Numerical Methods for Stochastic Differential Equations
with Applications to Finance

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Abstract

The pricing of financial derivatives is used to help investors to increase the expected returns and minimise the risk associated with an investment. Options, in particular, offer benefits such as limited risk and leverage and for this reason much research has been dedicated to the development of models that accurately price options. The most celebrated model for option pricing is the Black-Scholes model, however when reduced to the heat equation, the simplified assumptions lead to the mispricing of options. Thus, extensions of this model have been developed, that consider a variation of the volatility and of the interest rate. In this work, we review the classical approach using finite difference schemes for the heat equation, and then we apply numerical schemes for stochastic differential equations - Euler-Maruyama and Milstein schemes, using Monte Carlo simulations, which will be the main focus of this thesis.

We start by considering asset models where the volatility and the interest rate are time-dependent coefficients and then extend these models to coefficients that also depend on the price of the underlying asset. We will see that as long as sufficient conditions on the coefficients are satisfied, the numerical approximations will converge in weak and strong sense, to the price of European call options. We conclude with an implementation of the Heston model for stochastic volatility, showing that will also present good convergence rates.

1 Introduction

In this paper we will study some extensions to the Black-Scholes model that relax some of the assumptions underlying this model. We start with an overview of the classical Black-Scholes model providing important results and then we introduce some extensions of this model. In section 3 we recall the finite difference schemes for the Black-Scholes PDE transformed into the heat equation and in section 4 discuss the Euler-Maruyama and Milstein methods as well as their orders of convergences. In section 5 we present some numerical experiments involving finite difference schemes for the Black-Scholes PDE and Monte Carlo simulations using numerical methods for SDE's. The paper concludes with a discussion of the numerical results.

2 Black-Scholes model and extensions

2.1 Classical Black-Scholes model

In the Black-Scholes model, the price of the underlying asset S, follows a geometric Brownian motion, which evolves according to the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

Using the concept of delta hedging, that is, the idea of eliminating (or reducing) a portfolio's exposure to the price of an underling asset and applying Itô's lemma to equation (1) the Black-Scholes PDE can be easily derived (see [4] and [2])

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{2}$$

Without any further conditions this equation has many different solutions. In order to find a unique solution for equation (2) we must now define some boundary and final conditions. The boudary and final conditions for a vanilla European call option are

$$C(S,T) = \max\{S-K,0\}, \quad C(0,t)=0, \quad C(S,t) \to S \ \ as \ \ S \to \infty$$

and for a European put option are

$$P(S,T) = \max\{K-S,0\}, \quad P(0,t)=Ke^{-r(T-t)}, \quad P(S,t) \to 0 \ \ as \ \ S \to \infty$$

It can be shown that under an appropriate change of variables, the Black-Scholes equation can be transformed into the heat equation. Consider the following change of variables used by Willmot [7]

$$S = Ke^{x}, \quad t = T - \frac{2\tau}{\sigma^2}, \quad C(S,t) = Ke^{\alpha x + \beta\tau}u(x,\tau)$$

where $\alpha = -\frac{1}{2}(c-1)$, $\beta = -\frac{1}{4}(c+1)^2$ and $c = \frac{2r}{\sigma^2}$. This change of variables ensures that the domain of the new variable $u(x,\tau)$ is $D_u = \{(x,\tau) : -\infty < x < \infty, \ \ 0 \le \tau \le \frac{\sigma^2}{2}T\}$. Under the new change of variables the initial and boundary conditions for a European call option are given by

$$u(x,0) = \max\{e^{\frac{1}{2}(c+1)x} - e^{\frac{1}{2}(c-1)x}, 0\}, \quad \lim_{x\to-\infty} u(x,\tau) = 0, \quad \lim_{x\to+\infty} u(x,\tau) - e^{\frac{1}{2}(c+1)x + \frac{1}{4}(c+1)^2\tau} = 0 \tag{3}$$

and the initial and boundary conditions for a European put option are given by

$$u(x,0) = \max\{e^{\frac{1}{2}(c-1)x} - e^{\frac{1}{2}(c+1)x}, 0\}, \quad \lim_{x\to-\infty} u(x,\tau) - e^{\frac{1}{2}(c-1)x + \frac{1}{4}(c-1)^2\tau} = 0, \quad \lim_{x\to+\infty} u(x,\tau) = 0 \tag{4}$$

For European call and put options it is possible to compute the exact solution to the Black-Scholes formula. One approach is to solve equation (2) subjected to the boundary conditions. For a European call the exact solution is given by

$$C(S,t) = \mathcal{N}(d_1)S - \mathcal{N}(d_2)Ke^{-r(T-t)} \tag{5}$$

where $\mathcal{N}(x) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-z^2}{2}}dz$ is the normal cumulative distribution function,
$d_1 = \dfrac{\log(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}$ and $d_2 = \dfrac{\log(S/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}$

2.2 Extensions to the Black-Scholes model

2.2.1 Time-dependent parameters

We will start by assuming that both the volatility and the expected return on the stock are continuous deterministic functions of time. The stock price dynamics are similar to the ones of the Black-Scholes model but now the stock price follows the stochastic differential equation

