h(s,V_s^{t,v})}^\infty|\text{(B.51)}|\times(1+w^k+e^{kV_s^{t,v}})\,dw\\
&\quad\le Cxe^v\frac{\partial\gamma}{\partial v}(t,s,v)\Big(1+\frac{1}{\gamma(t,s,v)}\Big)\\
&\qquad\times\int_{\hat h(s,V_s^{t,v})}^\infty(1+w^k+e^{kV_s^{t,v}})\bigg(1+\bigg|\frac{1}{\gamma(t,s,v)}\bigg(\log\Big(\frac{w}{x}\Big)-r(s-t)+\frac12\gamma^2(t,s,v)\bigg)\bigg|^3\bigg)\\
&\qquad\qquad\times\frac{1}{\sqrt{2\pi}\gamma^2(t,s,v)wx}\exp\bigg\{-\frac{1}{2\gamma^2(t,s,v)}\bigg(\log\Big(\frac{w}{x}\Big)-r(s-t)+\frac12\gamma^2(t,s,v)\bigg)^2\bigg\}\,dw\\
&\quad\le\frac{C}{x\gamma(t,s,v)}(1+x^k+e^{kv})
\end{aligned}
$$

because the estimate we derived above for the integral in (B.49) is valid for any $\nu\ge0$, and using also that $\frac{\partial\gamma}{\partial v}(t,s,v)\big(1+\frac{1}{\gamma(t,s,v)}\big)\le C(1+e^{kv})$; consequently,

$$
xe^v\int_t^T\int_{\hat h(s,V_s^{t,v})}^\infty\big|\text{(B.51)}\big|(1+w^k+e^{kV_s^{t,v}})\,dw\,ds\le C\sqrt{T-t}(1+x^k+e^{kv}).
$$

The inequality $xe^v\int_t^T\int_{\hat h(s,V_s^{t,v})}^\infty\big|\text{(B.52)}\big|(1+w^k+e^{kV_s^{t,v}})\,dw\,ds\le C\sqrt{T-t}(1+x^k+e^{kv})$ is obtained in an analogous fashion. Hence

$$
|\text{(B.45)}|\le C\sqrt{T-t}(1+x^k+e^{kv})
$$

and we finally obtain the desired conclusion:

$$
\big|\mathcal{L}_1\hat f_1(t,x,v)\big|\le C\sqrt{T-t}(1+x^k+e^{kv}).
\tag{B.53}
$$

B.5 The function $\mathcal{L}_2\hat{f}_1$

Our next goal is to derive an upper bound for

$$\mathcal{L}_2\hat{f}_1(t,x,v) = \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\int_t^T e^{-r(s-t)}\int_{\hat{h}(s,V_s^{t,v})}^{\infty} G^+(w,s,x,t;v)\mathcal{L}_1\hat{f}_0(s,w,V_s^{t,v})\,dw\,ds\right)$$

$$= \frac{1}{2}\int_t^T e^{-r(s-t)}\left(\int_{\hat{h}(s,V_s^{t,v})}^{\infty} \frac{\partial^2 G^+}{\partial v^2}(w,s,x,t;v)\mathcal{L}_1\hat{f}_0(s,w,V_s^{t,v})\,dw \right. \tag{B.54}$$

$$+ 2\int_{\hat{h}(s,V_s^{t,v})}^{\infty} \frac{\partial G^+}{\partial v}(w,s,x,t;v)\frac{\partial \mathcal{L}_1\hat{f}_0}{\partial v}(s,w,V_s^{t,v})\frac{\partial V_s^{t,v}}{\partial v}\,dw \tag{B.55}$$

$$+ \int_{\hat{h}(s,V_s^{t,v})}^{\infty} G^+(w,s,x,t;v)\frac{\partial^2 \mathcal{L}_1\hat{f}_0}{\partial v^2}(s,w,V_s^{t,v})\left(\frac{\partial V_s^{t,v}}{\partial v}\right)^2 dw \tag{B.56}$$