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dW_t \tag{6}$$

where σ and μ are continuous deterministic functions. To obtain the explicit solution to equation (6) we compute $d\log S_t$ by applying Itô's lemma with $f(t,S) = \log S$, $\mu(t,S) = \mu(t)S$ and $\sigma(t,S) = \sigma(t)S$

$$S_T = S_t e^{\int_t^T (\mu(x) - \frac{1}{2}\sigma^2(x))dx + \int_t^T \sigma(x)dW_x} \tag{7}$$

which means that even when both the interest rate and volatility are functions of time, the log of the stock returns has a normal distribution, i.e. $\log\left(\frac{S_T}{S_t}\right) \sim \mathcal{N}\left[\left(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2\right)(T-t), \bar{\sigma}^2(T-t)\right]$, where $\bar{\mu} = \frac{1}{T-t}\int_t^T \mu(x)dx$ and $\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma^2(x)dx$.

The derivation of the Black-Scholes equation with time-dependent parameters is identical to the one with constant parameters. The following result can be found in Wilmott [7].

$$\frac{\partial V}{\partial t} + r(t)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} - r(t)V = 0 \tag{8}$$

2.2.2 Stochastic volatility - Heston model

The price dynamics of the Heston model are similar to those of the Black-Scholes model but they also include a stochastic behaviour for the volatility process. The price and variance dynamics are given by

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t}S_t dW_t^S \\ d\nu_t = k(\theta - \nu_t)dt + \xi\sqrt{\nu_t}dW_t^{\nu} \\ dW_t^S dW_t^{\nu} = \rho dt \end{cases} \tag{9}$$

where S_t is the stock price at time t, $\nu(t)$ is the instantaneous variance, μ is the rate of return of the asset, θ is the long-run mean, k is the rate at which V_t reverts to θ, ξ is the volatility of volatility and W_t^S and W_t^ν are Wiener processes with correlation ρ.

The derivation of the Black Scholes PDE is similar to the one for the constant parameters. However, in order to have a riskless portfolio we now need another derivative written on the same underlying asset to account for the new source of randomness introduced by the stochastic volatility. The two derivatives differ by the maturity date or the strike price. For the derivation of the following result see Rouah [6].

$$\frac{\partial V}{\partial t}+\frac{1}{2}\nu S^2\frac{\partial V}{\partial S^2}+\rho\xi\nu S\frac{\partial^2 V}{\partial\nu\partial S}+\frac{1}{2}\xi^2\nu\frac{\partial^2 V}{\partial\nu^2}-rV+rS\frac{\partial V}{\partial S}+[k(\theta-\nu)-\lambda(S,\nu,t)]\frac{\partial V}{\partial\nu}=0 \tag{10}$$

3 Finite difference methods for the Black-Scholes model

In this section we will use finite difference schemes to approximate the Black-Scholes partial differential equation when reduced to the heat equation. We follow the works [7] and [1]. The idea in the finite difference method is to find a solution for the differential equation by approximating every partial derivative numerically.

Consider now the following initial-boundary value problem with homogeneous Dirichlet boundary condition for the heat equation

$$\frac{\partial u}{\partial\tau}(x,\tau)=\frac{\partial^2 u}{\partial x^2}(x,\tau) \tag{11}$$

where $u(x,\tau)$ is defined in $-\infty<x<\infty$ and $0\le\tau\le\frac{1}{2}\sigma^2T$.

$$\begin{cases}\frac{\partial u}{\partial\tau}(x,\tau)=\frac{\partial^2 u}{\partial x^2}(x,\tau) & (x,\tau)\in(x_{N^-},x_{N^+})\times(\tau_0,\tau_M)\\ u(x,0)=u_0(x) & x\in(x_{N^-},x_{N^+})\\ u(x_{N^-},\tau)=0 & \tau\in(\tau_0,\tau_M)\\ u(x_{N^+},\tau)=0 & \tau\in(\tau_0,\tau_M)\end{cases} \tag{12}$$

Where the first equation is the heat equation, and the other equations correspond to the boundary conditions. For the finite difference approximation we need to define a rectangular region in the domain of u and partition it to form a mesh of equally spaced points. The discretisation steps Δx and $\Delta\tau$ are defined as

$$\Delta x=\frac{x_n-x_{N^-}}{n},\qquad n=0,1,...,N-1,N$$

$$\Delta\tau=\frac{\tau_m-\tau_0}{m},\qquad m=0,1,...,M-1,M$$

where $\tau_0=0$, $x_0=x_{N^-}$ and $x_N=x_{N^+}$.

A θ scheme is a convex combination of an explicit and an implicit scheme, which takes the form $\theta\text{scheme}=(1-\theta)\text{explicit}+\theta\text{implicit}$, where θ is a parameter in [0,1]. Therefore, for the value $\theta=0$ we recover the explicit scheme, for $\theta=1$ the fully implicit scheme and for $\theta=\frac{1}{2}$ we recover the Crank-Nicolson scheme. Moreover, when $\theta\neq0$ we have an implicit scheme. In matrix form $M_{I,1-\theta}u_{m+1}=M_{E,\theta}u_m+b_{m+\theta}$, where

$$M_{I,1-\theta}=\begin{bmatrix}1+2\chi\theta & -\theta\chi & 0 & \cdots & 0\\ -\theta\chi & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\theta\chi\\ 0 & \cdots & 0 & -\theta\chi & 1+2\chi\theta\end{bmatrix}\qquad b_{m+\theta}=\begin{bmatrix}\chi(1-\theta)u_{N^-,m}+\chi\theta u_{N^-,m+1}\\ 0\\ \vdots\\ 0\\ \chi(1-\theta)u_{N^+,m}+\chi\theta u_{N^+,m+1}\end{bmatrix}$$

$$M_{E,\theta}=\begin{bmatrix}1-2\chi(1-\theta) & (1-\theta)\chi & 0 & \cdots & 0\\ (1-\theta)\chi & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & (1-\theta)\chi\\ 0 & \cdots & 0 & (1-\theta)\chi & 1-2\chi(1-\theta)\end{bmatrix}\qquad u_m=\begin{bmatrix}u_{N^-+1,m}\\ \vdots\\ u_{0,m}\\ \vdots\\ u_{N^+-1,m}\end{bmatrix}$$

$b_{m+\theta}$ is the vector with the boundary conditions. It can be shown [1] that a scheme is unconditionally stable for $\theta \geq \frac{1}{2}$ and for $\theta < \frac{1}{2}$ it is stable if $(1-2\theta)\chi \leq \frac{1}{2}$ holds, where $\chi = \frac{\Delta\tau}{(\Delta x)^2}$. Furthermore, a scheme is consistent of order 2 when $\theta = \frac{1}{2}$ and 1 consistent of order in the other cases ($\theta = 0$ and $\theta = 1$), which implies convergence of the same order as long as the scheme is stable as well.

4 Numerical methods for stochastic differential equations

4.1 Euler-Maruyama method

The Euler-Maruyama method is a generalisation of the Euler method for ordinary differential equations to stochastic differential equations and may be applied to an equation of the form

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad X(0) = X_0, \quad 0 \leq t \leq T. \tag{13}$$

where a and b are scalar functions and the initial condition $X(0)$ is a random variable. .

To find an approximate solution on the interval $[0, T]$ we discretise it into L equal subintervals of width Δt and approximate X values $X_0 < X_1 < ... < X_L$ at the respective t points $0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_L = T$. The explicit method takes the form

$$X_j = X_{j-1} + a(\tau_{j-1}, X_{j-1})\Delta t + b(\tau_{j-1}, X_{j-1})\Delta W_j \qquad j = 1, 2, ..., L \tag{14}$$

where X_j denotes our approximation, $\Delta t = \frac{T}{L}$ and $\Delta W_j = W(\tau_j) - W(\tau_{j-1})$.

Like in the case of ODE's, it is also possible to define implicit schemes, however, in this paper we will only consider semi-implicit schemes, that is, schemes in which only the non-random coefficients are implicit. The family of semi-implicit Euler schemes for SDE's is given by

$$X_j = X_{j-1} + [\theta a(\tau_j, X_j) + (1-\theta)a(\tau_{j-1}, X_{j-1})]\,\Delta t + b(\tau_{j-1}, X_{j-1})\Delta W_j \qquad j = 1, 2, ..., L \tag{15}$$

where $\theta \in [0, 1]$ represents the degree of implicitness. When $\theta = 0$ we have the explicit scheme (14), when $\theta = 1$ we recover the fully semi-implicit scheme and when $\frac{1}{2}$ we have a generalisation of the deterministic trapezoidal method.

4.2 Order of convergence

Now that we have introduced the Euler-Maruyama scheme for SDE's we want to determine if the method converges to the true solution. The definition of convergence of numerical methods for SDE's is very similar to that of ODE's. However, because of the random component we have more than one way of analysing convergence. In the first one we are interested in computing the difference between the approximate and exact solutions at specific mesh points, therefore this type of convergence is path dependent. In the second we are only interested in convergence in distribution, that is, in approximating the expectations of the Ito process. In this section we will follow Higham [3] and Kloeden and Platen [5].

4.2.1 Strong convergence

The strong order of convergence gives the rate at which the mean of the errors decreases as the time step tends to zero. In our numerical experiments we are interested in the error only at maturity T.

A general time discrete approximation converges strongly to the solution at time T if

$$\lim_{\Delta t \to 0} E[|X(T) - \tilde{X}(T)|] = 0 \tag{16}$$

where E denotes the expected value and $\tilde{X}(T)$ is the approximation of $X(t)$ at time $t = T$ computed with constant step Δt.

Further, we denote the error at final time T in the strong sense as

$$e_{\Delta t}^{\text{strong}} := E[|X(T) - \tilde{X}(T)|] \tag{17}$$

In order to be able to compare the accuracy of different numerical schemes we must introduce the concept of rate of convergence, which is similar to the concept for ODE's.