$$\left. + \int_{\hat{h}(s,V_s^{t,v})}^{\infty} G^+(w,s,x,t;v)\frac{\partial \mathcal{L}_1\hat{f}_0}{\partial v}(s,w,V_s^{t,v})\frac{\partial^2 V_s^{t,v}}{\partial v^2}\,dw\right)ds \tag{B.57}$$

where we have employed again the integral representation (B.44) for $\hat{f}_1(t,x,v)$, as well as the fact that $G^+(\hat{h}(s,V_s^{t,v}),s,x,t;v) = 0 = \frac{\partial G^+}{\partial v}(\hat{h}(s,V_s^{t,v}),s,x,t;v)$. Now we compute the derivative of (3.37) with respect to v:

$$\frac{\partial G^+}{\partial v}(w,s,x,t;v) =$$

$$= \frac{1}{\sqrt{2\pi}\gamma(t,s,v)w}\left[\frac{\partial\gamma}{\partial v}(t,s,v)\left(\left\{\frac{1}{\gamma^2(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right) - 1\right\}\right.\right.$$

$$\left.\times \frac{1}{\gamma(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right) - \frac{1}{\gamma(t,s,v)}\right)$$

$$\times \exp\left\{-\frac{1}{2\gamma^2(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)^2\right\}$$

$$+\left(\left\{\frac{\partial\gamma}{\partial v}(t,s,v)\left(1 - \frac{1}{\gamma^2(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)\right)\right.\right.$$

$$\left. + \frac{2}{\gamma(t,s,v)}\frac{\partial \log\hat{h}}{\partial v}(s,V_s^{t,v})\frac{\partial V_s^{t,v}}{\partial v}\right\}$$

$$\times \frac{1}{\gamma^2(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)$$

$$\left. + \frac{1}{\gamma(t,s,v)}\frac{\partial\gamma}{\partial v}(t,s,v) - \hat{h}^{2\beta}(s,V_s^{t,v})\frac{\partial \hat{h}^{-2\beta}}{\partial v}(s,V_s^{t,v})\right)$$

$$\left.\times \left(\frac{w}{\hat{h}(s,V_s^{t,v})}\right)^{2\beta}\exp\left\{-\frac{1}{2\gamma^2(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)^2\right\}\right]$$

Using the estimates we obtained so far, it is easy to see that we can decompose $\frac{\partial G^+}{\partial v}(w,s,x,t;v)$ into a sum of terms, each of which with an absolute value that is upper bounded by a function of the type

$$\frac{1+w^k+e^{kv}}{\sqrt{2\pi}\gamma^2(t,s,v)wx}\left|\frac{1}{\gamma(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)\right|^{\nu}\times$$

$$\times \exp\left\{\frac{1}{2\gamma^2(t,s,v)}\left(\log\left(\frac{w}{x}\right) - r(s-t) + \frac{1}{2}\gamma^2(t,s,v)\right)^2\right\}$$

or

$$\frac{1+w^k+e^{kv}}{\sqrt{2\pi}\gamma^2(t,s,v)wx}\left|\frac{1}{\gamma(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right)-r(s-t)+\frac{1}{2}\gamma^2(t,s,v)\right)\right|^\nu\times$$
$$\times\exp\left\{-\frac{1}{2\gamma^2(t,s,v)}\left(\log\left(\frac{\hat{h}^2(s,V_s^{t,v})}{wx}\right)-r(s-t)+\frac{1}{2}\gamma^2(t,s,v)\right)^2\right\}$$

where $\nu\geq 0$. But we have already established that $\frac{\partial\mathcal{L}_1\hat{f}_0}{\partial v}(t,x,v)\leq C(1+x^k+e^{kv})$, so we just need to use the same method from our estimation of (B.46) to conclude that $\left|\text{(B.55)}\right|\leq C\sqrt{T-t}(1+x^k+e^{kv})$. The derivation of the inequality $\left|\text{(B.57)}\right|\leq C\sqrt{T-t}(1+x^k+e^{kv})$ is analogous.