A general time discrete approximation is said to strongly converge with order γ at time T if there exists a constant C such that

$$e_{\Delta t}^{\text{strong}} \leq C\Delta t^{\gamma} \tag{18}$$

It can be show that under some conditions on a and b [5] , the family of Euler schemes has a strong order of convergence of $\frac{1}{2}$.

4.2.2 Weak convergence

The strong order of convergence is quite demanding to implement, as it requires that the whole path is known. However, we do not always need that much information, if we are interested, for instance, in just knowing the probability distribution of the solution $X(t)$. In this case it would suffice to know the rate at which the error of the means decreases as the time step tends to zero.

A method has weak convergence at time T if

$$\lim_{\Delta t \to 0} |E[f(X(T))] - E[f(\tilde{X}(T))]| = 0 \tag{19}$$

for all functions f in the polynomial class. Moreover, f needs to be smooth and display polynomial growth.

Like we did for the strong convergence we define the error at the final time T as

$$e_{\Delta t}^{\text{Weak}} := |E[f(X(T))] - E[f(\tilde{X}(T))]| \tag{20}$$

A method is said to weakly converge with order γ at time T if there exists a constant C such that

$$e_{\Delta t}^{\text{Weak}} \leq C \Delta t^{\gamma} \tag{21}$$

It can be shown that, under certain conditions on a and b, the family of Euler methods has weak order of convergence of 1.

4.3 Milstein method

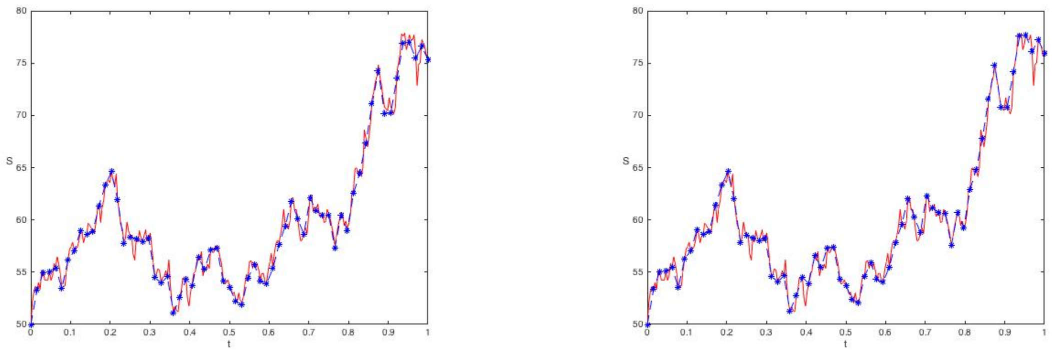
The Milstein's method uses Itô's lemma to add the second order term to the Euler-Maruyama scheme and increase the approximation's accuracy to 1. It has both strong and weak order of convergence equal to 1, under the usual assumptions on a and b. The Milstein's method is also applied to an equation of the form (13) and takes the form

$$X_j = X_{j-1} + a(\tau_{j-1}, X_{j-1})\Delta t + b(\tau_{j-1}, X_{j-1})\Delta W_j + \frac{1}{2} b(\tau_{j-1}, X_{j-1}) b'(\tau_{j-1}, X_{j-1}) \left[\Delta W_j^2 - \Delta t\right] \tag{22}$$

where $b' = \dfrac{\partial b}{\partial X}$. Similarly to the Euler-Maruyama, we can also define a family of semi-implicit Milstein schemes

$$\begin{aligned} X_j = X_{j-1} &+ \left[(1-\theta) a(\tau_j, X_j) + \theta a(\tau_{j-1}, X_{j-1})\right] \Delta t + b(\tau_{j-1}, X_{j-1})\Delta W_j + \\ &+ \frac{1}{2} b(\tau_{j-1}, X_{j-1}) b'(\tau_{j-1}, X_{j-1}) \left[\Delta W_j^2 - \Delta t\right] \end{aligned} \tag{23}$$

where $b' = \dfrac{\partial b}{\partial X}$ and $\theta \in [0, 1]$ is the degree of implicitness. When $\theta = 0$ we recover the explicit scheme (22), when $\theta = 1$ we obtain the fully semi-implicit scheme and when $\theta = \frac{1}{2}$ we obtain the generalisation of the deterministic trapezoidal method.



(a) True solution and Euler-Maruyama approximation (b) True solution and Milstein approximation

Figure 1: True solution (in red) and approximations (in blue) using Euler-Maruyama and Milstein methods.

Figure 1 shows the true and the approximate solution of equation (1) using the Euler-Maruyama and Milstein methods. The Brownian motion is sampled on the interval $[0, 1]$ and has a time step of $\delta t = 2^{-8}$, while the time step for the approximations is $\Delta t = 2^{-7}$. The values of the parameters are $\mu = 0.06$, $\sigma = 0.25$ and $S_0 = 50$.

4.4 Monte Carlo method for European option pricing

The idea of the Monte Carlo method when used in option pricing is to estimate the value of an option by simulating a large number of sample values of S_T, calculate the payoffs and find the estimated option price as the average of the discounted simulated payoffs. To approximate the price of the underlying asset at maturity, S_T, we will implement Euler-Maruyama and Milstein schemes.

In option pricing, the Monte Carlo method uses the risk neutral valuation, that is, uses $\mu = r$. Therefore, under the risk neutral measure, the value of an option at the present time is given by

$$V(S_0, 0) = E^{\mathbb{Q}}[h(S_0, 0)] \tag{24}$$

where $E^{\mathbb{Q}}[h(S_0, 0)]$ is the expected discounted payoff under the risk neutral measure. It is worth noting that we define (24) in a general way because when we consider models where the volatility and interest rate also depend on the price of the underlying asset the discount rate must be estimated as well. In fact, the main difference between the models we will study in this paper is in the way the payoffs of the options are discounted to the present.

5 Numerical experiments

Table 1 displays the approximate prices for European call and put options as well as the relative errors, at the point $V(S_0, t_0)$. The values of the parameters are the same as the ones used in the next table to allow comparisons between finite difference schemes and Monte Carlo simulations. In our choice of parameters we always use $T = 1$ to allow faster computations, specially when doing Monte Carlo simulations. For the time and space steps we chose $\Delta\tau = 0.001$ and $\Delta x = 0.05$, respectively.

Table 1: Relative errors in the approximation of the function V using finite differences, for $T = 1$, $r = 0.07$, $\sigma = 0.3$, $K = 100$, and $S_0 = \{80, 100, 120\}$

Scheme	Initial stock price	Approx. call price	True call price	Relative error	Approx. put price	True put price	Relative error
Explicit		5.02058		0.001585	18.26275		0.000588
Implicit	80	5.02862	5.01263	0.003189	18.26121	18.25201	0.000504
C-Nicolson		5.02441		0.002350	18.26180		0.000536
Explicit		15.19732		0.000866	8.44019		0.001147
Implicit	100	15.15414	15.21050	0.003706	8.38503	8.44988	0.007675
C-Nicolson		15.17587		0.002277	8.41275		0.004394
Explicit		30.29774		0.000491	3.54131		0.005380
Implicit	120	30.27447	30.28288	0.000278	3.50365	3.522260	0.005282
C-Nicolson		30.28600		0.000103	3.52239		0.000036

For the same parameters, we now compute Monte Carlo simulations to approximate the price V of European call and put options. The underlying asset is approximated using Euler-Maruyama (14) and Milstein (22) applied to equation $dS_t = rS_t dt + \sigma S_t dW_t$. The Brownian motion was discretised as $\Delta W = \sqrt{\delta t} z_i$ and the variables z_i were computed using the pseudorandom number generator *randn* from *Matlab*, which produces an independent pseudorandom number from the standard normal distribution. To generate the random paths we created an array with dimensions $1 \times N$ using $randn(1, N)$ and scaled by $\sqrt{\delta t}$. In order to be able to repeat the experiments, we set the initial state of the random number generator arbitrarily to 10 and used it for all the experiments to allow a better comparison of the results.

Table 2 displays the prices of call and put European options approximated using the Euler-Maruyama (14) and the Milstein (22) schemes with a time step $\Delta t = 2^{-7}$ and the relative weak errors in the approximation of their price.

Table 2: Relative errors in the approximation of the function V, for $T = 1$, $r = 0.07$, $\sigma = 0.3$ and $K = 100$, $M = 500\ 000$

Scheme	Initial stock	Approx Call	True Call	Relative error	Approx Put	True Put	Relative error
Euler-M	80	5.02412	5.03275	0.001716	18.2234	18.2289	0.000303
Milstein		5.02731		0.001082	18.2250		0.000215
Euler-M	100	15.23101	15.23283	0.000119	8.4203	8.4182	0.000245
Milstein		15.22552		0.000480	8.4128		0.000642
Euler-M	120	30.32419	30.32099	0.000105	3.5034	3.4956	0.002247
Milstein		30.31428		0.000221	3.4911		0.001265

Comparing the results on tables 1 and 2, the finite difference schemes and the Monte Carlo methods have similar performances, although the statistical results from the Monte Carlo simulations were slightly more accurate. In fact, we used 500 000 sample paths in the Monte Carlo simulations, which suggests that the error associated with the standard error should be in the order of 10^{-3} and the one associated with the discretisation should be in the order of 10^{-3} as we used a time step of 0.0078. On the other hand, in the finite difference schemes, as we used a time step of 0.001 and a space step of 0.05, the errors should be in the order of 10^{-3}, even for the Crank-Nicolson scheme, because the time term becomes too small. However, the Monte Carlo method converges rather slowly and needs a high number of paths to produce good results. Nonetheless, it allow to relax some assumptions, such as the constant parameters, as we will see next.