Moreover, if we compute $\frac{\partial^2 G^+}{\partial v^2}(w,s,t,x;v)$ by differentiating again with respect to v, we obtain an expression which we can also decompose into a sum of terms. The absolute value of each of these summands is again upper bounded by functions of the same type; accordingly, the estimate $\left|\text{(B.54)}\right|\leq C\sqrt{T-t}(1+x^k+e^{kv})$ holds as well. (For the sake of brevity we are omitting the lengthy analytical expression for $\frac{\partial^2 G^+}{\partial v^2}(w,s,t,x;v)$.)

To assure that the same upper bound holds for the term (B.56), it is clearly enough to show that $\frac{\partial^2\mathcal{L}_1\hat{f}_0}{\partial v^2}(t,x,v)\leq C(1+x^k+e^{kv})$. Once again, we show that the absolute value of the second derivative with respect to v of the terms (B.30) and (B.31) is upper bounded by $C(1+x^k+e^{kv})$, the other terms being handled similarly:

$$\begin{aligned}\left|\frac{\partial^2}{\partial v^2}\text{(B.30)}\right| = \frac{|1+2\beta|}{x^{2+2\beta}}\bigg|&\frac{\partial^3\hat{h}^{2+2\beta}}{\partial v^3}(t,v)\,\mathcal{N}(d_3(t,x,v))+2\,\frac{\partial^2\hat{h}^{2+2\beta}}{\partial v^2}(t,v)\,\frac{\partial d_3}{\partial v}(t,x,v)\,n(d_3(t,x,v))\\&+\frac{\partial\hat{h}^{2+2\beta}}{\partial v}(t,v)\left\{\frac{\partial^2 d_3}{\partial v^2}(t,x,v)-\left(\frac{\partial d_3}{\partial v}(t,x,v)\right)^2 d_3(t,x,v)\right\}n(d_3(t,x,v))\bigg|\\
\left|\frac{\partial^2}{\partial v^2}\text{(B.31)}\right| = \frac{|1+2\beta|}{x^{2+2\beta}}\bigg|&\frac{\partial^2\hat{h}^{2+2\beta}}{\partial v^2}(t,v)\,\frac{\partial d_3}{\partial v}(t,x,v)\\&+2\frac{\partial\hat{h}^{2+2\beta}}{\partial v}(t,v)\left\{\frac{\partial^2 d_3}{\partial v^2}(t,x,v)-\left(\frac{\partial d_3}{\partial v}(t,x,v)\right)^2 d_3(t,x,v)\right\}\\&+\hat{h}^{2+2\beta}(t,v)\bigg\{\frac{\partial^3 d_3}{\partial v^3}(t,x,v)-3\frac{\partial^2 d_3}{\partial v^2}(t,x,v)\,\frac{\partial d_3}{\partial v}(t,x,v)\,d_3(t,x,v)\\&\qquad+\left(\frac{\partial d_3}{\partial v}(t,x,v)\right)^3\left(d_3^2(t,x,v)-1\right)\bigg\}\bigg|\,n(d_3(t,x,v))\end{aligned}$$

From the inequalities (B.24)–(B.26) it follows that

$$\left|\frac{\partial^2}{\partial v^2}\text{(B.30)}\right|\leq C(1+x^k+e^{kv})\qquad\text{and}\qquad\left|\frac{\partial^2}{\partial v^2}\text{(B.31)}\right|\leq C(1+x^k+e^{kv}).$$

Therefore $\left|\text{(B.56)}\right|\leq C\sqrt{T-t}(1+x^k+e^{kv})$, which allows us to conclude that

$$\left|\mathcal{L}_2\hat{f}_1(t,x,v)\right|\leq C\sqrt{T-t}(1+x^k+e^{kv})\tag{B.58}$$

as required.