5.1 Time-dependent parameters

We now analyse the strong and weak orders of convergence of the Euler-Maruyama (14) and Milstein (22) schemes in the approximation of the value $V(S_0, 0)$ with initial stock price $S_0 = 80$, strike price $K = 100$ and maturity $T = 1$. We start by considering that the stock price follows the SDE $dS_t = r(t)S_t dt + \sigma(t)S_t dW_t$, where $\sigma(t) = 0.1 + 0.3t + 0.03\sin(30t)$ and $r(t) = 0.01 + 0.03t + 0.03\sin(60t)$ are sinusoidal functions, because both market volatility and the interest rate tend to exhibit oscillatory behaviour. The approximations were computed for seven different time steps using Monte Carlo simulations. The analysis of the order of convergence of the numerical schemes is carried out by using an approximation of the true solution with a small time step of 2^{-11}.

The strong error was then computed as the expectation of the absolute value of the difference of the discounted payoffs for an approximation with a time step 2^{-11} and approximations with time steps Δt, where $\Delta t = \{2^{-3}, 2^{-4}, ..., 2^{-9}\}$ that is

$$e^{strong} = E[|h(\tilde{S}_0, 0) - h(\tilde{S}_0, 0)^{\Delta t}|] \tag{25}$$

where $h(\tilde{S}_0, 0)$ is the discounted payoff computed using the reference time step of 2^{-11}.

The reference values for the true solution are 3.9247 and 3.9252 for the strong convergence and 4.0065 and 4.0066 for the weak convergence, using the Euler-Maruyama and Milstein schemes, respectively.

As strong convergence is computationally costlier we only sampled through 5000 sample paths. The weak error was computed over 500 000 sample paths, by taking the absolute value of the difference between the expected values of the discounted payoffs, for meshes of size Δt and $\frac{\Delta t}{2}$, i.e.

$$e^{weak} = |E[h(\tilde{S}_0, 0)] - E[h(\tilde{S}_0, 0)]^{\Delta t}| \tag{26}$$

Figure 2 shows the plotted values of the strong and weak errors for the Euler-Maruyama and Milstein methods on a loglog scale, which are represented by the blue asterisk. The red dashed line is a linear regression, estimated using a least squares fit, and the green dashed line is the reference slope for the theoretical order of convergence.

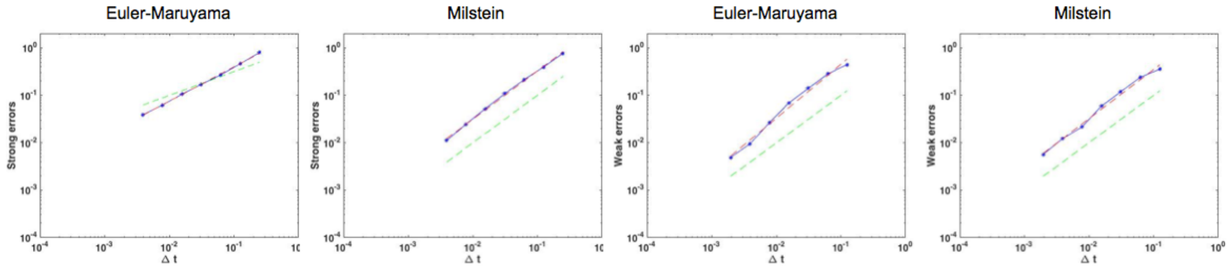


Figure 2: Strong (pictures on the left) and weak (pictures on the right) convergence of the Euler-Maruyama and Milstein schemes when the interest rate and volatility are functions of time.

Figure 2 suggests that the experimental strong order of convergence agrees with the theoreticall value for the Milstein scheme and it is above the theoretical value for the Euler-Maruyama scheme. Further, the slope of the estimated linear regression, which gives the experimental weak order of convergence, is $q = 1.1321$ for the Euler-Maruyama scheme and $q = 1.0346$ for the Milstein scheme. This means that, for both schemes, the experimental orders of convergence are above or in accordance with the theory.

5.2 Time and asset price dependent parameters

We now assume that the stock price follows a stochastic differential equation $dS_t = r(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$ where the interest rate and volatility functions are $r(t, S_t) = \frac{0.05}{1+t} + \frac{0.05}{1+S_t}$ and $\sigma(t, S_t) = \frac{0.2}{1+t} + \frac{0.2}{1+S_t}$, respectively. Figure 3 shows the plotted values of the strong and weak errors for the Euler-Maruyama and Milstein methods on a loglog scale, which are represented by the blue asterisk. The red dashed line is a linear regression, estimated using a least squares fit, and the green dashed line is the reference slope for the theoretical order of convergence. The strong and weak errors are computed as in (26) and (25), using 50 000 sample paths for the weak error and 5000 for the strong error. The price of the underlying asset is approximated using explicit Euler-Maruyama (14) and Milstein (22) schemes, then the payoffs are computed for each S_T^i and discounted to the present as $h(S_0^i, 0) = \prod_{n=0}^{L-1} e^{-r_n(t_{n+1}-t_n)} h(S_T^{i,L}, T), \quad i = 1, 2, ..., M$ to obtain the price of the option.

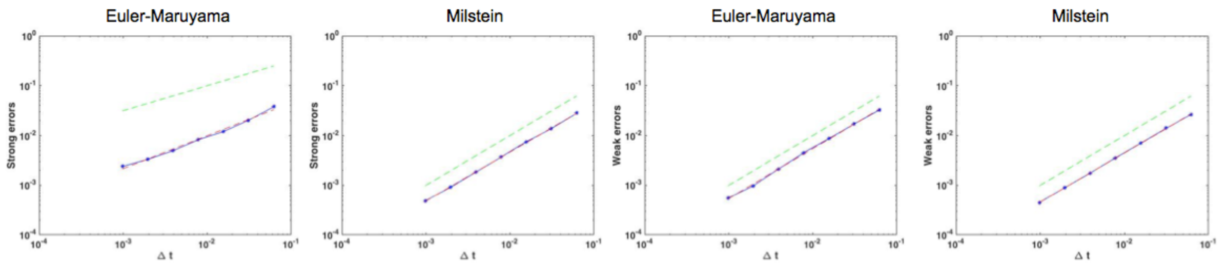


Figure 3: Strong (pictures on the left) and weak (pictures on the right) convergence of the Euler-Maruyama and Milstein schemes when the interest rate and volatility are functions of time and of the price of the underlying asset.

The estimated regression for the strong convergence in the Euler-Maruyama method has a slope of $q = 0.8251$, which is above the theoretically predicted value and for the Milstein scheme it is $q = 1.0588$, which is in good accordance with the theoretical value. Moreover, the slope of the estimated regressions for the weak convergence in the Euler-Maruyama and Milstein schemes are $q = 1.0448$ and $q = 1.0430$, respectively, which means that the experimental weak orders of convergence of the schemes agree with the theoretical order of convergence.

Now to approximate the price of the stock at each time step we used the predictor-corrector method with the explicit Euler-Maruyama method $\bar{S}_j = S_{j-1} + r(\tau_j, S_j)S_j\Delta t + \sigma(\tau_{j-1}, S_{j-1})S_{j-1}\sqrt{\Delta t}z_i$ as the predictor and the generalisation of the deterministic trapezoidal method $S_j = S_{j-1} + \frac{1}{2}\left[r(\tau_j, \bar{S}_j)\bar{S}_j + r(\tau_{j-1}, S_{j-1})S_{j-1}\right]\Delta t + \sigma(\tau_{j-1}, S_{j-1})S_{j-1}\sqrt{\Delta t}z_i$ as the corrector method. We do the same using the Milstein schemes. The strong and weak errors were computed as in (25) and (26), respectively and the number of sample paths is only 500 due to computational costs.

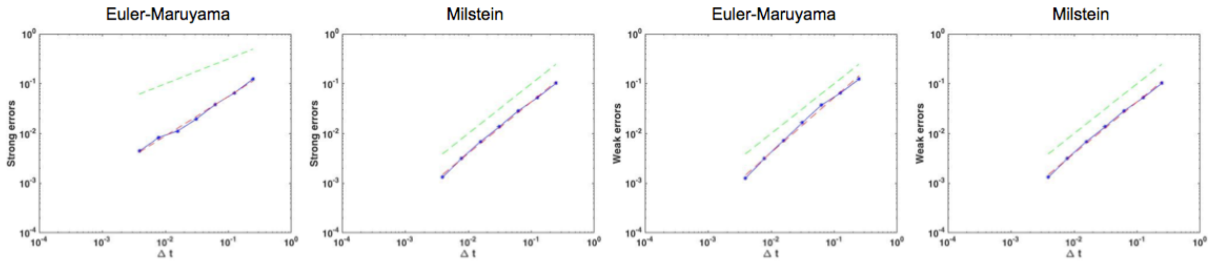


Figure 4: Strong (pictures on the left) and weak (pictures on the right) convergence of the Euler and Milstein predictor-corrector schemes, when the interest rate and volatility are functions of time and of the price of the underlying asset.

The Euler-Maruyama predictor-corrector method has an experimental strong order of convergence of $q = 0.8166$ and an experimental weak order of convergence of $q = 1.1044$ while the Milstein predictor-corrector method has an experimental strong order of convergence $q = 1.0361$ and an experimental weak order of convergence of $q = 1.0360$. So, we can conclude that the orders of convergence of both numerical schemes are in good accordance with the theory and that the strong order of convergence for the Euler-Maruyama scheme is actually above the theoretical value.

5.3 Heston model

To numerically test the Heston model, we chose the following parameters $T = 1$, $r = 0.0015$, $\nu_0 = 0.2$, $\theta = 0.2$, $\xi = 1.4$, $k = 6$, $\rho = -0.7$, $S_0 = 100$, $K = 100$, which satisfy the Feller condition $\frac{2k\theta}{\xi^2} > 1$. One of the difficulties of the Heston model is how to choose the parameters, as they influence the shape of the volatility smile and can induce skewness in the distribution of the stock returns. Furthermore, the prices approximated by the model are quite parameter sensitive, so small changes in the parameters values lead to considerably different results.

Applying the Euler-Maruyama scheme (14) to equations (9) we get the following discretisations $S_t = S_{t-1} + rS_{t-1}\Delta t + \sqrt{\nu_{t-1}}S_{t-1}\sqrt{\Delta t}Z_t^s$ and $\nu_t = \nu_{t-1} + k(\theta - \nu_{t-1})\Delta t + \xi\sqrt{\nu_{t-1}}\sqrt{\Delta t}Z_t^\nu$, where $\{Z_t^s\}_{t\geq 0}$ and $\{Z_t^\nu\}_{t\geq 0}$ are standard normal random variables with correlation ρ. These variables can be expressed as a function of independent standard random variables $Z_t^s = Z_t^1$ and $Z_t^\nu = \rho Z_t^1 + \sqrt{1-\rho^2}Z_t^2$, where $\{Z_t^1\}_{t\geq 0}$ and $\{Z_t^2\}_{t\geq 0}$ are two independent standard normal random variables. The discretisation using the Milstein method is similar.

Figure 5 shows the plotted values of the strong and weak errors for the Euler-Maruyama and Milstein methods on a loglog scale, which are represented by the blue asterisk. The red dashed line is a linear regression, estimated using a least squares fit, and the green dashed line is the reference slope for the theoretical order of convergence. The strong and weak errors were computed as in (25) and (26) using 5000 and 500 000 sample paths, respectively

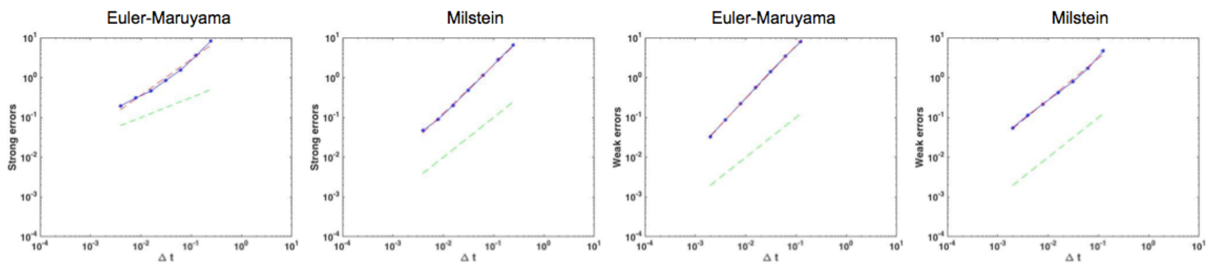


Figure 5: Strong (pictures on the left) and weak (pictures on the right) convergence of the explicit Euler-Maruyama and Milstein schemes when the volatility is modelled by the Heston model.

The estimated linear regression for the strong convergence rate of Euler-Maruyama has a slope of $q = 0.8976$ and the estimated linear regression for the weak convergence rate is $q = 1.3267$, which confirms that this method converges with a higher order of convergence than the one predicted in theory. Moreover, the Milstein scheme has an experimental strong order of convergence $q = 1.2090$ and an experimental weak order of convergence $q = 1.0363$. It is also interesting to note that the Euler-Maruyama scheme performs poorly for bigger time steps than the Milstein scheme, but has greater accuracy when the discretisation is more refined.

6 Conclusions

The approximations via finite difference schemes for PDE's were used to approximate the classical Black-Scholes equation, where the volatility and interest rate were constant parameters. The obtained results have good accuracy, with relative errors in the order of 10^{-3}. Moreover, the computation time was less than one second, even for the implicit schemes. However the use of constant volatility and interest rate is not realistic and the relaxation of these assumptions is better handled by numerical methods for SDE's in Monte Carlo simulations than by finite difference schemes for PDE's.

When compared to the finite difference schemes the accuracy of the methods were very similar, although slightly better in the Monte Carlo simulations. The use of Monte Carlo methods is justified when no reduction to PDEs is available. On the other hand the Monte Carlo simulations are very inefficient when compared to finite difference schemes, as it took almost one minute to run the algorithm and the results are not significantly better.

The strong and weak convergence results for the time-dependent model were above or in accordance with the theoretically predicted values. For the time and asset price dependent model the orders of convergence were also above or in accordance with the theoretically predicted values. For the Heston model, using the Euler-Maruyama method we obtained strong and weak orders of convergence above the value predicted by theory and the convergence results for the Milstein scheme were in good accordance with the theoretically predicted values. It is worth emphasizing that in all the models the strong orders of convergence for the Euler-Maruyama scheme were above the theoretical value.

Overall the Euler-Maruyama and the Milstein methods produced similar results in what concerns weak convergence. For this reason it is natural to choose the Euler-Maruyama scheme for smaller time steps, as it is simpler to implement and less burdensome and the Milstein scheme could be used for larger time steps. When it comes to strong convergence, the Milstein scheme has always a greater accuracy than the Euler-Maruyama scheme.

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